Surface Waves and Free Oscillations

Surface waves in an elastic half spaces: Rayleigh waves

- Potentials
- Free surface boundary conditions
- Solutions propagating along the surface, decaying with depth
- Lamb's problem

Surface waves in media with depth-dependent properties: Love waves

- Constructive interference in a low-velocity layer
- Dispersion curves
- Phase and Group velocity

Free Oscillations

- Spherical Harmonics
- Modes of the Earth
- Rotational Splitting

The Wave Equation: Potentials

Do solutions to the wave equation exist for an elastic half space, which travel along the interface? Let us start by looking at potentials:

These potentials are solutions to the wave equation

$$u_i = \nabla\Phi + \nabla \times \Psi$$

$$\nabla = (\partial_x, \partial_y, \partial_z)$$

$$\partial_t^2 \Phi = \alpha^2 \nabla^2 \Phi$$

$$\partial_t^2 \Psi_i = \beta^2 \nabla^2 \Psi_i$$

u_i	displacement
Φ	scalar potential
Ψ_i	vector potential
α	P-wave speed
β	Shear wave speed

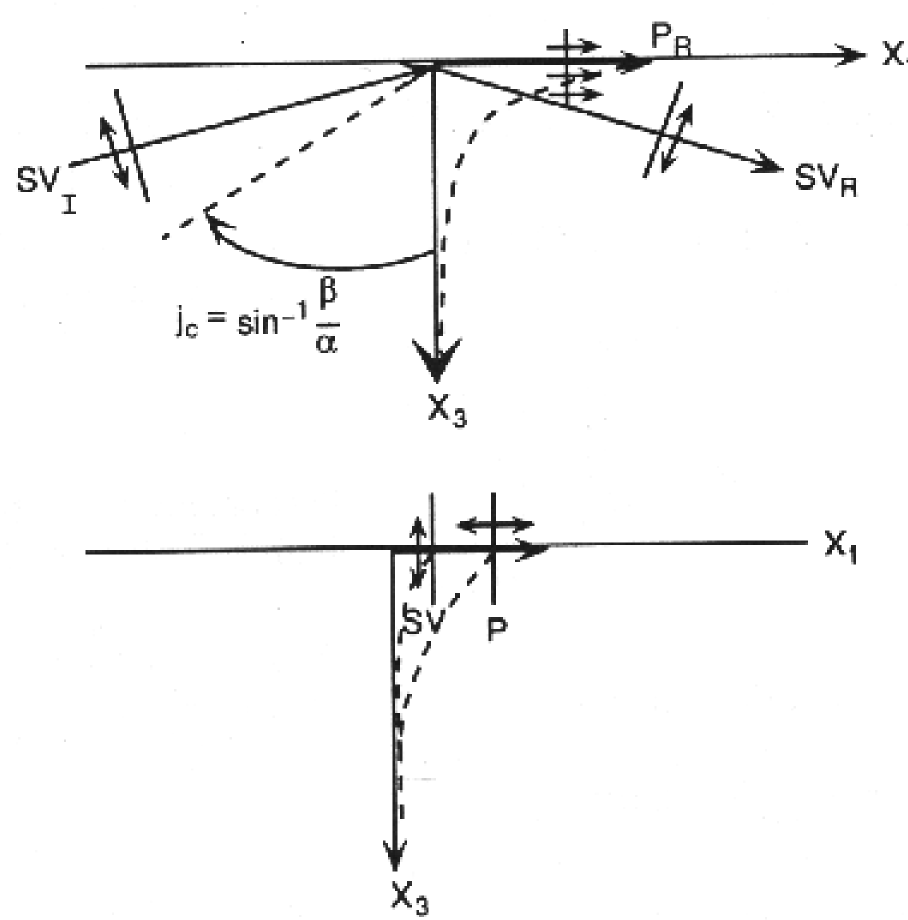
What particular geometry do we want to consider?

Rayleigh Waves

SV waves incident on a free surface: conversion and reflection

An evanescent P-wave propagates along the free surface decaying exponentially with depth. The reflected post-crticially reflected SV wave is totally reflected and phase-shifted. These two wave types can only exist together, they both satisfy the free surface boundary condition:

-> Surface waves



Surface waves: Geometry

We are looking for plane waves traveling along one horizontal coordinate axis, so we can - for example - set

$$\partial_y(.) = 0$$

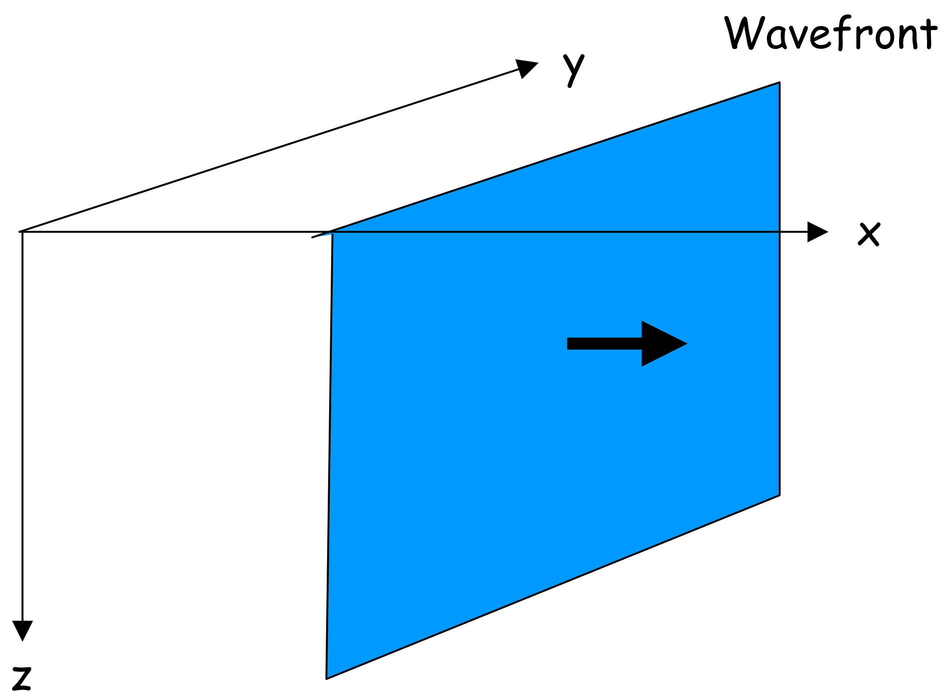
And consider only wave motion in the x,z plane. Then

$$u_x = \partial_x \Phi - \partial_z \Psi_y$$

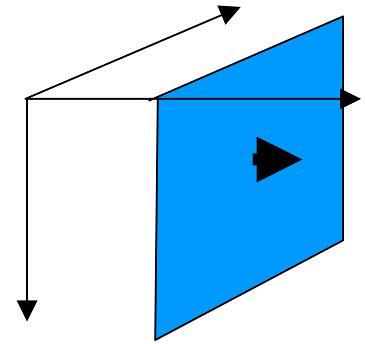
$$u_z = \partial_z \Phi + \partial_x \Psi_y$$

As we only require Ψ_y we set $\Psi_y = \Psi$ from now on. Our trial solution is thus

$$\Phi = A \exp[ik(ct \pm az - x)]$$



Surface waves: Disperion relation



With this trial solution we obtain for example coefficients a for which travelling solutions exist

$$a = \pm\sqrt{\frac{c^2}{\alpha^2} - 1}$$

In order for a plane wave of that form to decay with depth a has to be imaginary, in other words

$$c < \alpha$$

Together we obtain

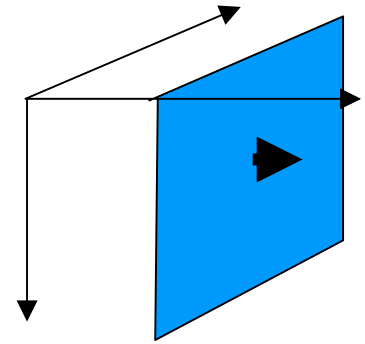
$$\Phi = A\exp[ik(ct \pm \sqrt{c^2/\alpha^2 - 1}z - x)]$$

$$\Psi = B\exp[ik(ct \pm \sqrt{c^2/\beta^2 - 1}z - x)]$$

So that

$$c < \beta < \alpha$$

Surface waves: Boundary Conditions



Analogous to the problem of finding the reflection-transmission coefficients we now have to satisfy the boundary conditions at the free surface (stress free)

$$\sigma_{xz} = \sigma_{zz} = 0$$

In isotropic media we have

$$\sigma_{zz} = \lambda(\partial_x u_x + \partial_z u_z) + 2\mu\partial_z u_z$$
$$\sigma_{xz} = 2\mu\partial_x u_z$$

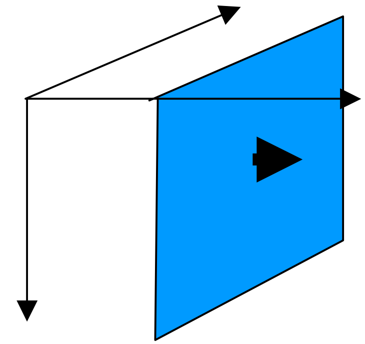
where

$$u_x = \partial_x \Phi - \partial_z \Psi_y$$
$$u_z = \partial_z \Phi + \partial_x \Psi_y$$

and

$$\Phi = A\exp[ik(ct \pm \sqrt{c^2/\alpha^2 - 1}z - x)]$$
$$\Psi = B\exp[ik(ct \pm \sqrt{c^2/\beta^2 - 1}z - x)]$$

Rayleigh waves: solutions



This leads to the following relationship for c, the phase velocity:

$$(2 - c^2/\beta^2)^2 = 4(1 - c^2/\alpha^2)^{1/2}(1 - c^2/\beta^2)^{1/2}$$

For simplicity we take a fixed relationship between P and shear-wave velocity $\alpha = \sqrt{3}\beta$

… to obtain

$$c^6/\beta^6 - 8c^4/\beta^4 + 56/3c^2/\beta^2 - 32/2 = 0$$

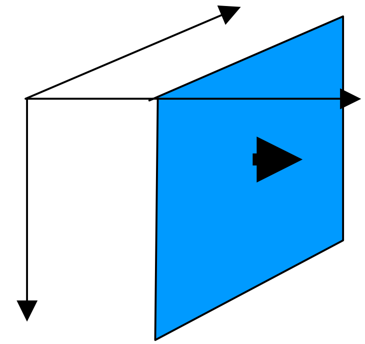
… and the only root which fulfills the condition $c < \beta$

is

$$\boxed{c = 0.9194\beta}$$

Displacement

Putting this value back into our solutions we finally obtain the displacement in the x-z plane for a plane harmonic surface wave propagating along direction x



$$u_x = C(e^{-0.8475kz} - 0.5773e^{-0.3933kz})\sin k(ct - x)$$

$$u_z = C(-0.8475e^{-0.8475kz} + 1.4679e^{-0.3933kz})\cos k(ct - x)$$

This development was first made by Lord Rayleigh in 1885. It demonstrates that YES there are solutions to the wave equation propagating along a **free surface**!

Some remarkable facts can be drawn from this particular form:

Lamb's Problem

-the two components are out of phase by π

– for small values of z a particle describes an ellipse and the motion is retrograde

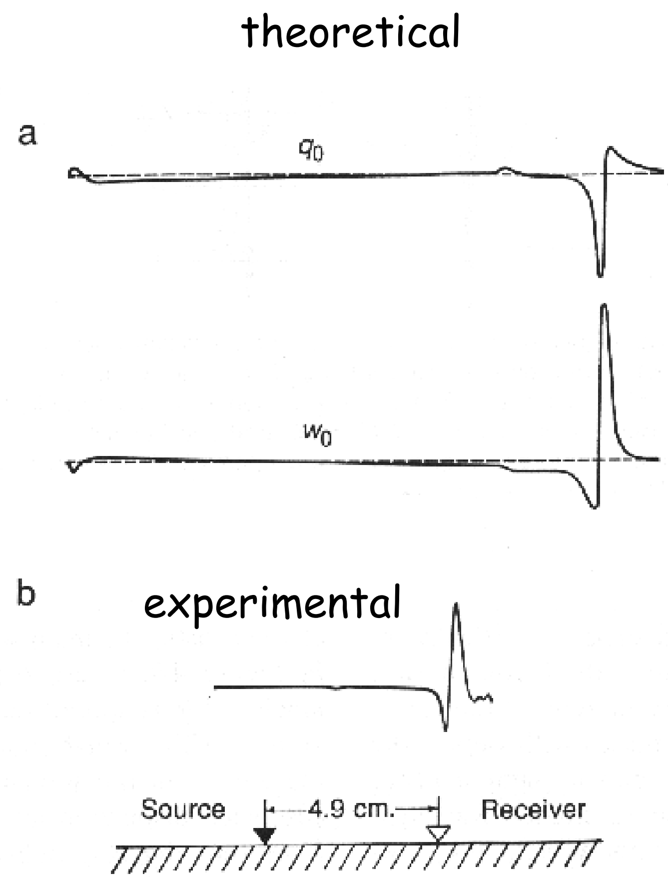
- at some depth z the motion is linear in z

- below that depth the motion is again elliptical but prograde

- the phase velocity is independent of k: **there is no dispersion** for a homogeneous half space

- the problem of a vertical point force at the surface of a half space is called **Lamb's problem** (after Horace Lamb, 1904).

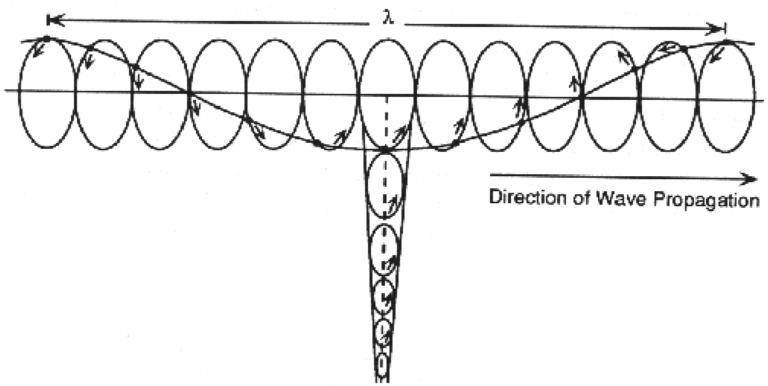
- Right Figure: radial and vertical motion for a source at the surface

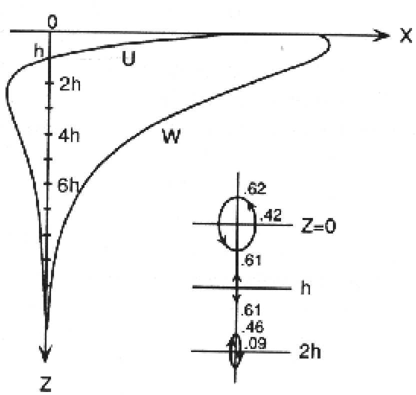


Particle Motion (1)

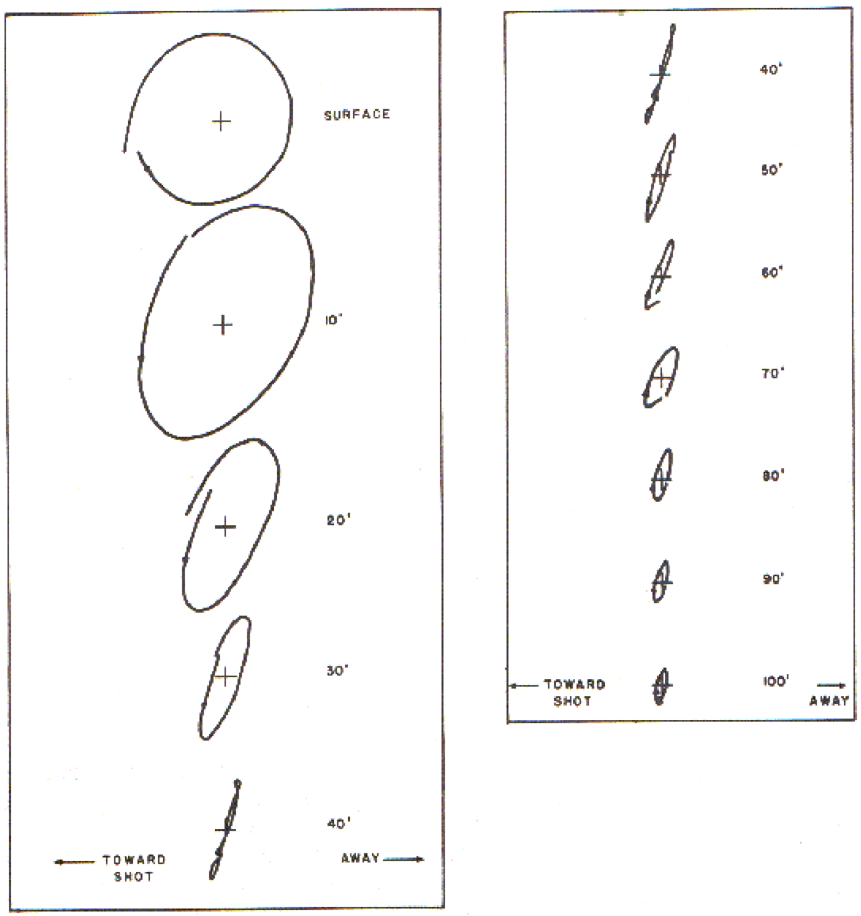
How does the particle motion look like?

theoretical

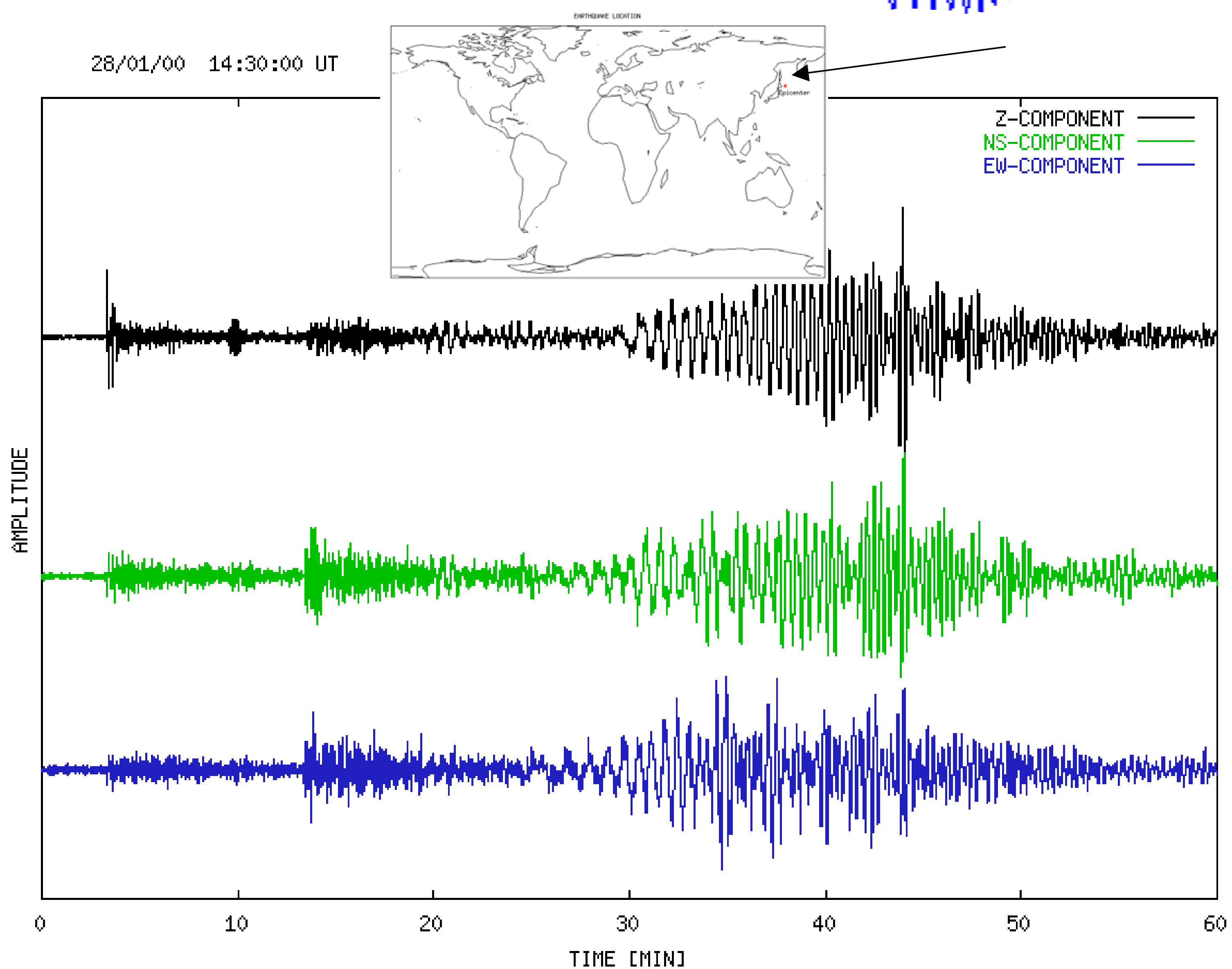




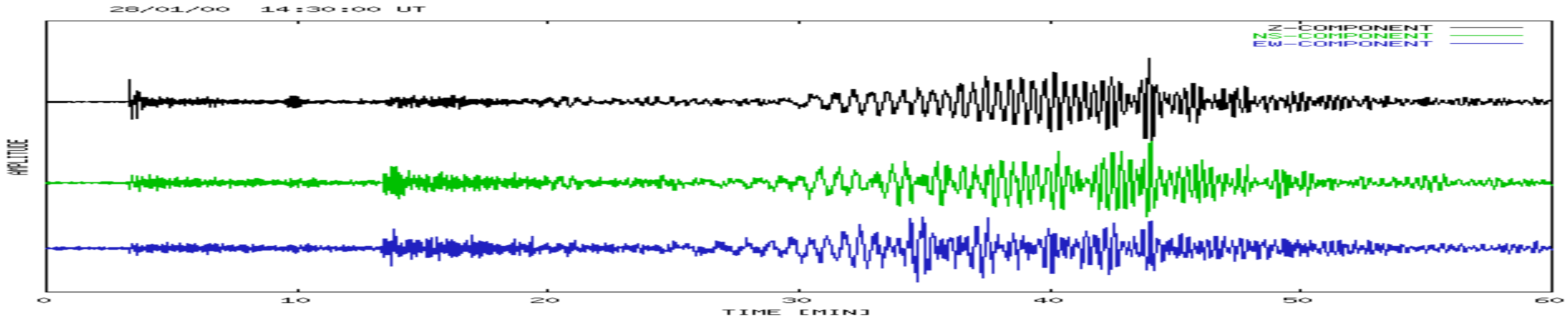
experimental



Data Example



Data Example



Question:

We derived that Rayleigh waves are non-dispersive!

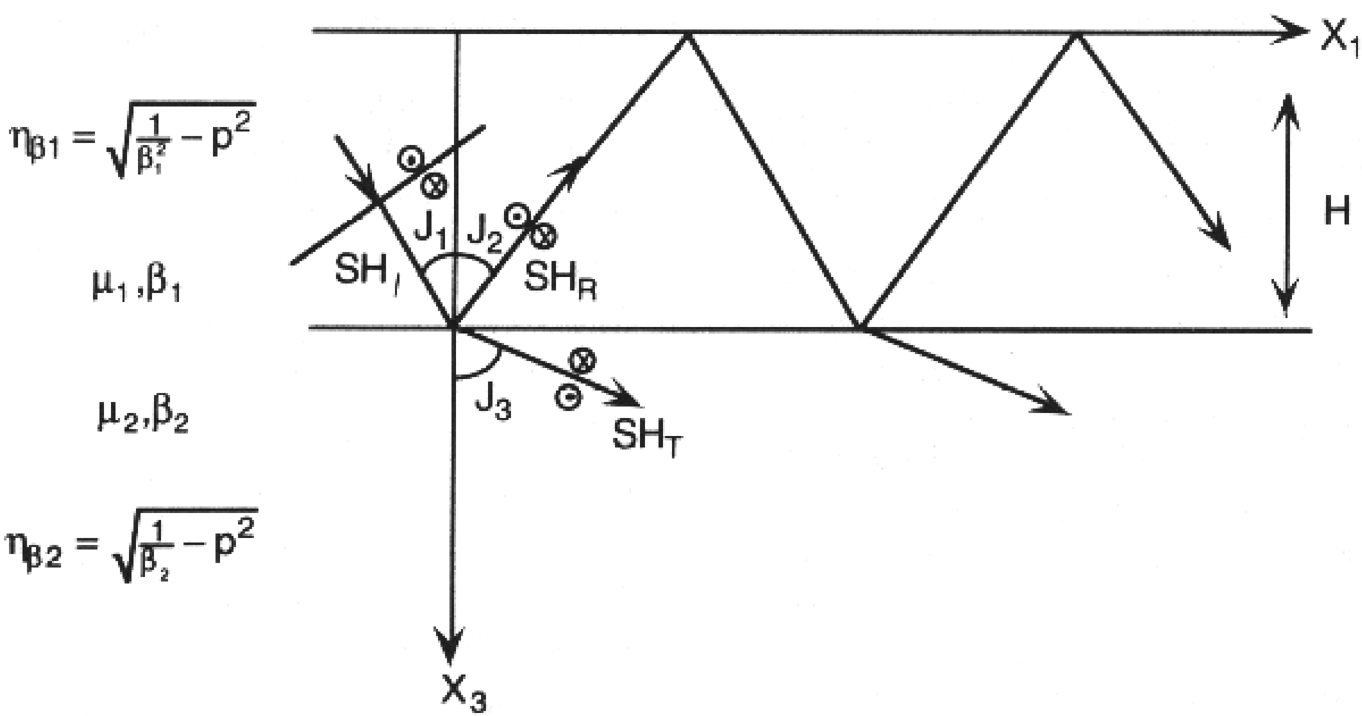
But in the observed seismograms we clearly see a highly dispersed surface wave train?

We also see dispersive wave motion on both horizontal components!

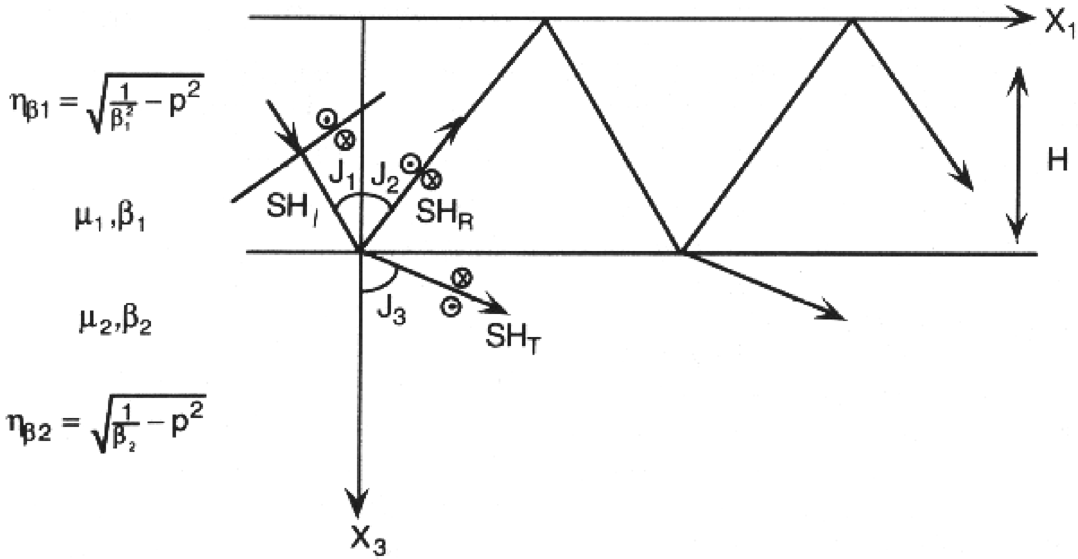
Do SH-type surface waves exist?
Why are the observed waves dispersive?

Love Waves: Geometry

In an elastic half-space no SH type surface waves exist. Why? Because there is total reflection and no interaction between an evanescent P wave and a phase shifted SV wave as in the case of Rayleigh waves. What happens if we have layer over a half space (Love, 1911) ?



Love Waves: Trapping



Repeated reflection in a layer over a half space.

Interference between incident, reflected and transmitted SH waves.

When the layer velocity is smaller than the halfspace velocity, then there is a critical angle beyon which SH reverberations will be totally trapped.

Love Waves: Trapping

The formal derivation is very similar to the derivation of the Rayleigh waves. The conditions to be fulfilled are:

1. Free surface condition
2. Continuity of stress on the boundary
3. Continuity of displacement on the boundary

Similary we obtain a condition for which solutions exist. This time we obtain a frequency-dependent solution a **dispersion** relation

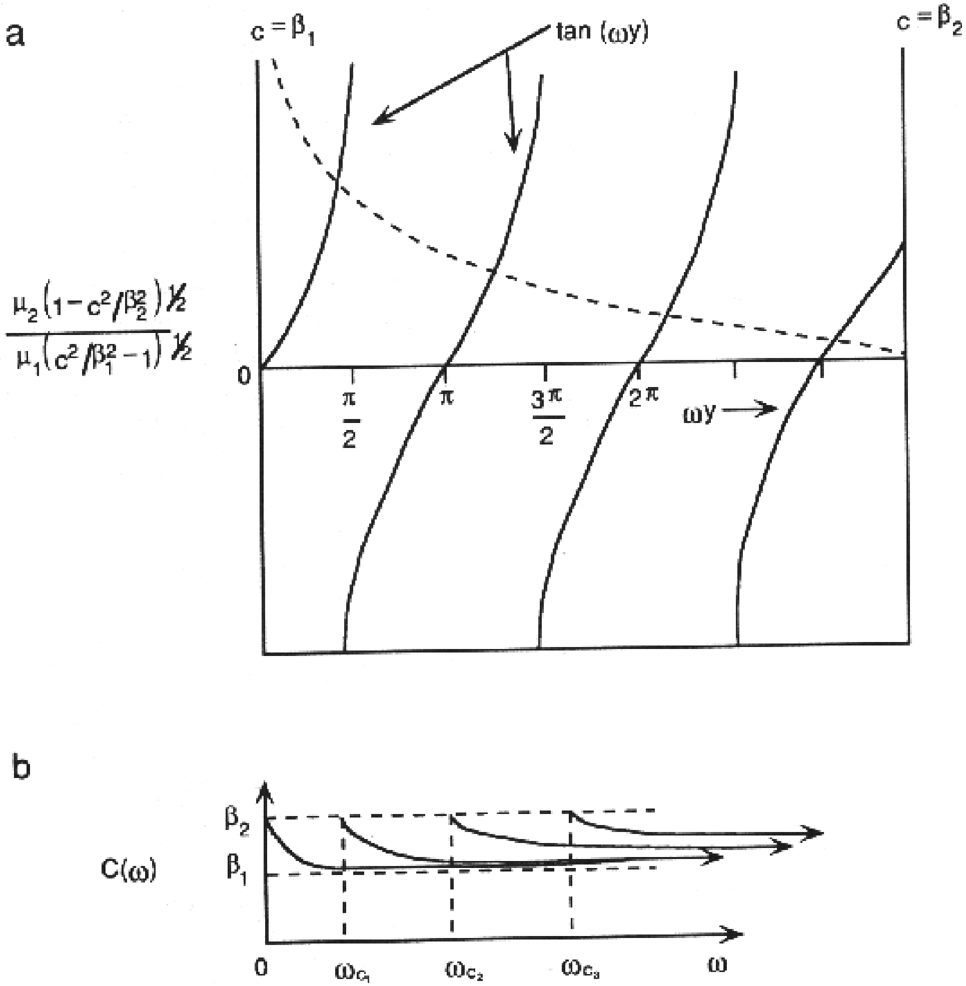
$$\tan\left(H\omega\sqrt{1/\beta_1^2 - 1/c^2}\right) = \frac{\mu_2\sqrt{1/c^2 - 1/\beta_2^{\;2}}}{\mu_1\sqrt{1/\beta_1^2 - 1/c^2}}$$

... indicating that there are only solutions if ...

$$\boxed{\beta < c < \beta_2}$$

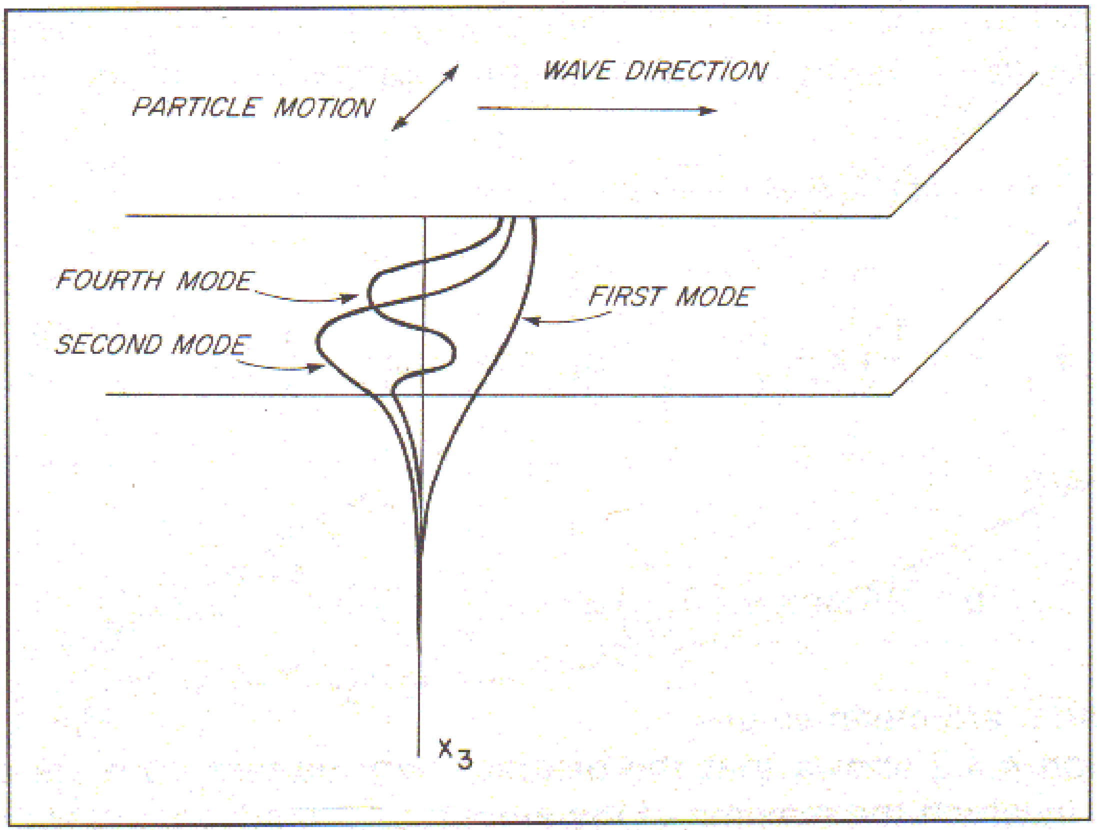
Graphical solution of the previous equation. Intersection of dashed and solid lines yield discrete modes.

Is it possible, now, to explain the observed dispersive behaviour?

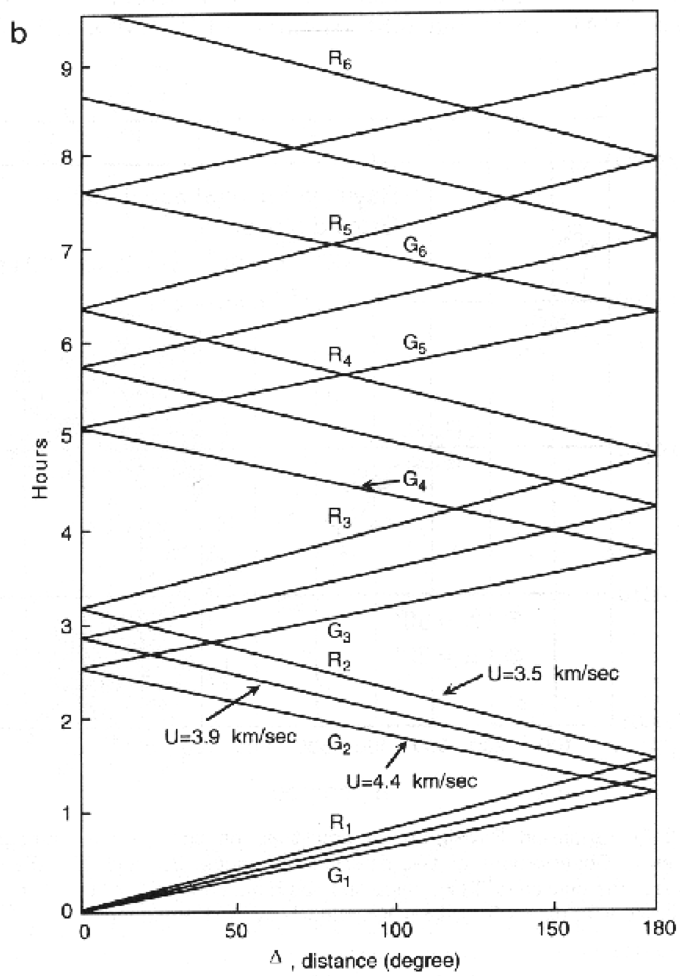


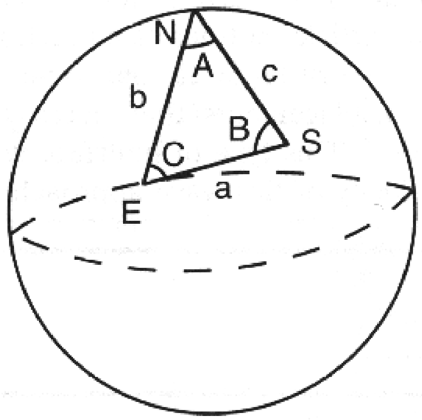
Love Waves: modes

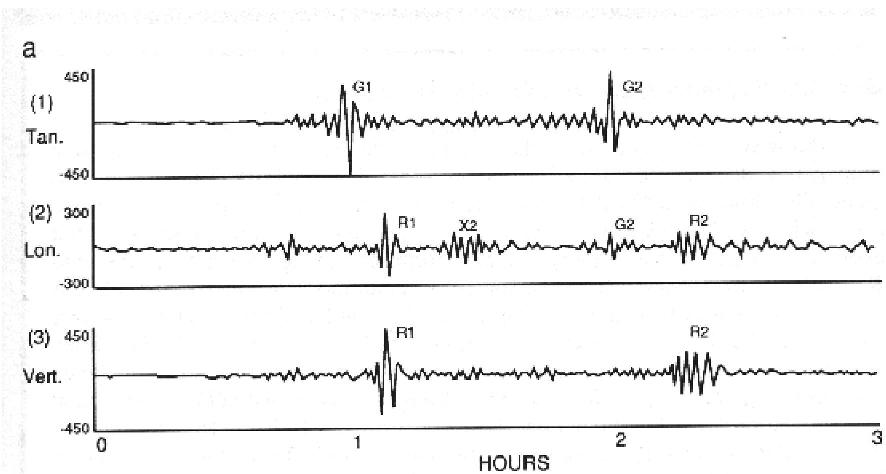
Some modes for Love waves



Waves around the globe

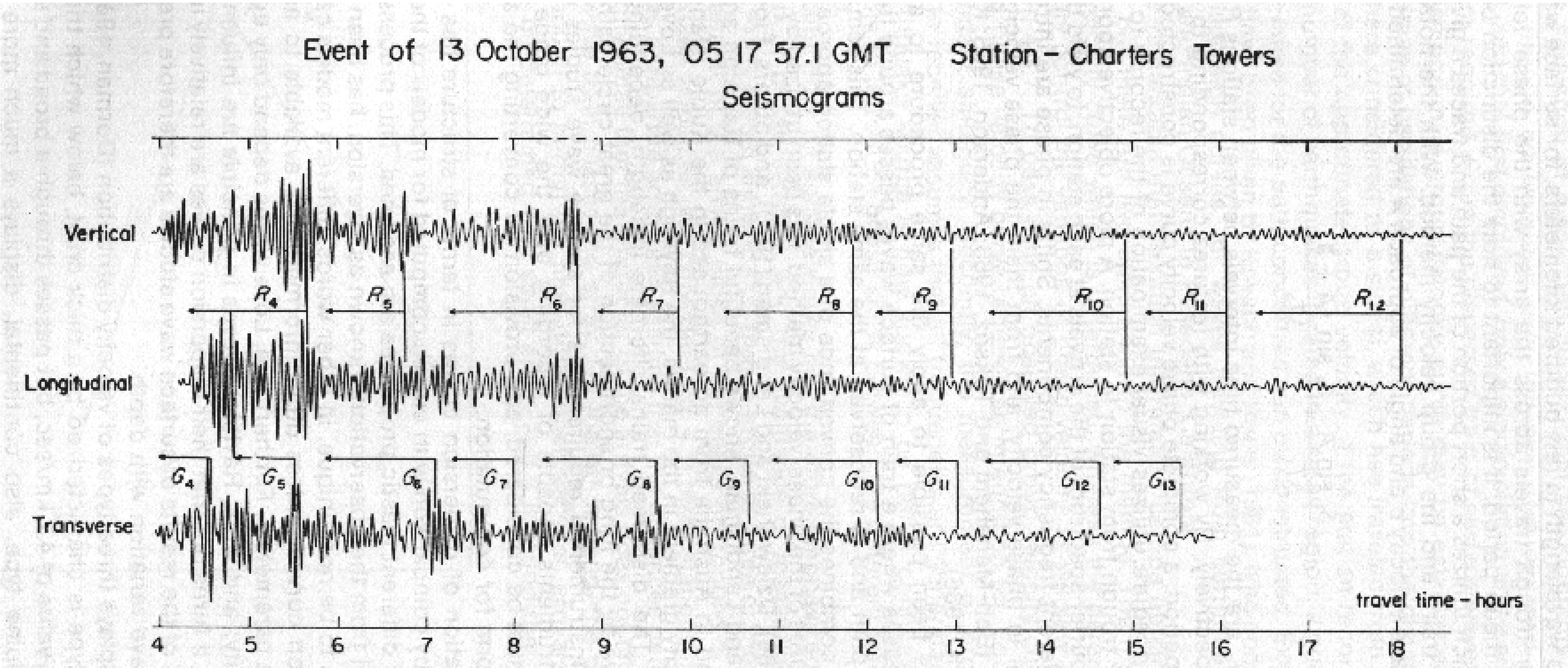






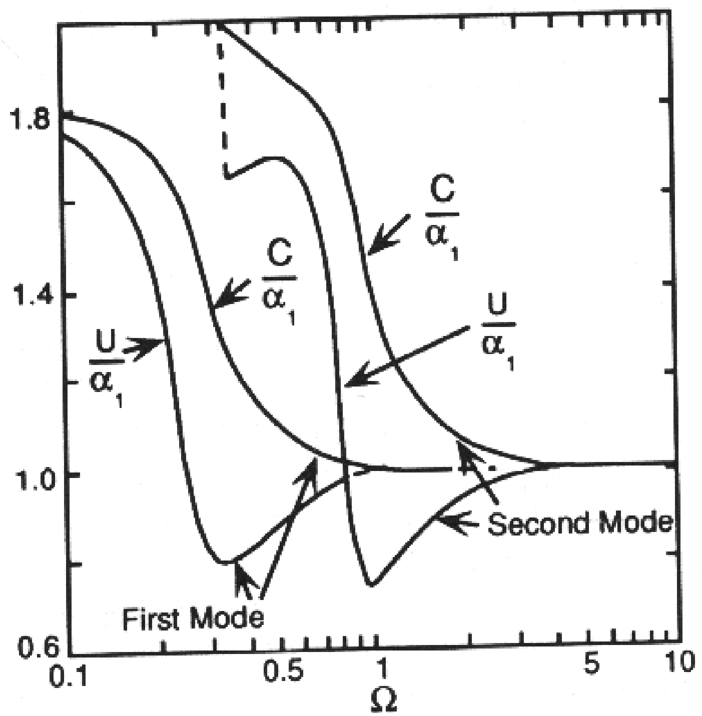
Data Example

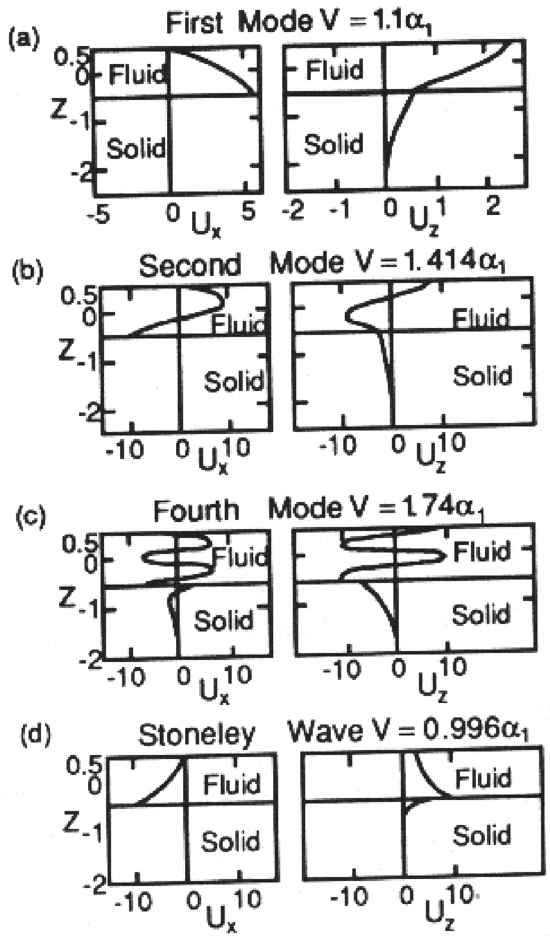
Surface waves travelling around the globe



Liquid layer over a half space

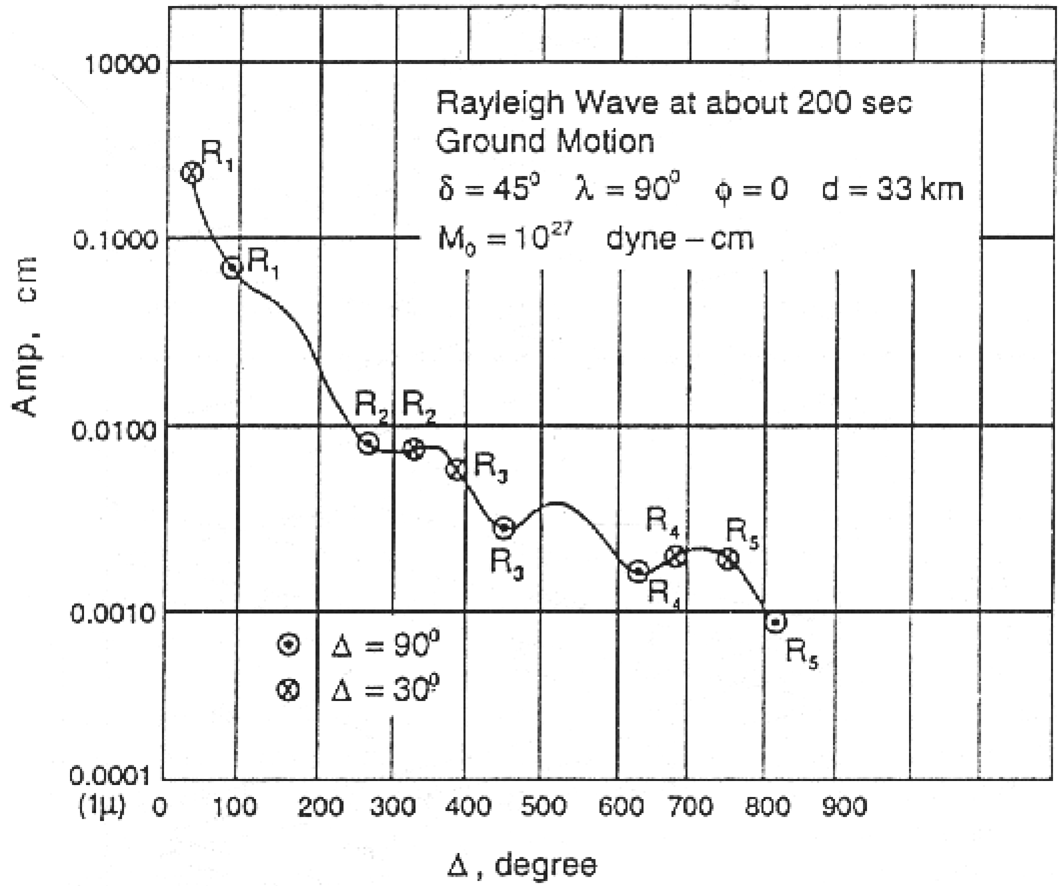
Similar derivation for Rayleigh type motion leads to dispersive behavior





Amplitude Anomalies

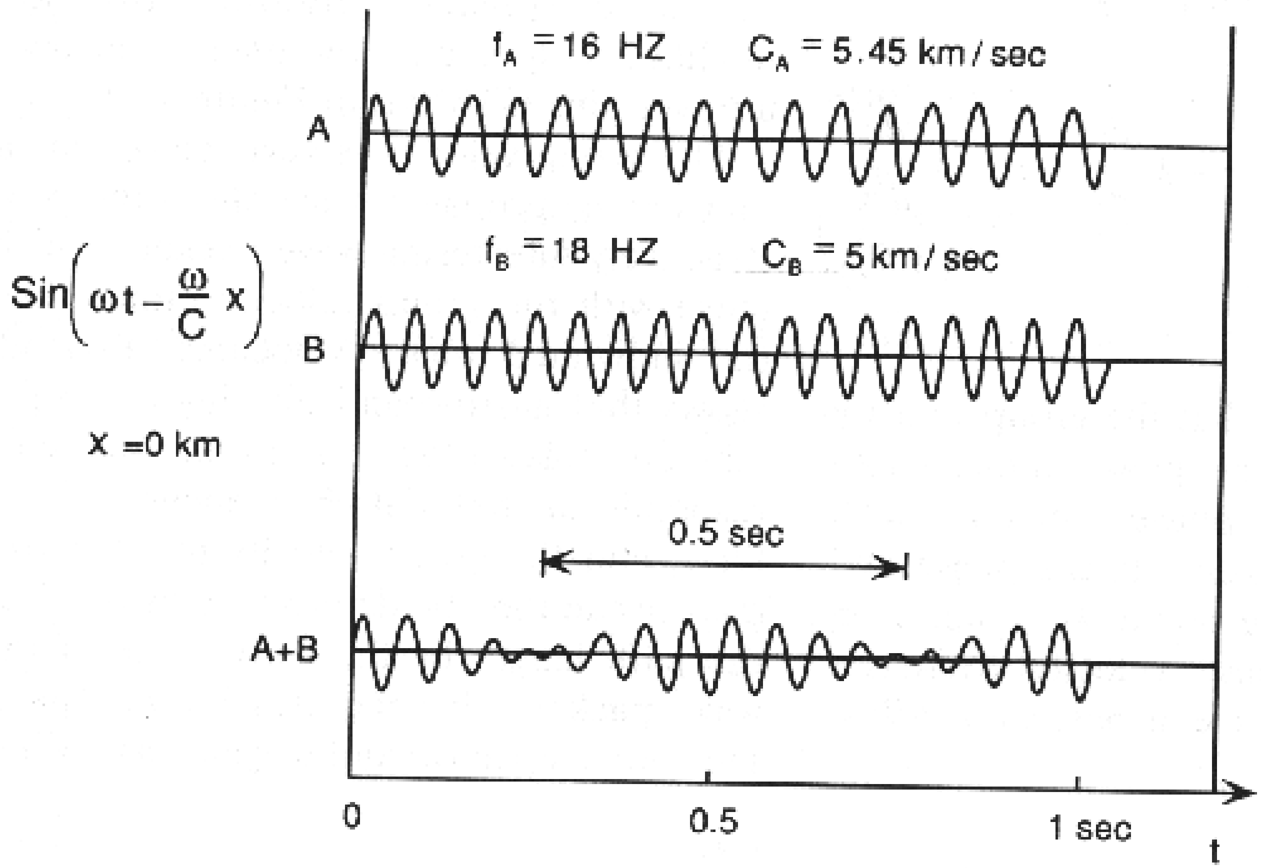
What are the effects on the amplitude of surface waves?



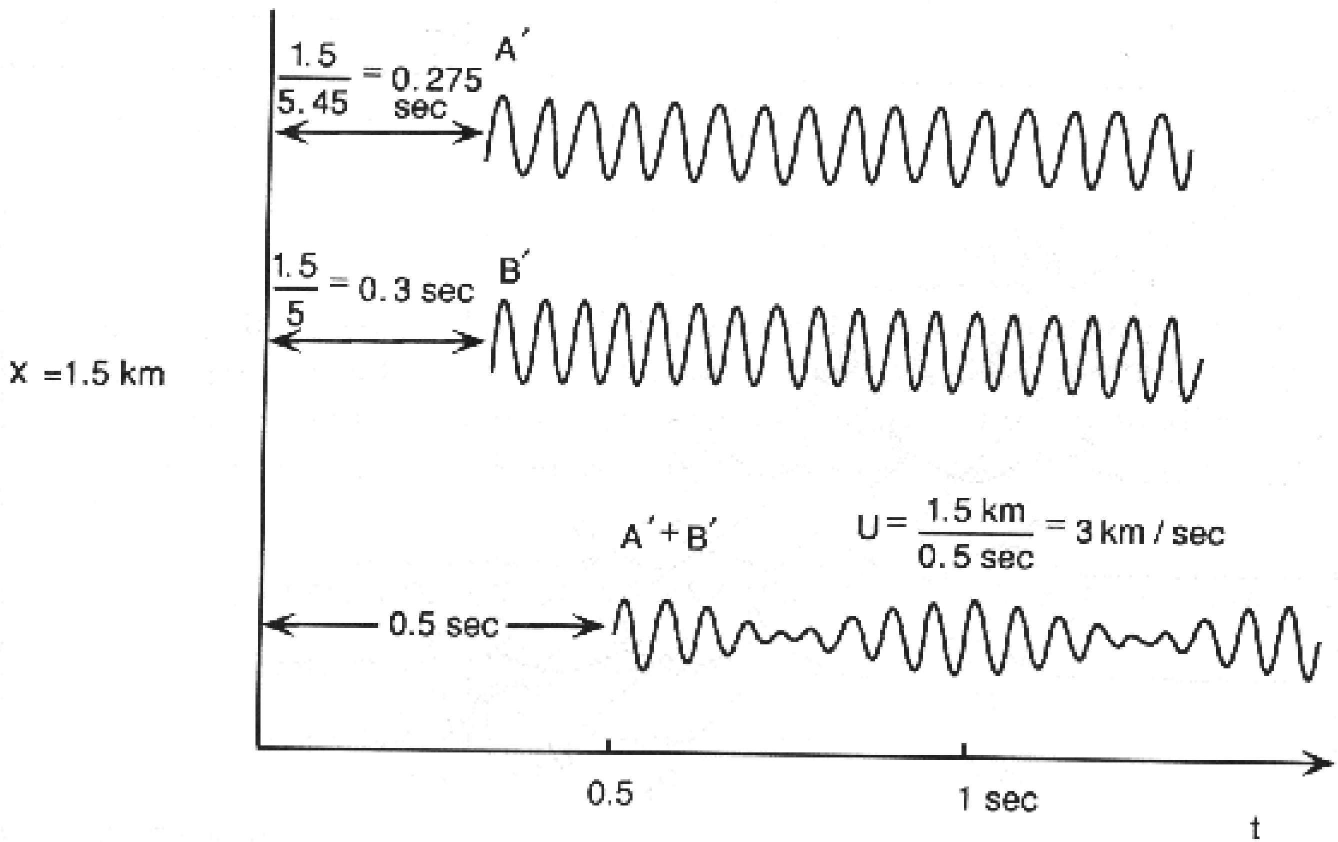
Away from source or antipode geometrical spreading is approx. prop. to $(\sin\Delta)^{1/2}$

Group-velocities

Interference of two waves at two positions (1)

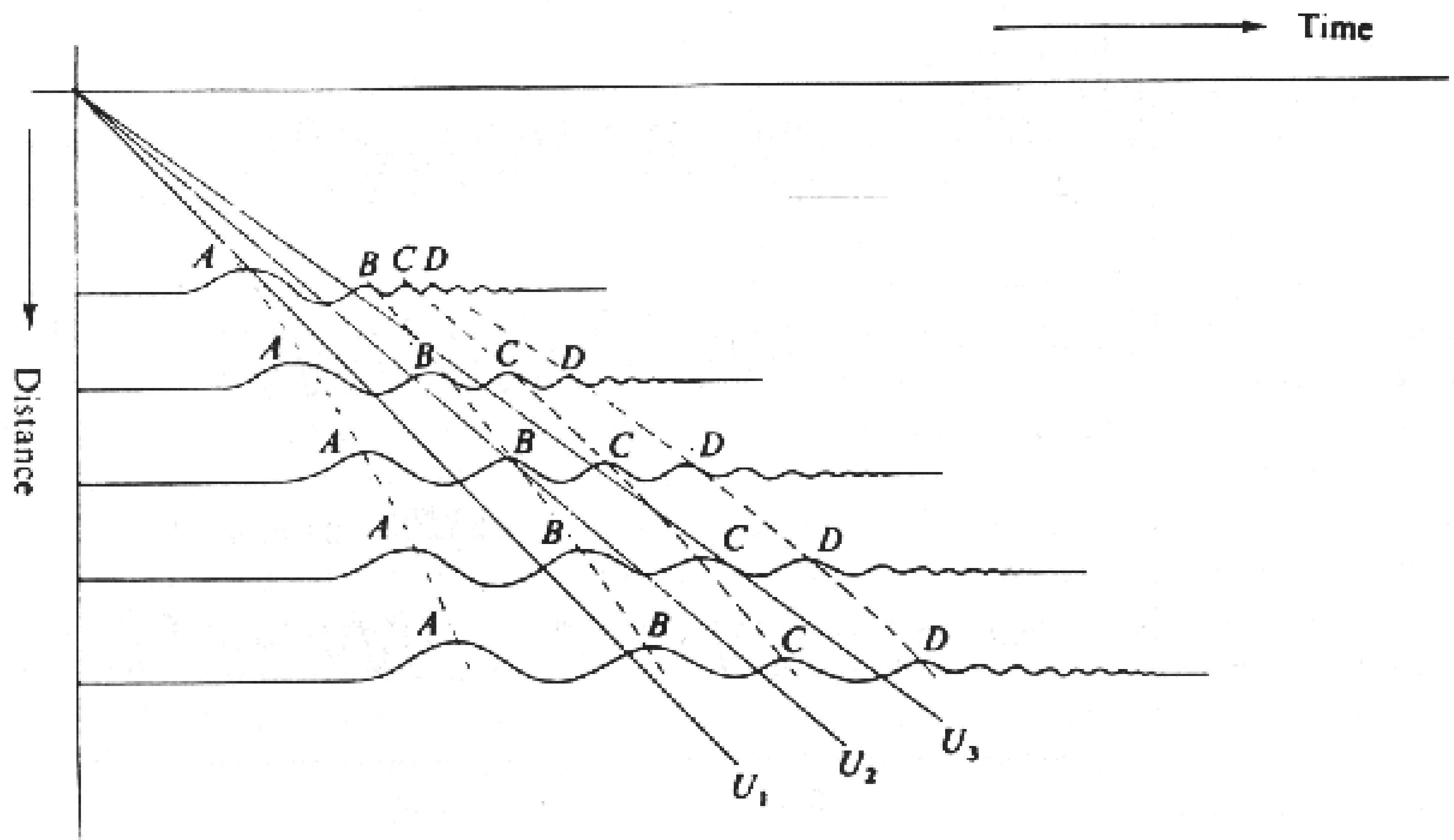


Interference of two waves at two positions (2)



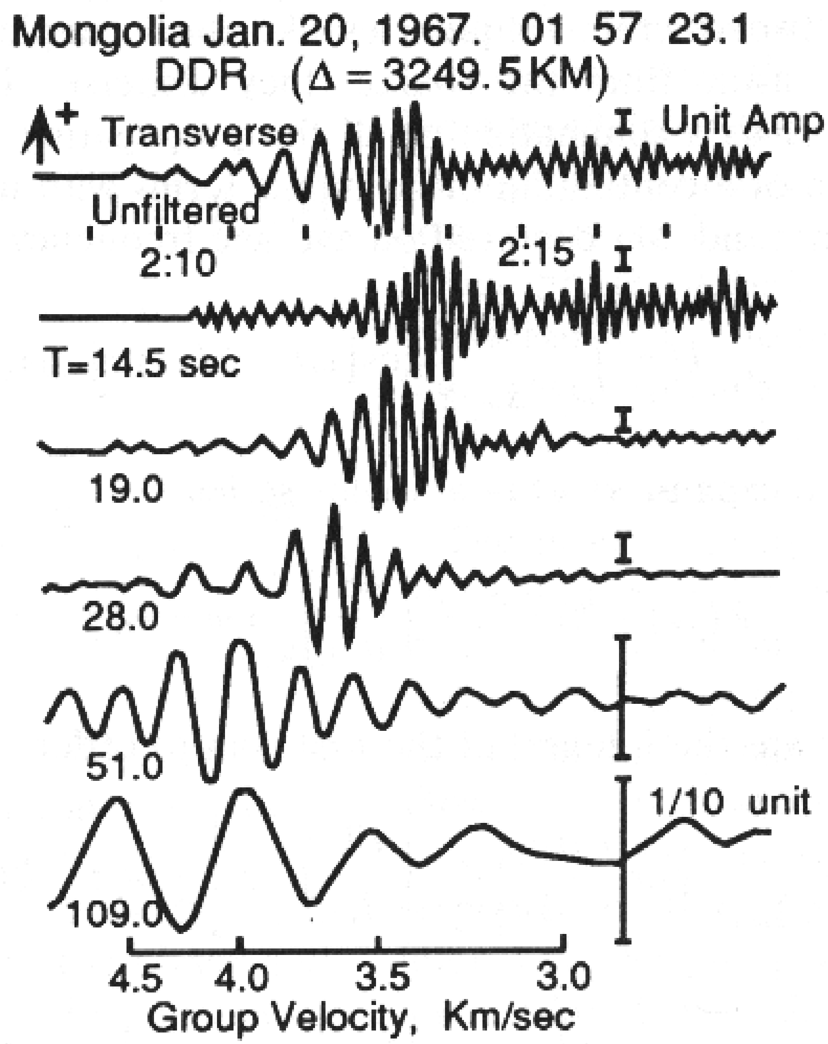
Dispersion

The typical dispersive behavior of surface waves
solid – group velocities; dashed – phase velocities



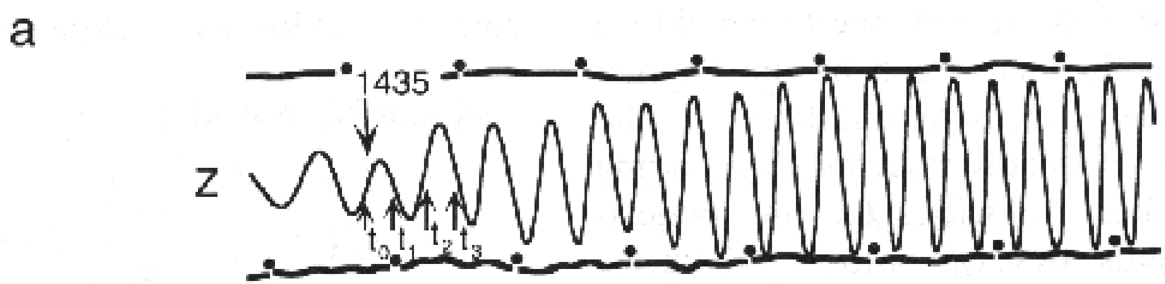
Wave Packets

Seismograms of a Love wave train filtered with different central periods. Each narrowband trace has the appearance of a wave packet arriving at different times.

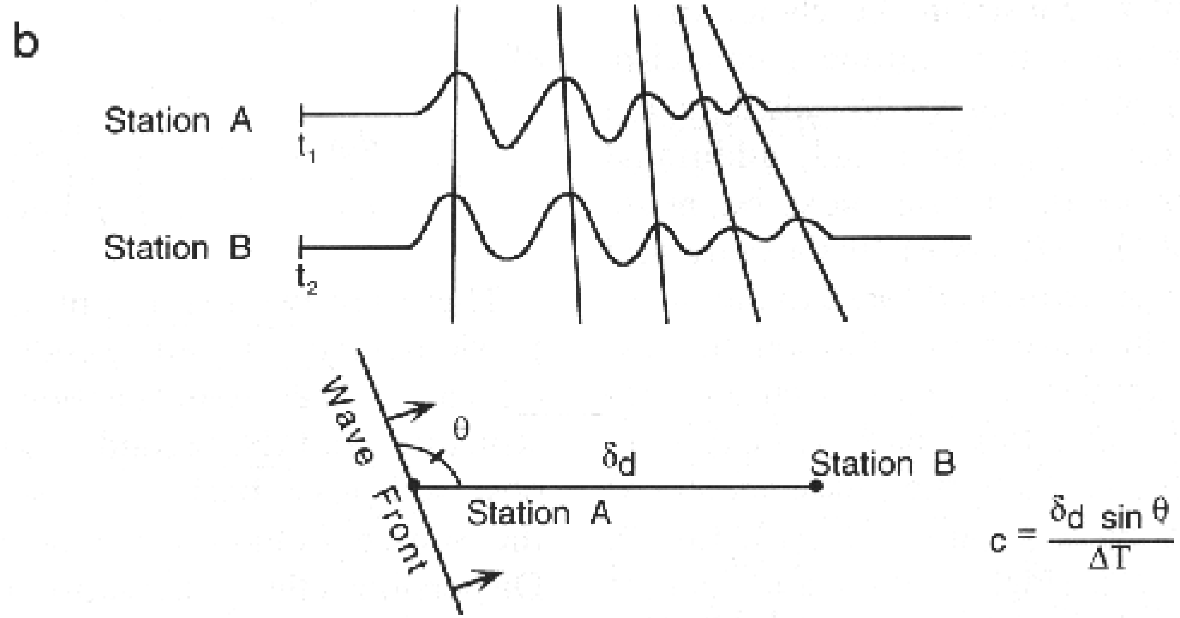


Wave Packets

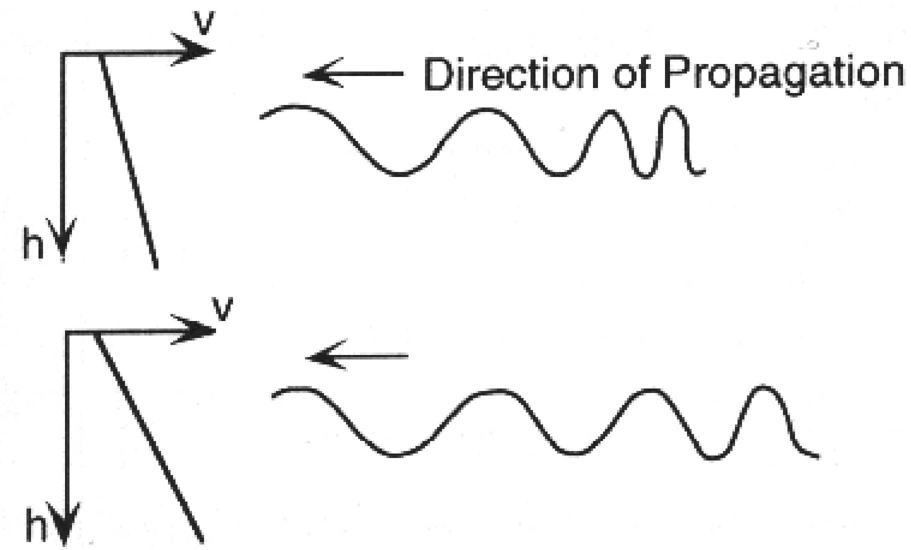
Group and phase velocity measurements peak-and-trough method



Phase velocities from array measurement



Dispersion

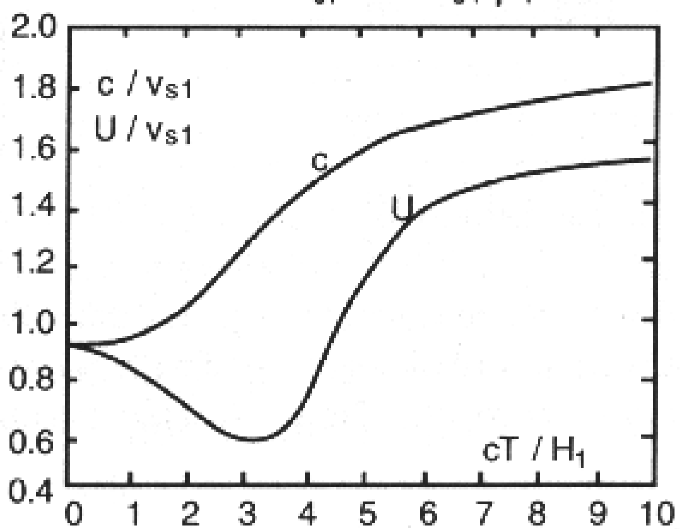


Stronger gradients cause greater dispersion

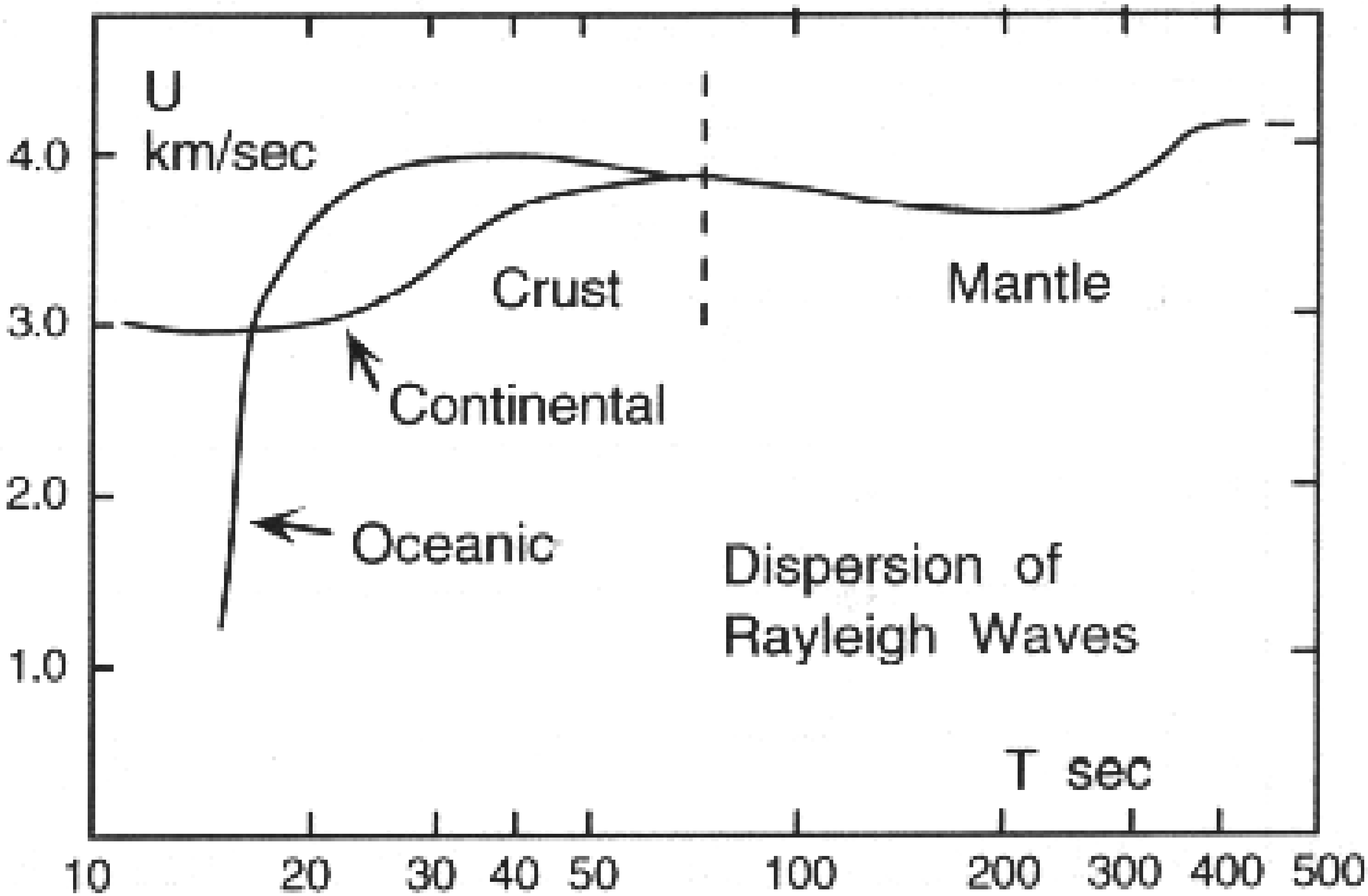
Dispersion

Fundamental Mode Rayleigh dispersion curve for
a layer over a half space.

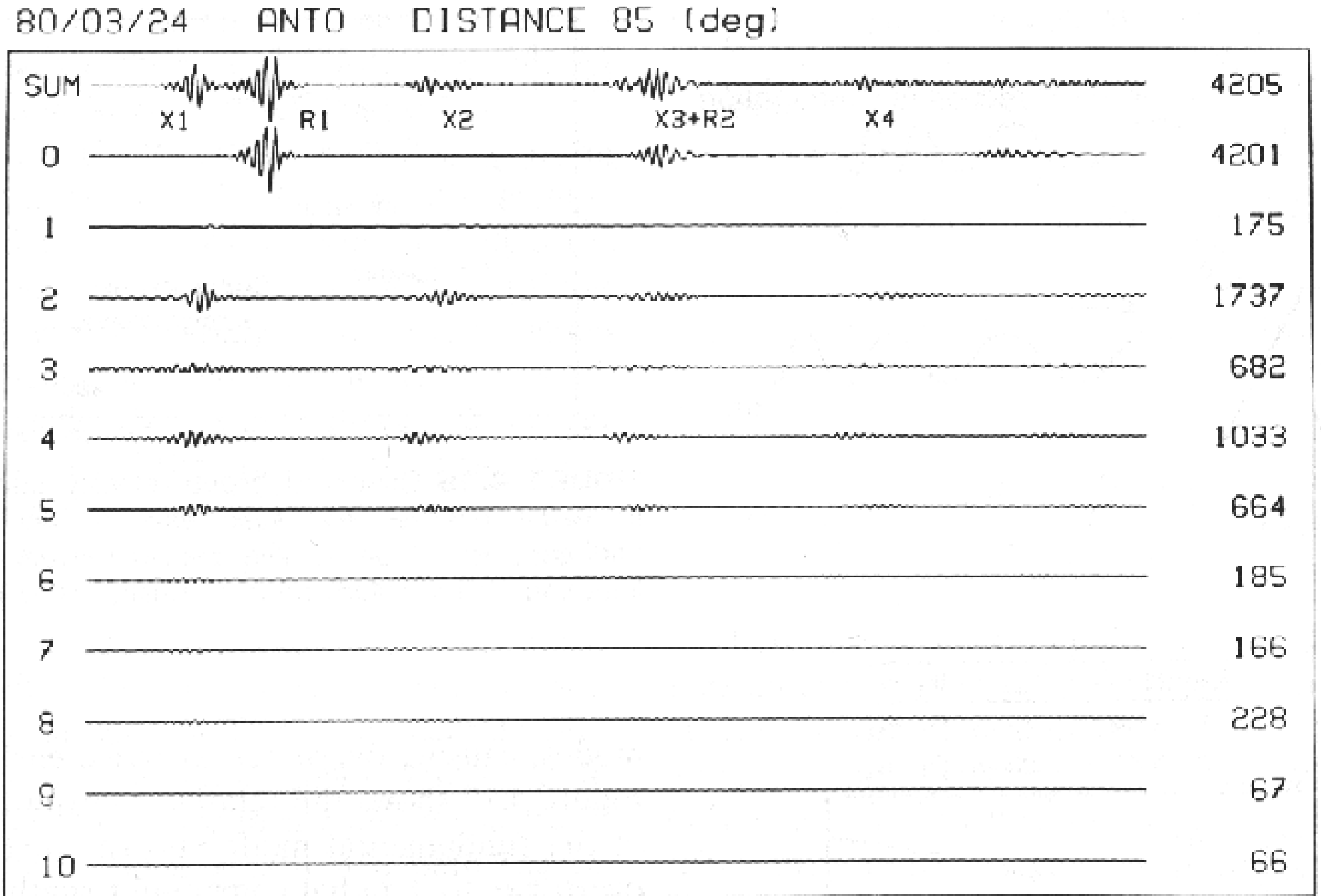
Layer	v_p	v_s	ρ	H
1	$1.732 v_{s1}$	v_{s1}	ρ_1	H_1
2	$3.873 v_{s1}$	$2.236 v_{s1}$	ρ_1	∞



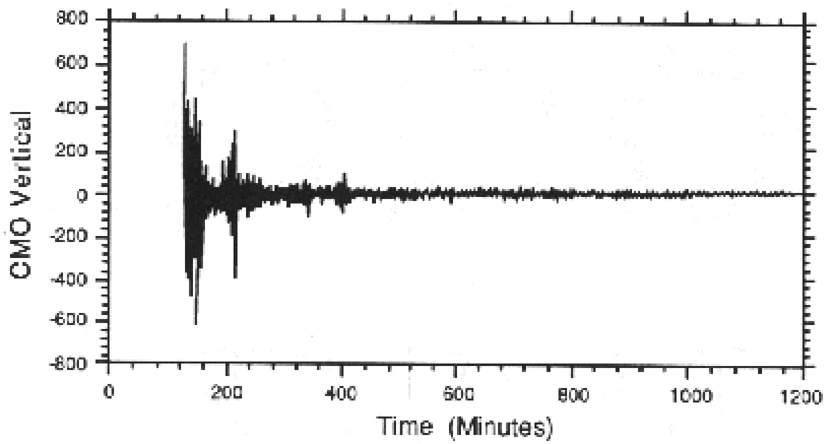
Observed Group Velocities (T< 80s)



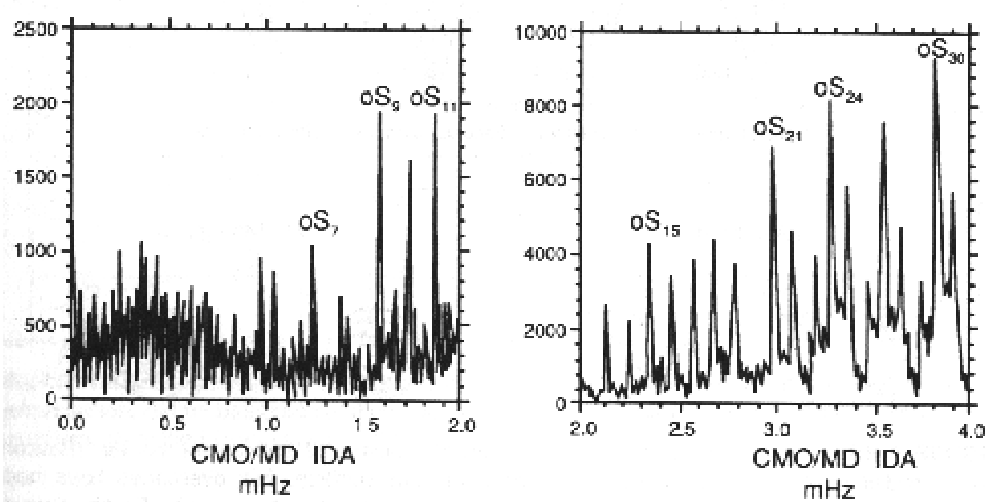
Modal Summation



Free oscillations - Data



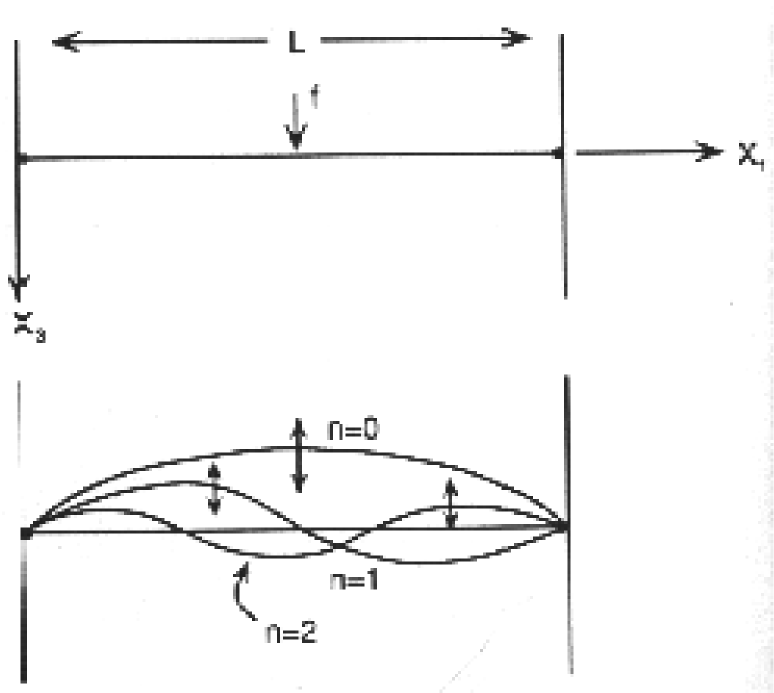
20-hour long recording of a gravimeter recordind the strong earthquake near Mexico City in 1985 (tides removed). Spikes correspond to Rayleigh waves.



Power Spectrum

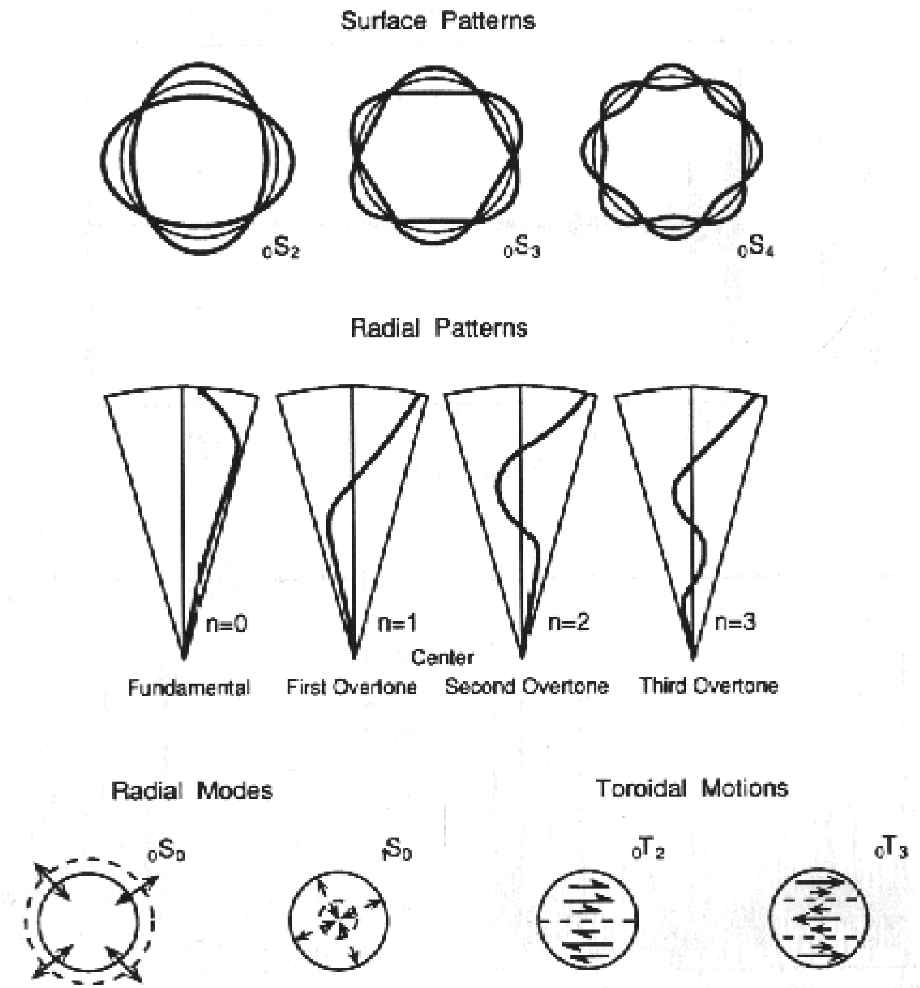
Spectra of the seismogram given above. Spikes at discrete frequencies correspond to eigenfrequencies of the Earth

Eigenmodes of a string



Geometry of a string undern tension with fixed end points. Motions of the string excited by any source comprise a weighted sum of the eigenfunctions (which?).

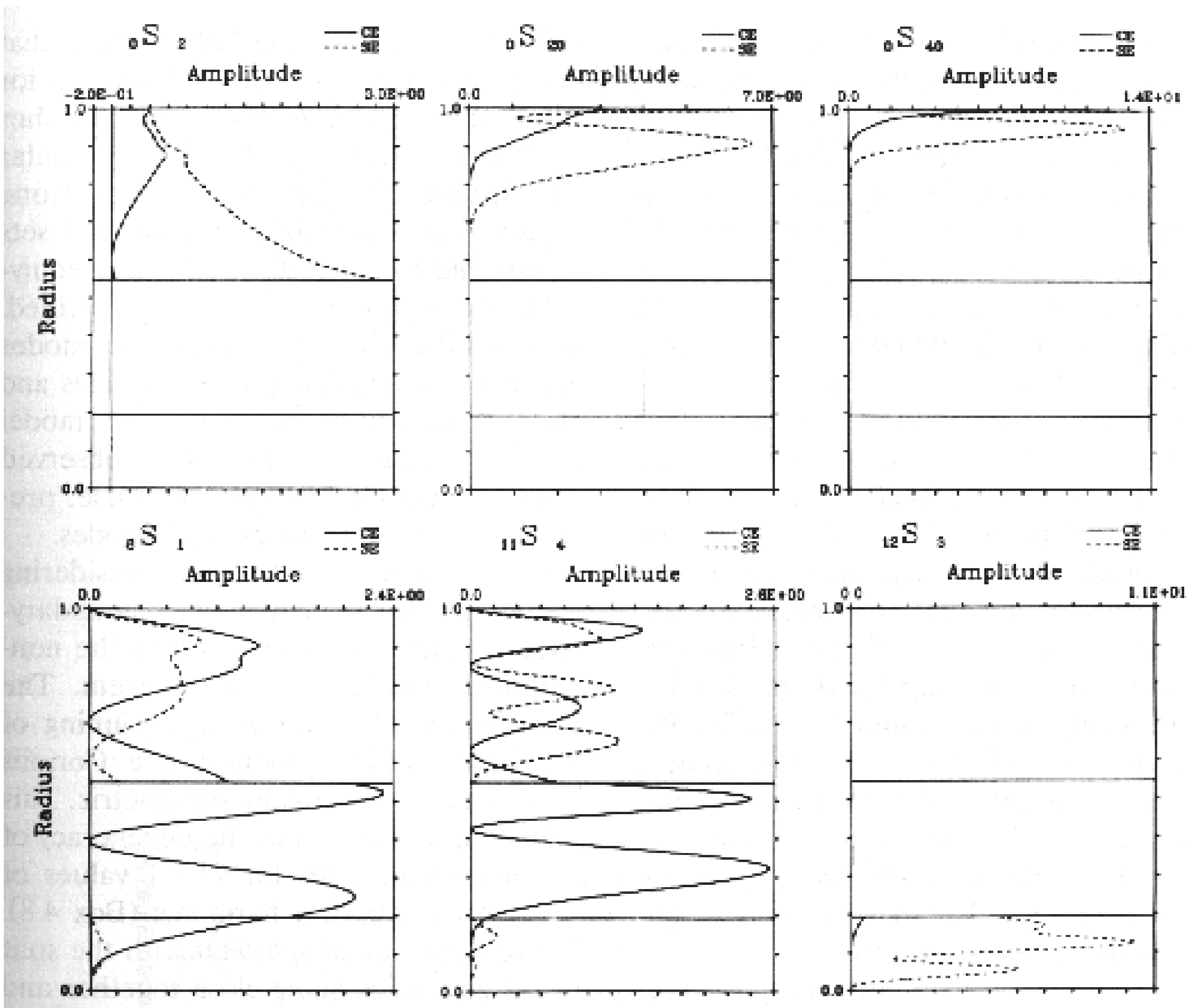
Eigenmodes of a sphere



Eigenmodes of a homogeneous sphere. Note that there are modes with only volumetric changes (like P waves, called spheroidal) and modes with pure shear motion (like shear waves, called toroidal).

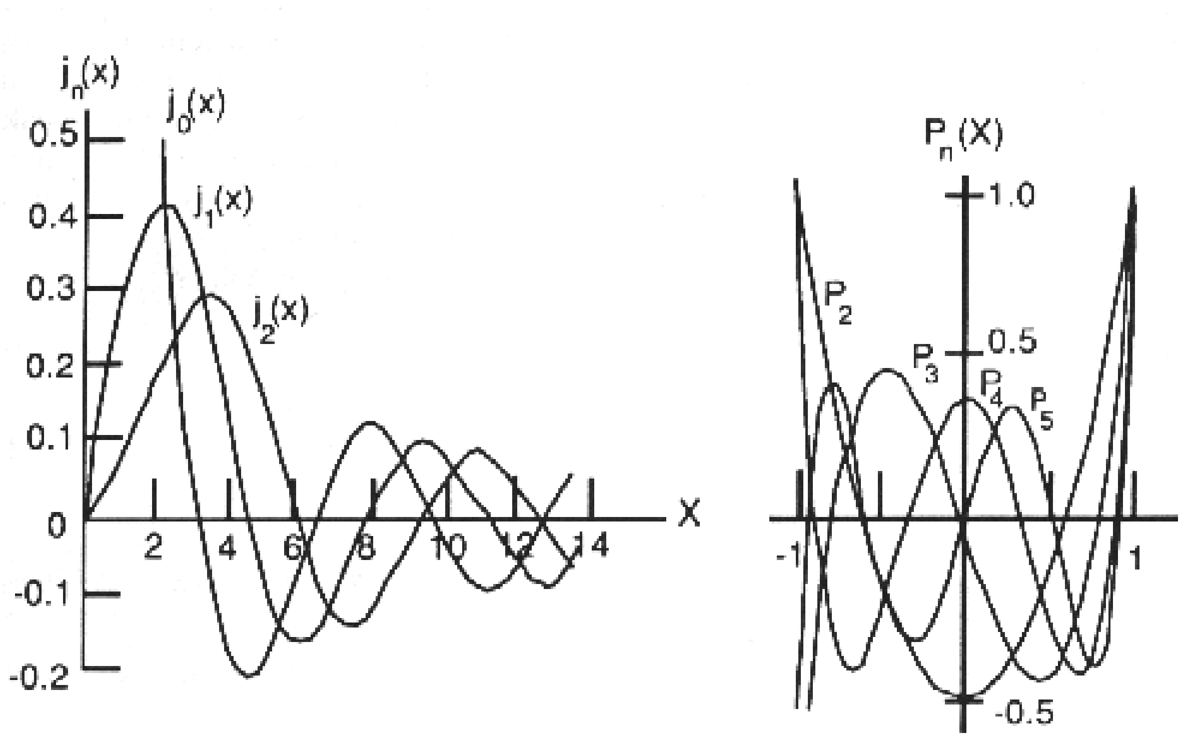
- pure radial modes involve no nodal patterns on the surface
- overtones have nodal surfaces at depth
- toroidal modes involve purely horizontal twisting
- toroidal overtones have nodal surfaces at constant radii.

Eigenmodes of a sphere



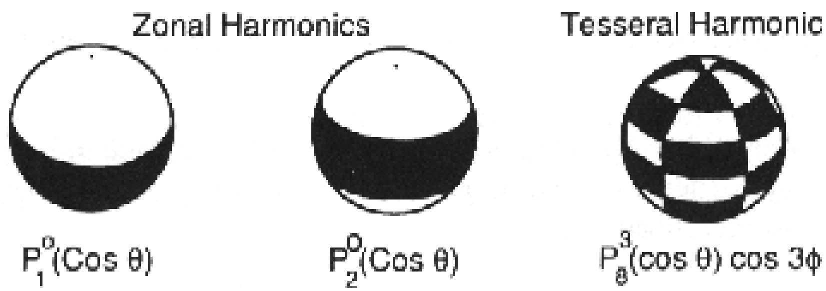
Compressional (solid) and shear (dashed) energy density for fundamental spheroidal modes and some overtones, sensitive to core structure.

Bessel and Legendre



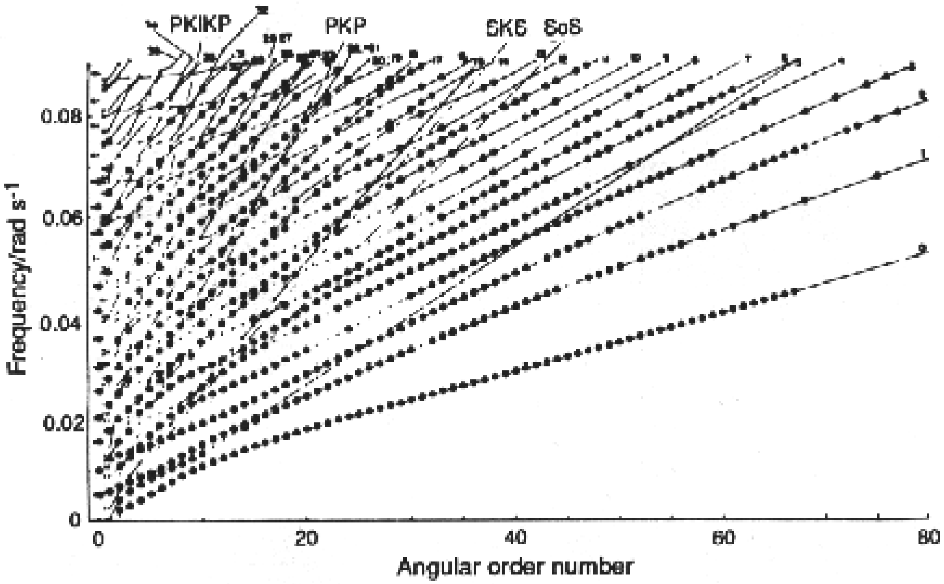
Solutions to the wave equation on spherical coordinates: Bessel functions (left) and Legendre polynomials (right).

Spherical Harmonics

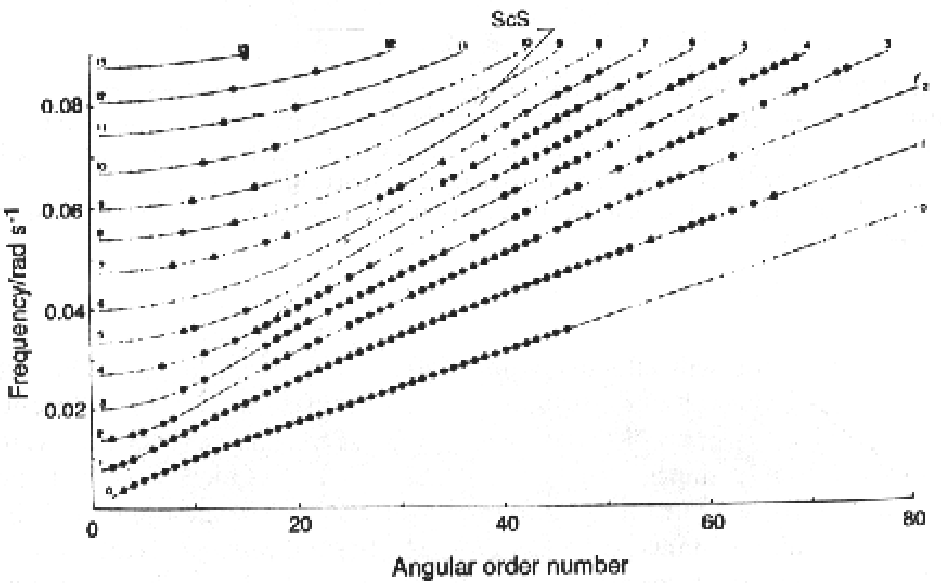


Examples of spherical surface harmonics. There are zonal, sectoral and tesseral harmonics.

The Earth's Eigenfrequencies

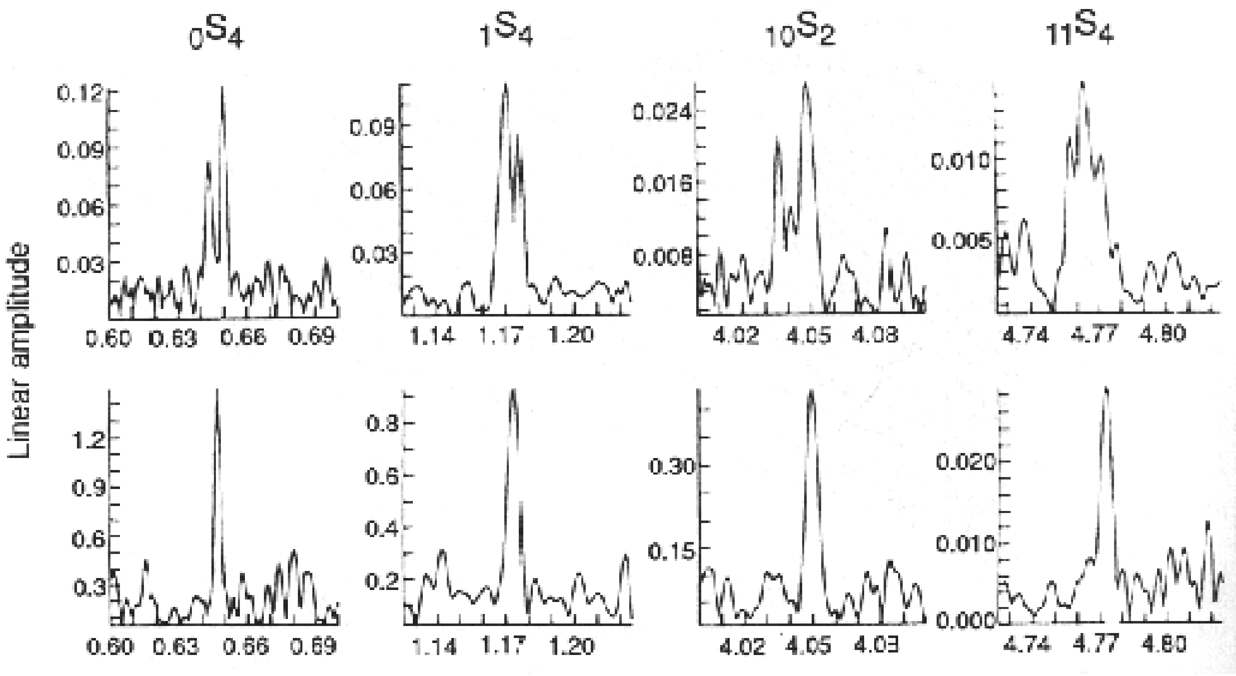


Spheroidal mode eigenfrequencies



Toroidal mode eigenfrequencies

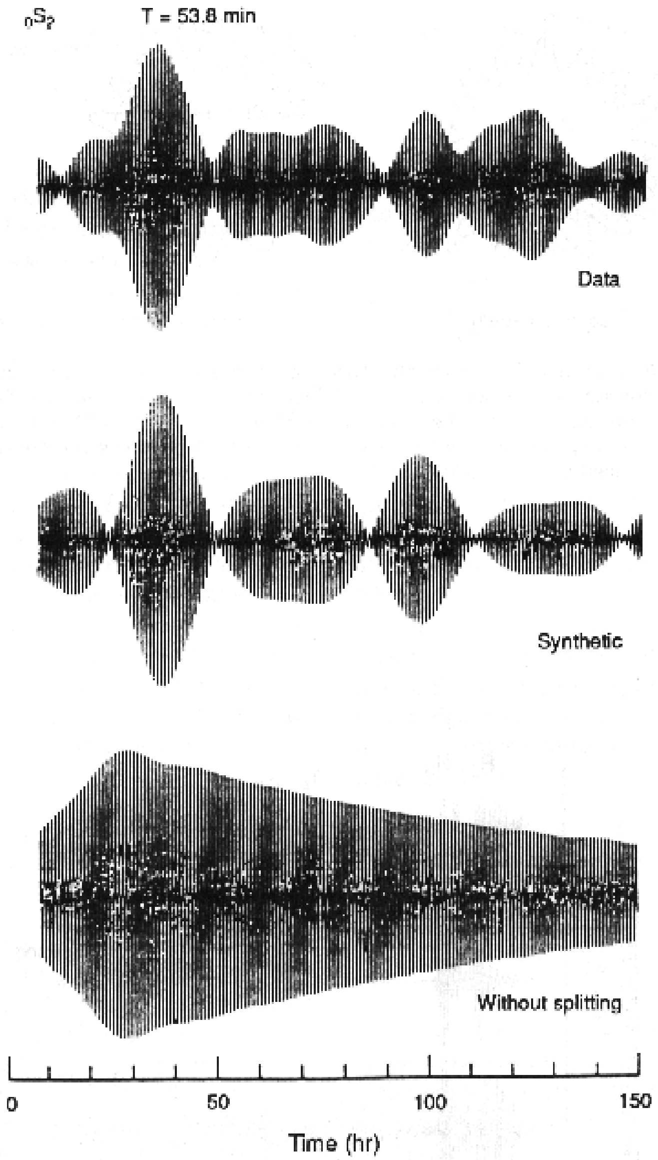
Effects of Earth's Rotation



non-polar latitude

polar latitude

Effects of Earth's Rotation: seismograms



observed

synthetic

synthetic no splitting

Lateral heterogeneity

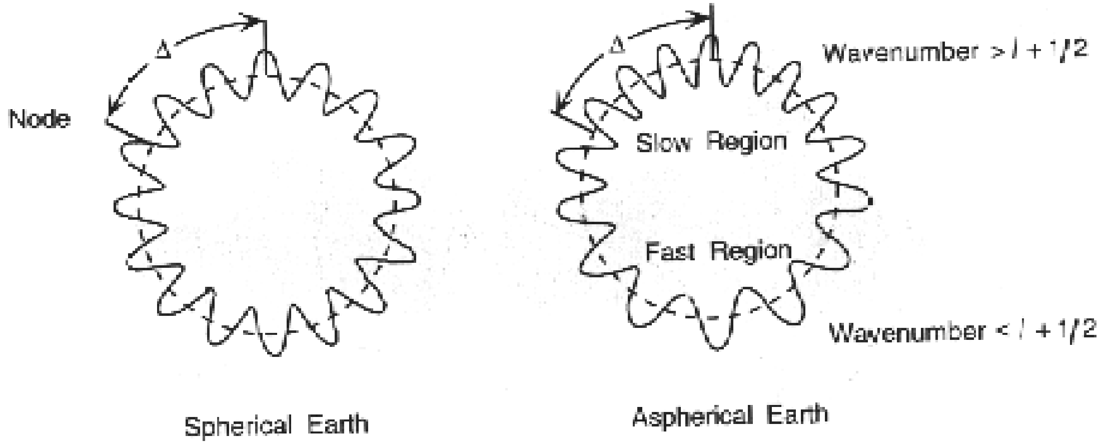


Illustration of the distortion of standing-waves due to heterogeneity. The spatial shift of the phase perturbs the observed multiplet amplitude

Surface Waves: Summary

Rayleigh waves are solutions to the elastic wave equation given a half space and a free surface. Their amplitude decays exponentially with depth. The particle motion is elliptical and consists of motion in the plane through source and receiver.

SH-type surface waves do not exist in a half space. However in layered media, particularly if there is a low-velocity surface layer, so-called Love waves exist which are dispersive, propagate along the surface. Their amplitude also decays exponentially with depth.

Free oscillations are standing waves which form after big earthquakes inside the Earth. Spheroidal and toroidal eigenmodes correspond are analogous concepts to P and shear waves.