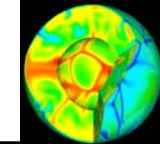
The Elastic Wave Equation



- Elastic waves in infinite homogeneous isotropic media

  Numerical simulations for simple sources

- Plane wave propagation in infinite media

  Frequency, wavenumber, wavelength

- Conditions at material discontinuities

  Snell's Law
  Reflection coefficients
  Free surface

- Reflection Seismology: an example from the Gulf of Mexico

Equations of motion

$$\boxed{\rho \partial_t^2 u_i = f_i + \partial_j \sigma_{ij}}$$

What are the solutions to this equation? At first we look at infinite homogeneous isotropic media, then:

$$\sigma_{ij} = \lambda \theta \delta_{ij} + 2\mu \varepsilon_{ij}$$

$$\sigma_{ij} = \lambda \partial_k u_k \delta_{ij} + \mu (\partial_i u_j + \partial_j u_i)$$

$$\rho \partial_t^2 u_i = f_i + \partial_j \left(\lambda \partial_k u_k \delta_{ij} + \mu (\partial_i u_j + \partial_j u_i) \right)$$

$$\rho \partial_t^2 u_i = f_i + \lambda \partial_i \partial_k u_k + \mu \partial_i \partial_j u_j + \mu \partial_j^2 u_i$$

Equations of motion – homogeneous media

$$\rho\partial_t^2 u_i = f_i + \lambda\partial_i\partial_k u_k + \mu\partial_i\partial_j u_j + \mu\partial_j^2 u_i$$

We can now simplify this equation using the curl and div operators

$$\nabla \bullet u = \partial_i u_i \qquad \nabla^2 = \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$$

and

$$\Delta\mathbf{u} = \nabla\nabla\bullet\mathbf{u} - \nabla\times\nabla\times\mathbf{u}$$

$$\boxed{\rho\partial_t^2\mathbf{u} = \mathbf{f} + (\lambda+2\mu)\nabla\nabla\bullet\mathbf{u} - \mu\nabla\times\nabla\times\mathbf{u}}$$

… this holds in any coordinate system …

This equation can be further simplified, separating the wavefield into curl free and div free parts

Equations of motion – P waves

$$\boxed{\rho \partial_t^2 \mathbf{u} = (\lambda + 2\mu)\nabla\nabla \bullet \mathbf{u} - \mu \nabla \times \nabla \times \mathbf{u}}$$

Let us apply the **div** operator to this equation, we obtain

$$\rho \partial_t^2 \theta = (\lambda + 2\mu)\Delta\theta$$

where

$$\theta = \nabla \mathbf{u} = \partial_{\mathbf{i}} u_i$$

or

Acoustic wave equation

$$\boxed{\partial_t^2 \theta = \frac{1}{\alpha^2}\Delta\theta}$$

P wave velocity

$$\boxed{\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}}$$

Equations of motion – shear waves

$$\boxed{\rho \partial_t^2 \mathbf{u} = (\lambda + 2\mu)\nabla\nabla \bullet \mathbf{u} - \mu \nabla \times \nabla \times \mathbf{u}}$$

Let us apply the curl operator to this equation, we obtain

$$\rho \partial_t^2 \nabla \times u_i = (\lambda + \mu)\nabla \times \nabla \theta + \mu \Delta (\nabla \times u_i)$$

we now make use of $\nabla \times \nabla \theta = 0$ and define

$$\nabla \times u_i = \varphi_i$$

to obtain

Wave equation for shear waves

$$\boxed{\partial_t^2 \varphi = \frac{1}{\beta^2} \Delta \varphi}$$

Shear wave velocity

$$\boxed{\beta = \sqrt{\frac{\mu}{\rho}}}$$

Elastodynamic Potentials

Any vector may be separated into scalar and vector potentials

$$u = \nabla\Phi + \nabla \times \Psi$$

where P is the potential for Φ waves and Ψ the potential for shear waves

$$\Rightarrow \theta = \Delta\Phi$$

$$\Rightarrow \varphi = \nabla \times u = \nabla \times \nabla \times \Psi = -\Delta\Psi$$

P-waves have no rotation

Shear waves have no change in volume

$$\boxed{\partial_t^2\theta = \alpha^2\Delta\theta}$$

$$\boxed{\partial_t^2\varphi = \beta^2\Delta\varphi}$$

Seismic Velocities

Material and Source	P-wave velocity (m/s)	shear wave velocity (m/s)
Water	1500	0
Loose sand	1800	500
Clay	1100-2500	
Sandstone	1400-4300	
Anhydrite, Gulf Coast	4100	
Conglomerate	2400	
Limestone	6030	3030
Granite	5640	2870
Granodiorite	4780	3100
Diorite	5780	3060
Basalt	6400	3200
Dunite	8000	4370
Gabbro	6450	3420

Solutions to the wave equation - general

Let us consider a region without sources

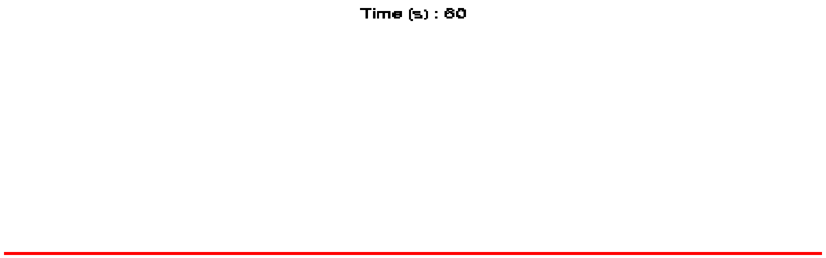
$$\boxed{\partial_t^2 \eta = c^2 \Delta \eta}$$

Where n could be either dilatation or the vector potential and c is either P- or shear-wave velocity. The general solution to this equation is:

$$\eta(x_i, t) = G(a_j x_j \pm ct)$$

Let us take a look at a 1-D example

Let us consider a region without sources



$$\boxed{\partial_t^2 \eta = c^2 \Delta \eta}$$

The most appropriate choice for G is of course the use of harmonic functions:

$$\boxed{u_i(x_i,t) = A_i \exp[ik(a_j x_j - ct)]}$$

Solutions to the wave equation - harmonic

… taking only the real part and considering only 1D we obtain

$$\boxed{u(x,t) = A\cos[k(x-ct)]}$$

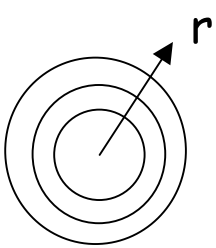
$$\boxed{k(x-ct) = kx - kct = \frac{2\pi}{\lambda}x - \omega t = \frac{2\pi}{\lambda}x - \frac{2\pi}{T}t}$$

c	wave speed
k	wavenumber
λ	wavelength
T	period
ω	frequency
A	amplitude

Spherical Waves

$$\boxed{\partial_t^2 \eta = c^2 \Delta \eta}$$

Let us assume that η is a function of the distance from the source



$$\Delta \eta = \partial_r^2 \eta + \frac{2}{r} \partial_r \eta = \frac{1}{c^2} \partial_2^t \eta$$

where we used the definition of the Laplace operator in spherical coordinates
let us define

$$\eta = \frac{\overline{\eta}}{r}$$

to obtain

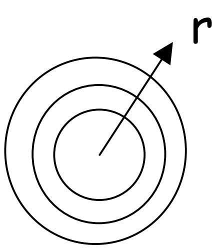
$$\boxed{\partial_t^2 \eta = c^2 \Delta \eta}$$

with the known solution $\overline{\eta} = f(r - \alpha t)$

Geometrical spreading

so a disturbance propagating away with spherical wavefronts decays like

$$\eta = \frac{1}{r} f(r - \alpha t) \qquad \eta \approx \frac{1}{r}$$



... this is the geometrical spreading for spherical waves, the amplitude decays proportional to 1/r.

If we had looked at cylindrical waves the result would have been that the waves decay as (e.g. surface waves)

$$\eta \approx \frac{1}{\sqrt{r}}$$

Plane waves

... what can we say about the direction of displacement, the **polarization** of seismic waves?

$$u = \nabla\Phi + \nabla\times\Psi \qquad \Rightarrow u = P + S$$

$$P = \nabla\Phi \qquad S = \nabla\times\Psi$$

... we now assume that the potentials have the well known form of plane harmonic waves

$$\Phi = A\exp i(\mathbf{k}\bullet\mathbf{x} - \omega t) \qquad \Psi = B\exp i(\mathbf{k}\bullet\mathbf{x} - \omega t)$$

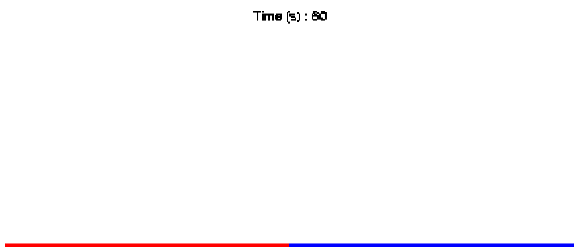
$$\boxed{P = A\mathbf{k}\exp i(\mathbf{k}\bullet\mathbf{x} - \omega t)} \qquad \boxed{S = \mathbf{k}\times B\exp i(\mathbf{k}\bullet\mathbf{x} - \omega t)}$$

P waves are longitudinal as P is parallel to k

shear waves are transverse because S is normal to the wave vector k

Heterogeneities

.. What happens if we have heterogeneities ?



Depending on the kind of reflection part or all of the signal is reflected or transmitted.

- What happens at a free surface?
- Can a P wave be converted in an S wave or vice versa?
- How big are the amplitudes of the reflected waves?

Boundary Conditions

... what happens when the material parameters change?

ρ_1 v_1

welded interface

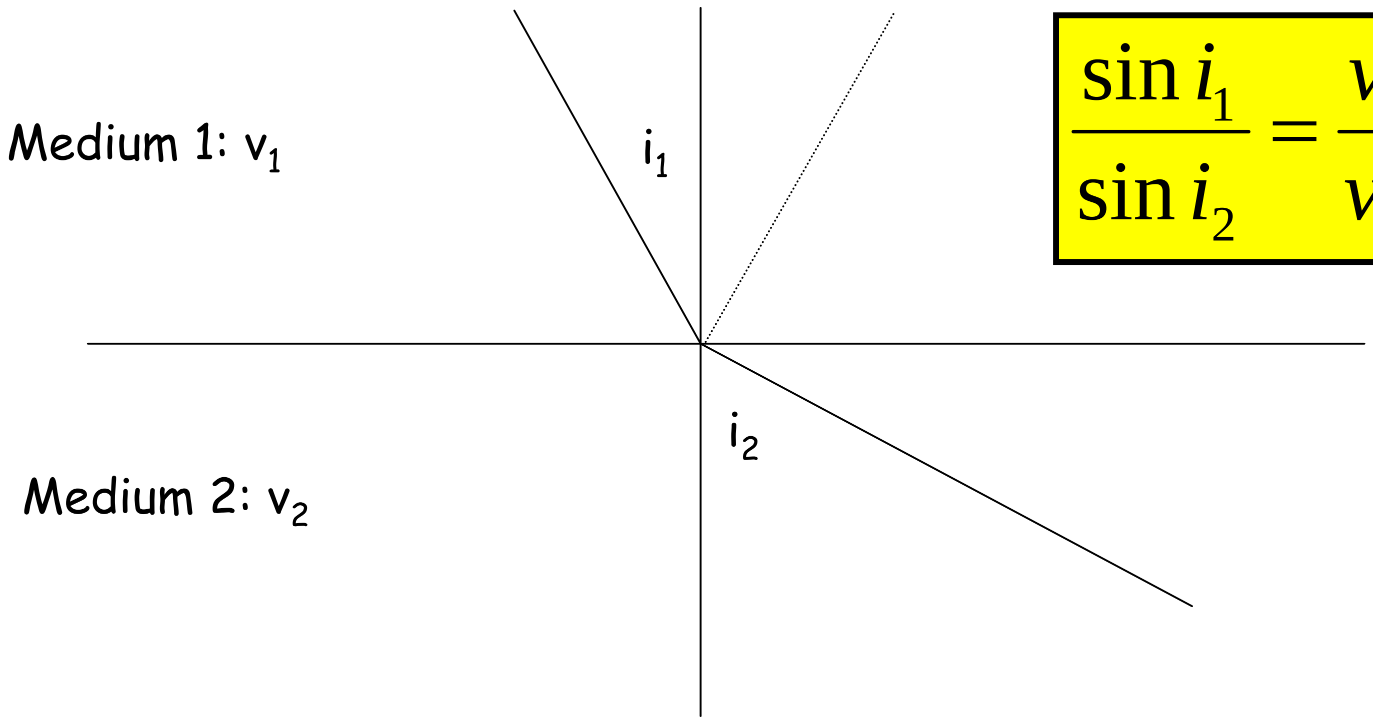
ρ_2 v_2

At a material interface we require continuity of displacement and traction

A special case is the free surface condition, where the surface tractions are zero.

Reflection and Transmission – Snell's Law

What happens at a (flat) material discontinuity?

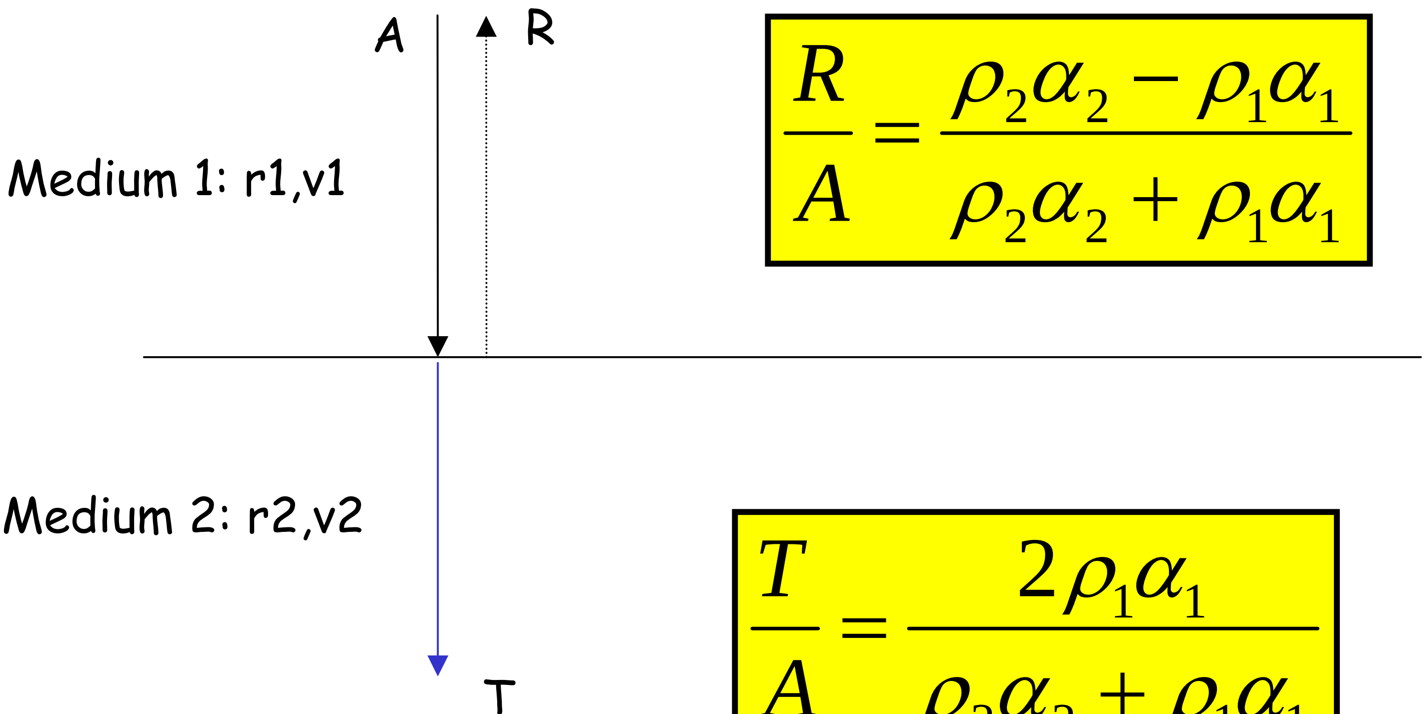


$$\boxed{\frac{\sin i_1}{\sin i_2} = \frac{v_1}{v_2}}$$

But how much is reflected, how much transmitted?

Reflection and Transmission coefficients

Let's take the most simple example: P-waves with normal incidence on a material interface



$$\boxed{\frac{R}{A} = \frac{\rho_2\alpha_2 - \rho_1\alpha_1}{\rho_2\alpha_2 + \rho_1\alpha_1}}$$

$$\boxed{\frac{T}{A} = \frac{2\rho_1\alpha_1}{\rho_2\alpha_2 + \rho_1\alpha_1}}$$

At oblique angles conversions from S-P, P-S have to be considered.

Reflection and Transmission – Ansatz

How can we calculate the amount of energy that is transmitted or reflected at a material discontinuity?

We know that in homogeneous media the displacement can be described by the corresponding potentials

$$u = \nabla\Phi + \nabla \times \Psi$$

in 2-D this yields

$$u_x = \partial_x \Phi - \partial_y \Psi_z$$

$$u_y = \partial_z \Psi_x - \partial_x \Psi_z$$

$$u_z = \partial_z \Phi + \partial_x \Psi_y$$

an incoming P wave has the form

$$\Phi = A_0 \exp i \frac{\omega}{\alpha}(a_j x_j - \alpha t)$$

Reflection and Transmission – Ansatz

... here a_i are the components of the vector normal to the wavefront : a_i=(cos e, 0, -sin e), where e is the angle between surface and ray direction, so that for the free surface

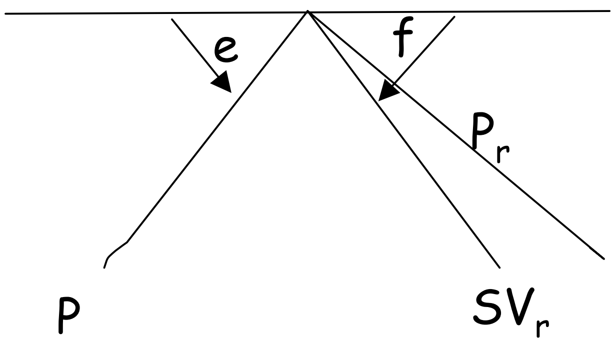
$$\Phi = A_0 \exp ik(x_1 - x_3 \tan e - ct) + A \exp ik(x_1 + x_3 \tan e - ct)$$

$$\Psi = B \exp ik'(x_1 + x_3 \tan f - c't)$$

where

$$c = \frac{\alpha}{\cos e} \qquad c' = \frac{\beta}{\cos f}$$

$$k = \frac{\omega}{\alpha} \cos e = \frac{\omega}{c} \qquad k' = \frac{\omega}{\beta} \cos f$$



what we know is that

$$\sigma_{xz} = 0$$

$$\sigma_{zz} = 0$$

Reflection and Transmission – Coefficients

... putting the equations for the potentials (displacements) into these equations leads to a relation between incident and reflected (transmitted) amplitudes

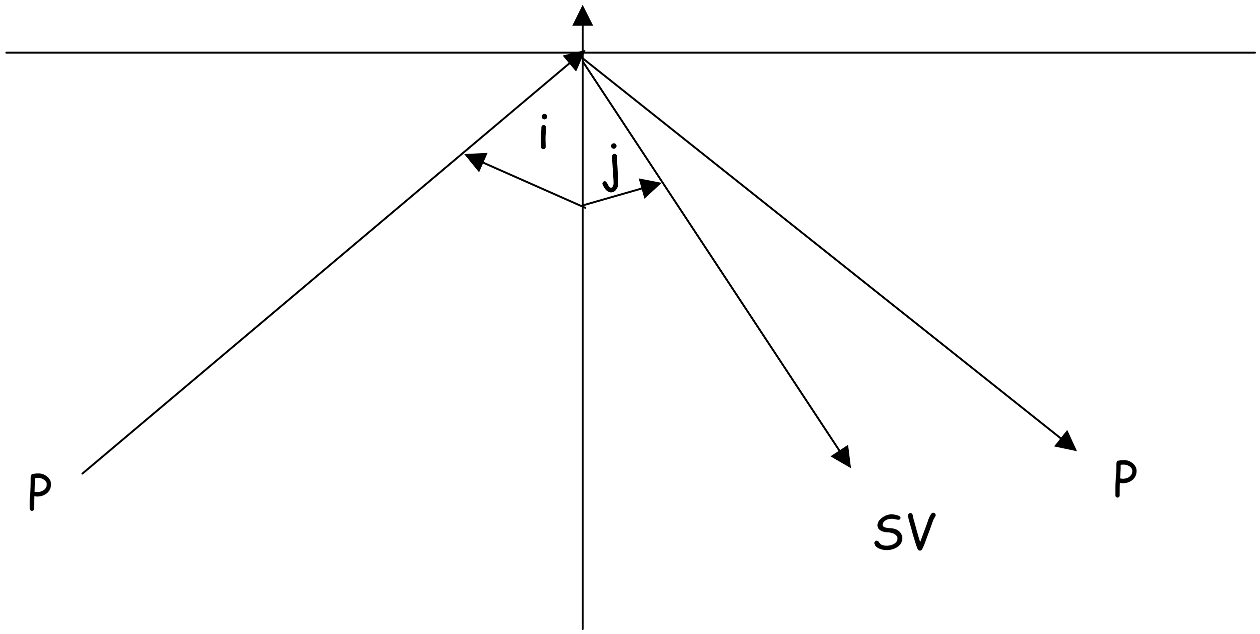
$$R_{PP}=\frac{A}{A_0}=\frac{4\tan e\tan f-(1-\tan^2 f)^2}{4\tan e\tan f+(1-\tan^2 f)^2}$$

$$R_{PS}=\frac{A}{A_0}=\frac{4\tan e-(1-\tan^2 f)}{4\tan e\tan f+(1-\tan^2 f)^2}$$

These are the reflection coefficients for a plane P wave incident on a free surface, and reflected P and SV waves.

Case 1: Reflections at a free surface

A P wave is incident at the free surface ...

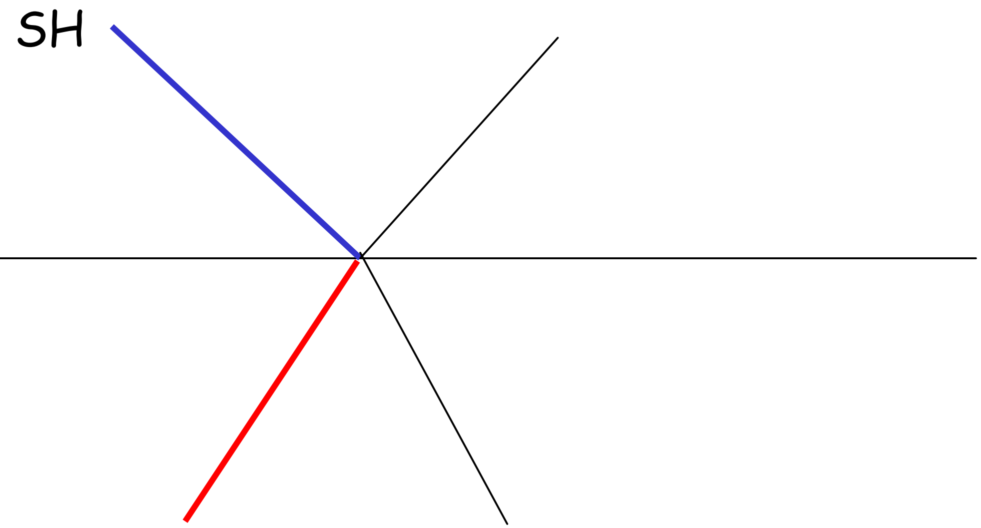


The reflected amplitudes can be described by the scattering matrix S

$$S = \begin{pmatrix} P_u P_d & S_u P_d \\ P_u S_d & S_u S_d \end{pmatrix}$$

Case 2: SH waves

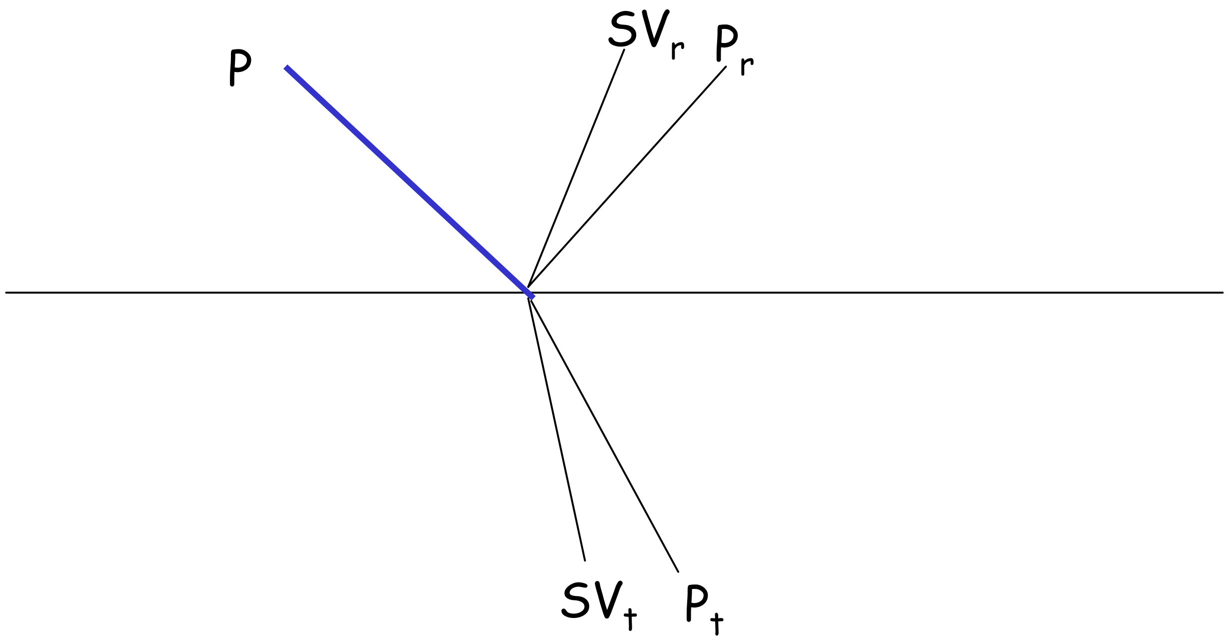
For layered media SH waves are completely decoupled from P and SV waves



There is no conversion only SH waves are reflected or transmitted

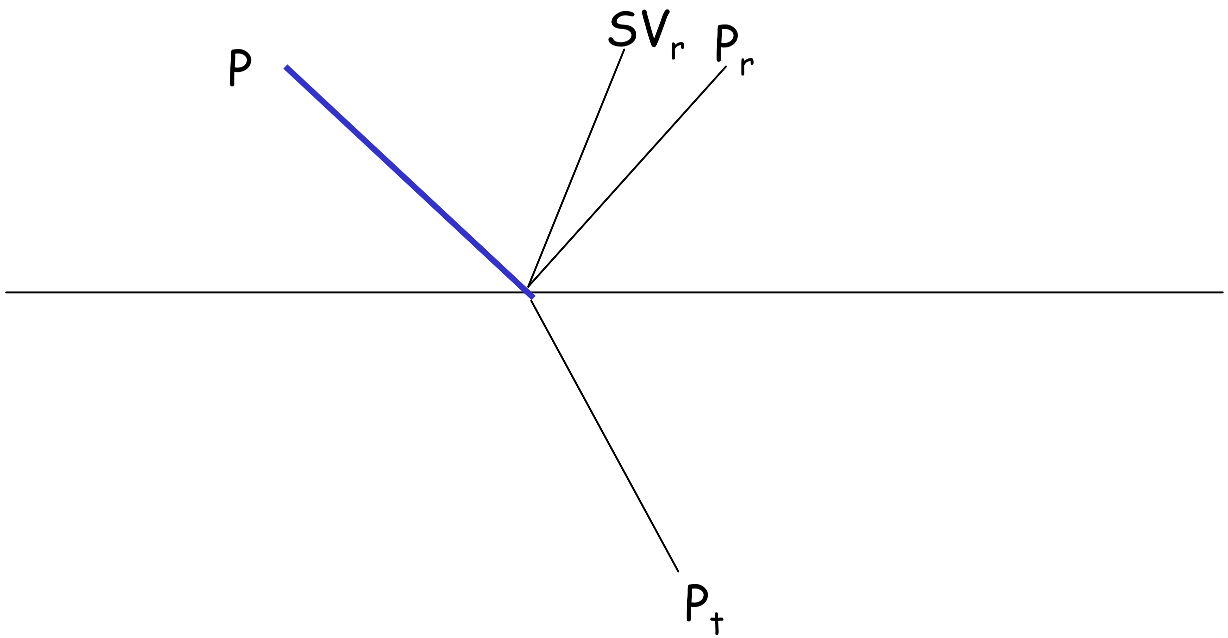
$$S = \begin{pmatrix} S_u S_d & S_u S_d \\ S_u S_d & S_u S_d \end{pmatrix}$$

Case 3: Solid-solid interface



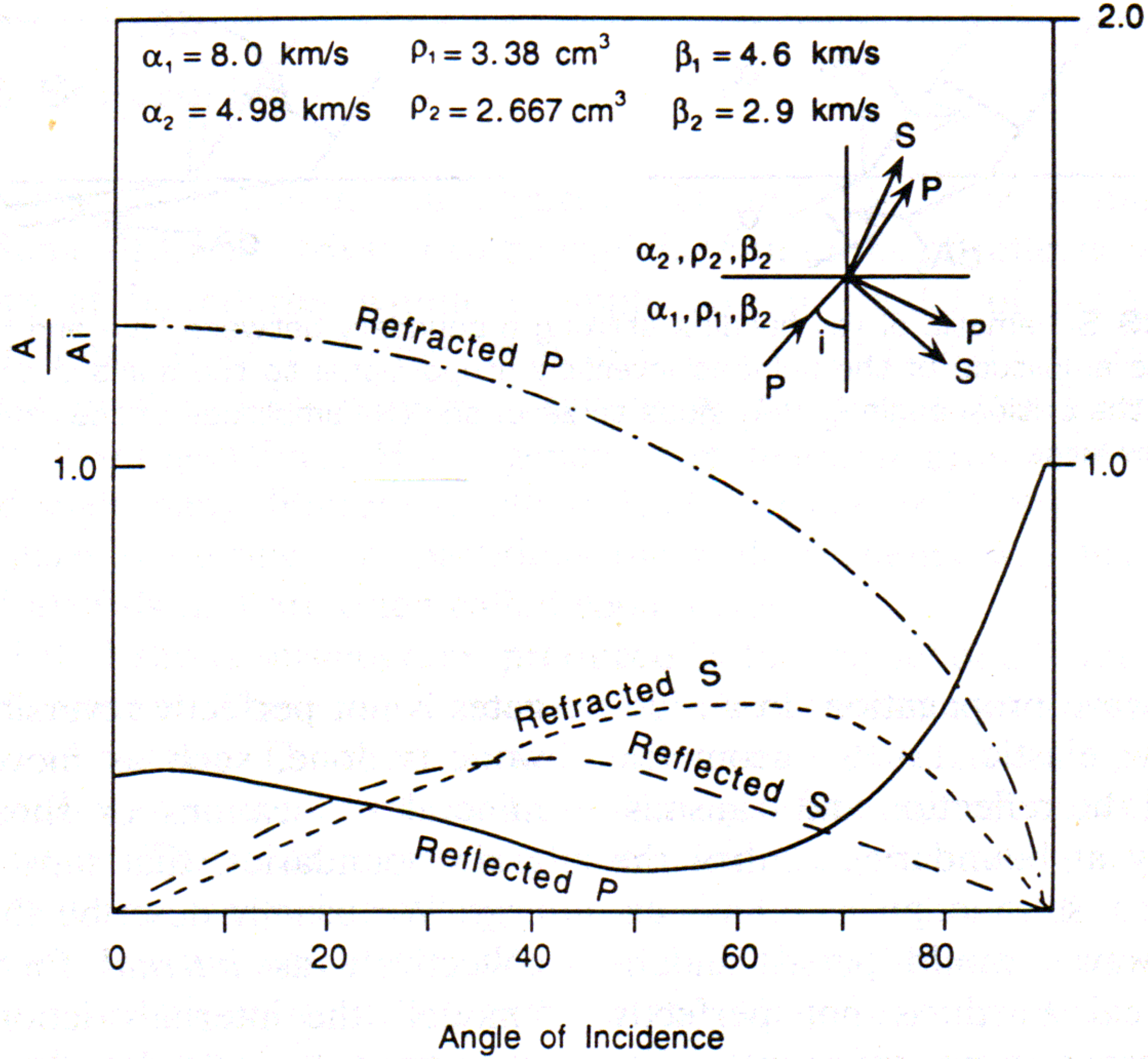
To account for all possible reflections and transmissions we need 16 coefficients, described by a 4x4 scattering matrix.

Case 4: Solid-Fluid interface

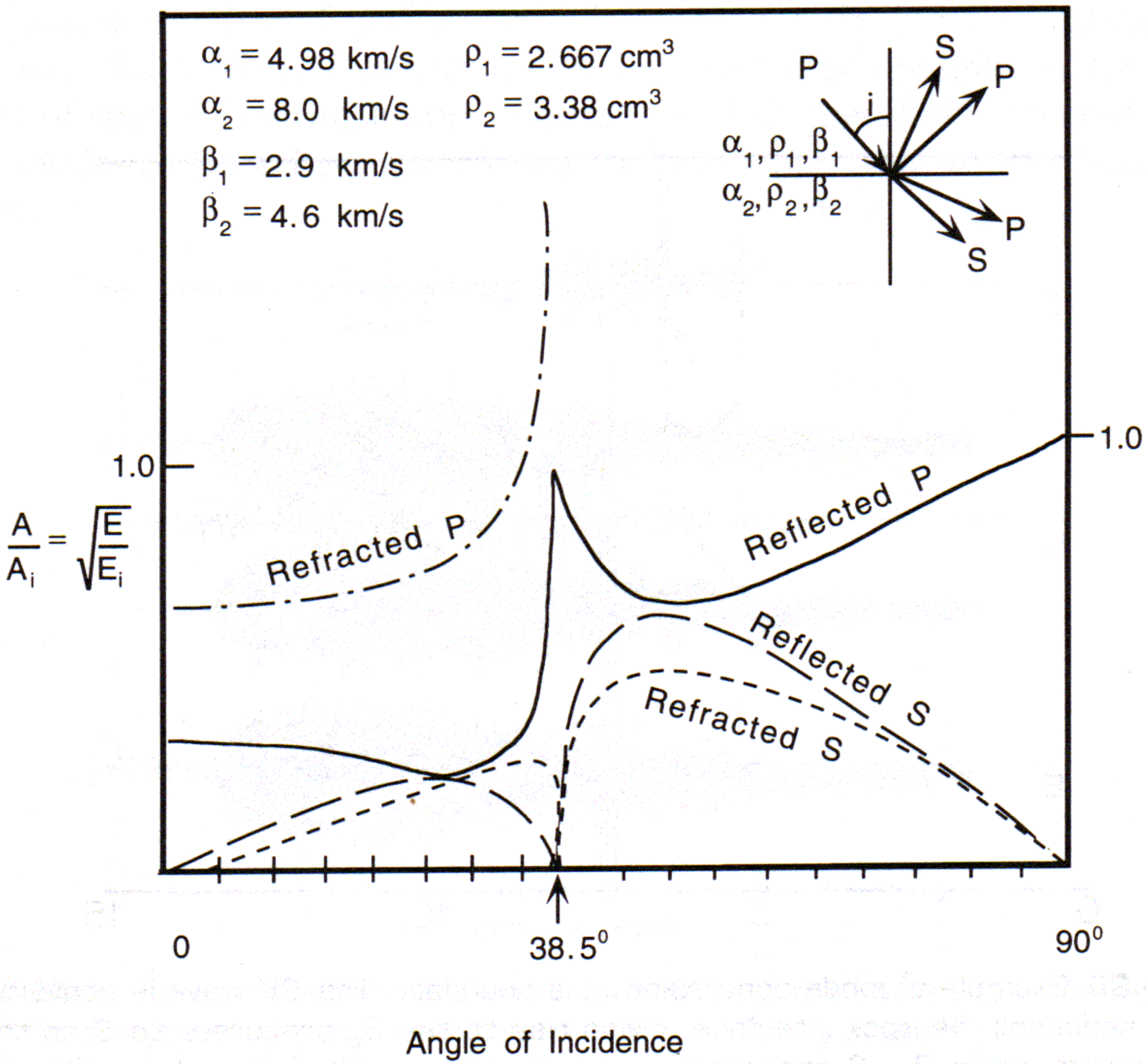


At a solid-fluid interface there is no conversion to SV in the lower medium.

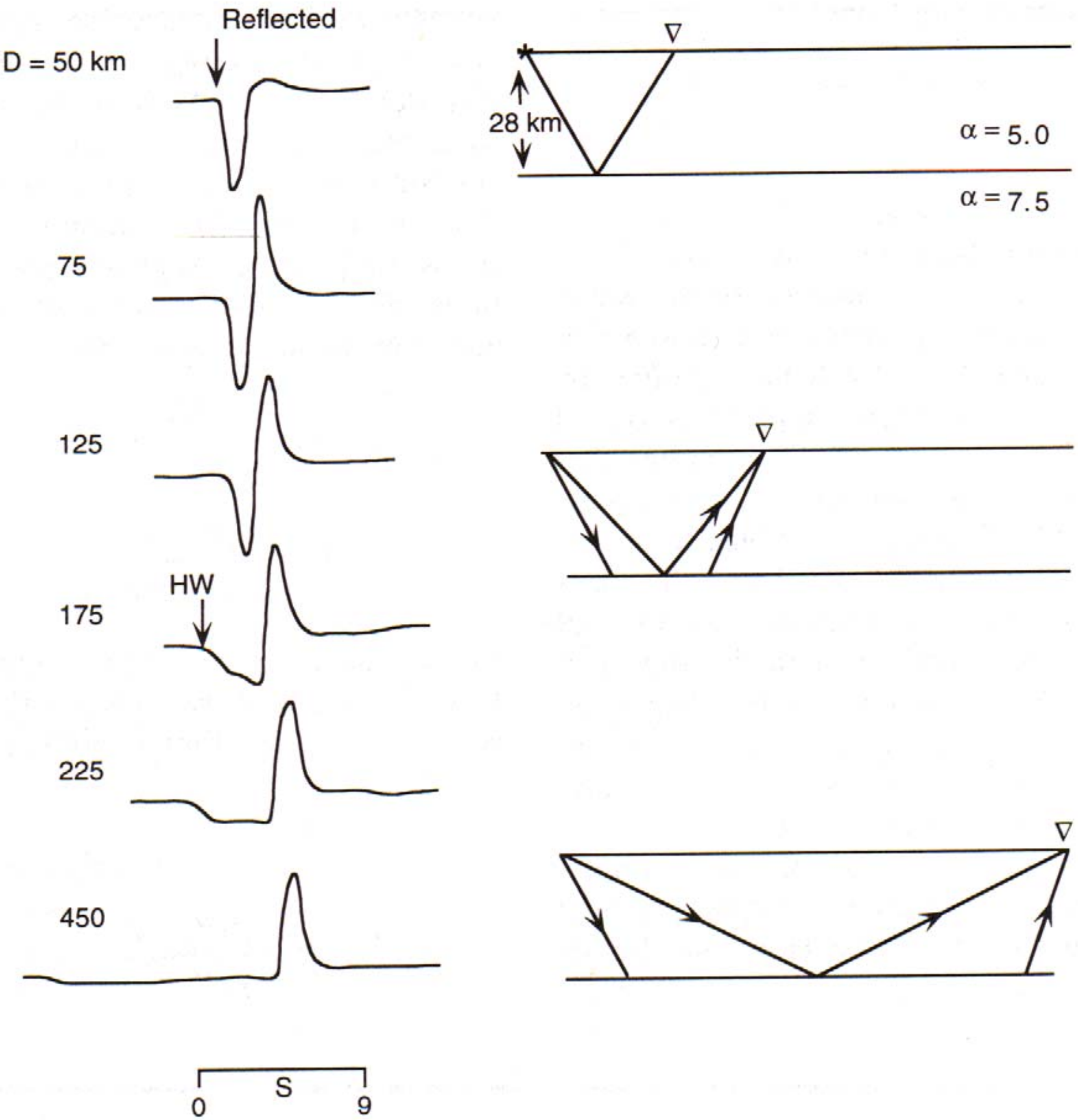
Reflection coefficients - example



Reflection coefficients - example



Refractions – waveform effects



Scattering and Attenuation

Propagating seismic waves loose energy due to

- geometrical spreading

  e.g. the energy of spherical wavefront emanating from a point source is distributed over a spherical surface of ever increasing size

- intrinsic attenuation

  elastic wave propagation consists of a permanent exchange between potential (displacement) and kinetic (velocity) energy. This process is not completely reversible. There is energy loss due to shear heating at grain boundaries, mineral dislocations etc.

- scattering attenuation

  whenever there are material changes the energy of a wavefield is scattered in different phases. Depending on the material properties this will lead to amplitude decay and dispersive effects.

Intrinsic attenuation

How can we describe intrinsic attenuation? Let us try a spring model:

The equation of motion for a damped harmonic oscillator is

$$m\ddot{x} + \gamma\dot{x} + kx = 0$$

$$\ddot{x} + \frac{\gamma}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\ddot{x} + \varepsilon\varpi_0\dot{x} + \omega_0^2 x = 0$$

$$\varepsilon = \frac{\gamma}{m\varpi_0}$$

$$\varpi_0 = \left(\frac{k}{m}\right)^{1/2}$$

where ε is the friction coefficient.

Q

The solution to this system is

$$x(t) = A_0 e^{-\varepsilon \varpi_0 t} \sin(\varpi_0 t \sqrt{1-\varepsilon^2})$$

so we have a time-dependent amplitude of

$$A(t) = A_0 e^{-\varepsilon \varpi_0 t} = A_0 e^{-\frac{\varpi_0 t}{2Q}}$$

and defining

$$\varepsilon = \frac{1}{2Q} \qquad \delta = \ln \frac{A_1}{A_2} \qquad Q = \frac{\pi}{\delta}$$

Q is the energy loss per cycle. Intrinsic attenuation in the Earth is in general described by Q.

Dispersion effects

What happens if we have frequency independent Q, i.e. each frequency looses the same amount of energy per cycle?

$$A(x) = A_0 e^{-(f\pi Qv)x}$$

high frequencies – more oscillations – more attenuation
low frequencies – less oscillations – less attenuation

Consequences:

- high frequencies decay very rapidly
- pulse broadening

In the Earth we observe that Q_p is large than $Q_{S.}$ This is due to the fact that intrinsic attenuation is predominantly caused by shear lattice effects at grain boundaries.

Q in the Earth

Rock Type	Q_p	Q_s
Shale	30	10
Sandstone	58	31
Granite	250	70-250
Peridotite Midmantle Lowermantle Outer Core	650 360 1200 8000	280 200 520 0

Scattering