Numerical Methods in Spherical Geometry

- The wave equation in spherical coordinates
- The axi-symmetric approximation
- Single domain - multi-domain methods
- Sources in the axi-symmetric approach
- \- Spherical sections
- \- Cubed Sphere
- \- Hybrid approach
- Examples

Wave equation - spherical coordinates

$$
\begin{aligned}
\rho\partial_t v_r &= \partial_r\sigma_{rr} + \frac{1}{r}\partial_\theta\sigma_{r\theta} + \frac{1}{r\sin\theta}\partial_\varphi\sigma_{r\varphi} \\
&\quad + \frac{1}{r}(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi} + \sigma_{r\theta}\cot\theta) + f_r \\
\rho\partial_t v_\theta &= \partial_r\sigma_{r\theta} + \frac{1}{r}\partial_\theta\sigma_{\theta\theta} + \frac{1}{r\sin\theta}\partial_\varphi\sigma_{\theta\varphi} \\
&\quad + \frac{1}{r}((\sigma_{\theta\theta} - \sigma_{\varphi\varphi})\cot\theta + 3\sigma_{r\theta}) + f_\theta \\
\rho\partial_t v_\varphi &= \partial_r\sigma_{r\varphi} + \frac{1}{r}\partial_\theta\sigma_{\theta\varphi} + \frac{1}{r\sin\theta}\partial_\varphi\sigma_{\varphi\varphi} \\
&\quad + \frac{1}{r}((3\sigma_{r\varphi} + 2\sigma_{\theta\varphi}\cot\theta)) + f_\varphi
\end{aligned}
$$

Equations of motion in spherical coordinates
velocity - stress formulation

Wave equation - stress - strain

$$
\begin{aligned}
\partial_t \epsilon_{rr} &= \partial_r v_r \\
\partial_t \epsilon_{\theta\theta} &= \frac{1}{r}\partial_\theta v_\theta + \frac{1}{r} v_r \\
\partial_t \epsilon_{\varphi\varphi} &= \frac{1}{r\sin\theta}\partial_\varphi v_\varphi + \frac{1}{r} v_r + \frac{\cot\theta}{r} v_\theta \\
\partial_t \epsilon_{r\theta} &= \frac{1}{2}(\frac{1}{r}\partial_\theta v_r + \partial_r v_\theta - \frac{1}{r} v_\theta) \\
\partial_t \epsilon_{\theta\varphi} &= \frac{1}{2}(\frac{1}{r\sin\theta}\partial_\varphi v_\theta + \frac{1}{r}\partial_\theta v_\varphi - \frac{\cot\theta}{r} v_\varphi) \\
\partial_t \epsilon_{r\varphi} &= \frac{1}{2}(\frac{1}{r\sin\theta}\partial_\varphi v_r + \partial_r v_\varphi - \frac{1}{r} v_\varphi)\ .
\end{aligned}
$$

Equations of motion in spherical coordinates
strain-displacement relation

Wave equation - Hooke's Law

$$
\begin{aligned}
\sigma_{rr} &= \lambda\Delta + 2\mu\epsilon_{rr} + M_{rr} \\
\sigma_{\theta\theta} &= \lambda\Delta + 2\mu\epsilon_{\theta\theta} + M_{\theta\theta} \\
\sigma_{\varphi\varphi} &= \lambda\Delta + 2\mu\epsilon_{\varphi\varphi} + M_{\varphi\varphi} \\
\sigma_{r\theta} &= 2\mu\epsilon_{r\theta} + M_{r\theta} \\
\sigma_{\theta\varphi} &= 2\mu\epsilon_{\theta\varphi} + M_{\theta\varphi} \\
\sigma_{r\varphi} &= 2\mu\epsilon_{r\varphi} + M_{r\varphi} \quad ,
\end{aligned}
$$

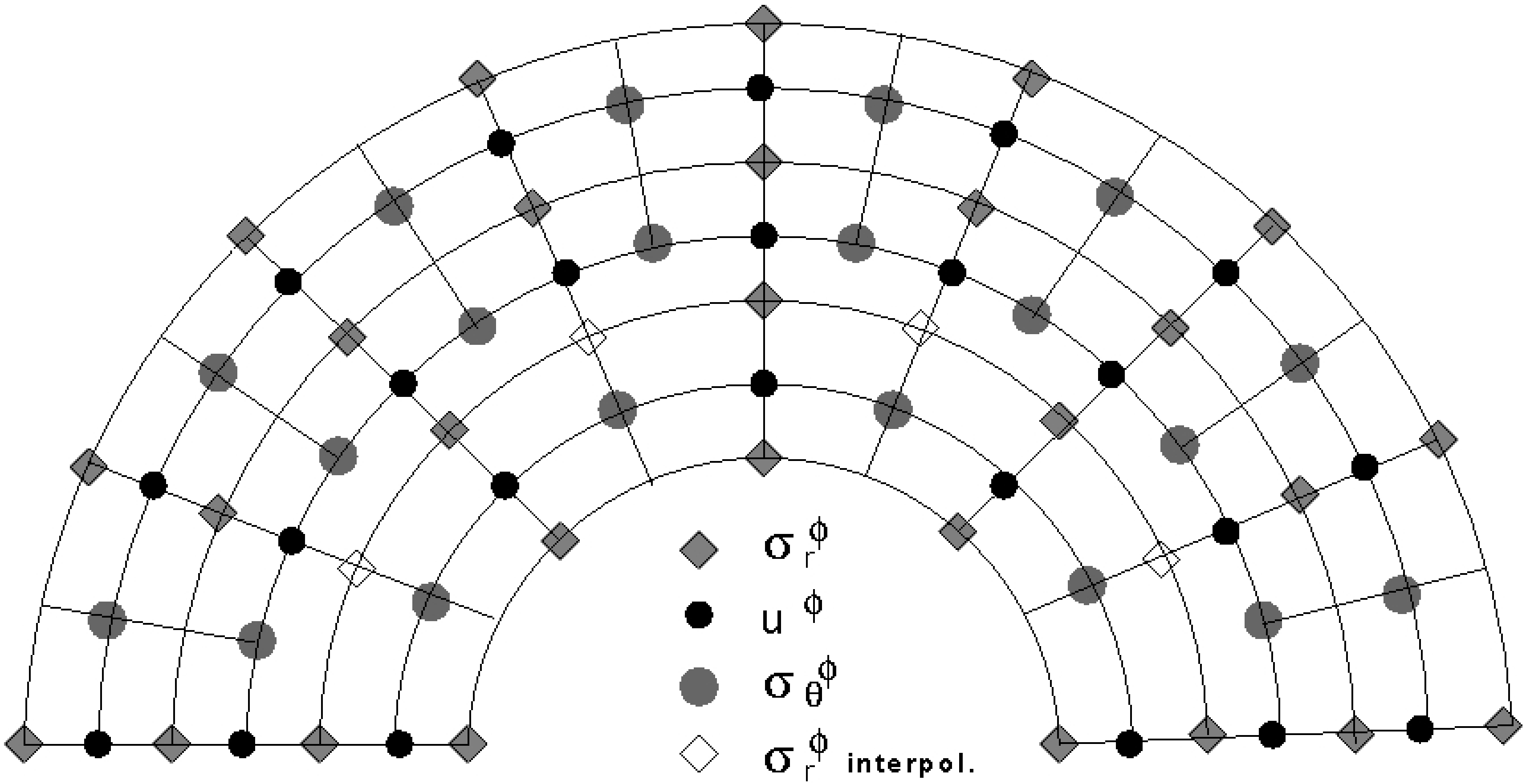
Equations of motion in spherical coordinates
stress-strain relation (isotropic media)

Axisymmetric approximation

$$
\begin{aligned}
\rho \partial_t v_r &= \partial_r \sigma_{rr} + \frac{1}{r}\partial_\theta \sigma_{r\theta} + \frac{1}{r\sin\theta}\boxed{\partial_\varphi \sigma_{r\varphi}} \\
&+ \frac{1}{r}(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi} + \sigma_{r\theta}\cot\theta) + f_r \\
\rho \partial_t v_\theta &= \partial_r \sigma_{r\theta} + \frac{1}{r}\partial_\theta \sigma_{\theta\theta} + \frac{1}{r\sin\theta}\boxed{\partial_\varphi \sigma_{\theta\varphi}} \\
&+ \frac{1}{r}((\sigma_{\theta\theta} - \sigma_{\varphi\varphi})\cot\theta + 3\sigma_{r\theta}) + f_\theta \\
\rho \partial_t v_\varphi &= \partial_r \sigma_{r\varphi} + \frac{1}{r}\partial_\theta \sigma_{\theta\varphi} + \frac{1}{r\sin\theta}\boxed{\partial_\varphi \sigma_{\varphi\varphi}} \\
&+ \frac{1}{r}((3\sigma_{r\varphi} + 2\sigma_{\theta\varphi}\cot\theta)) + f_\varphi
\end{aligned}
$$

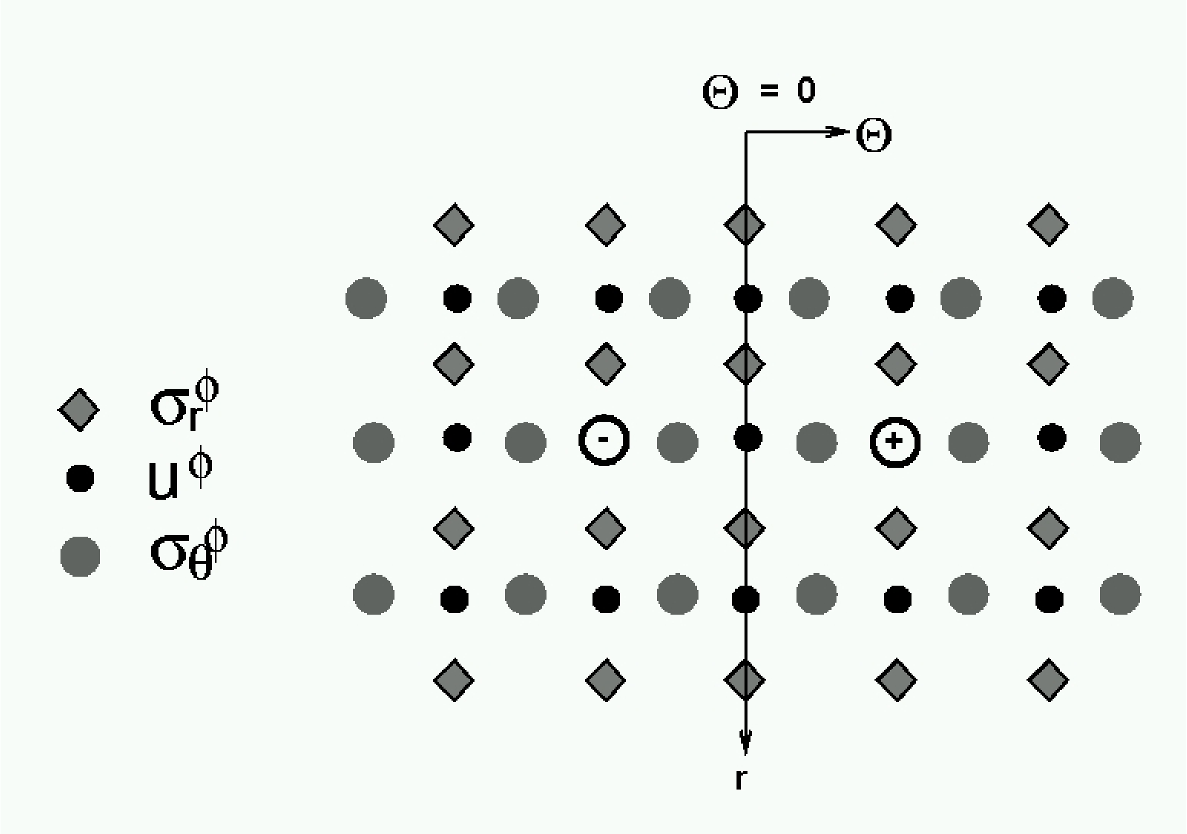
When all fields (and the model) are axisymmetric the terms in the red squares vanish.

Axisymmetric grids - SH case



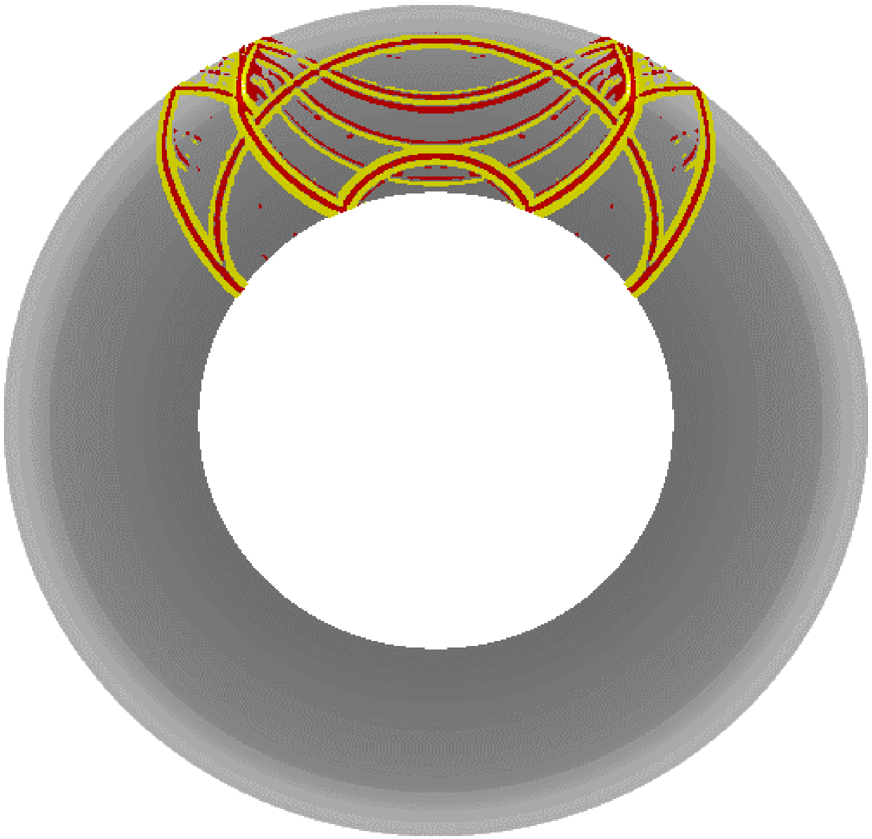
Two-domain method SH case

Axisymmetric grids - sources



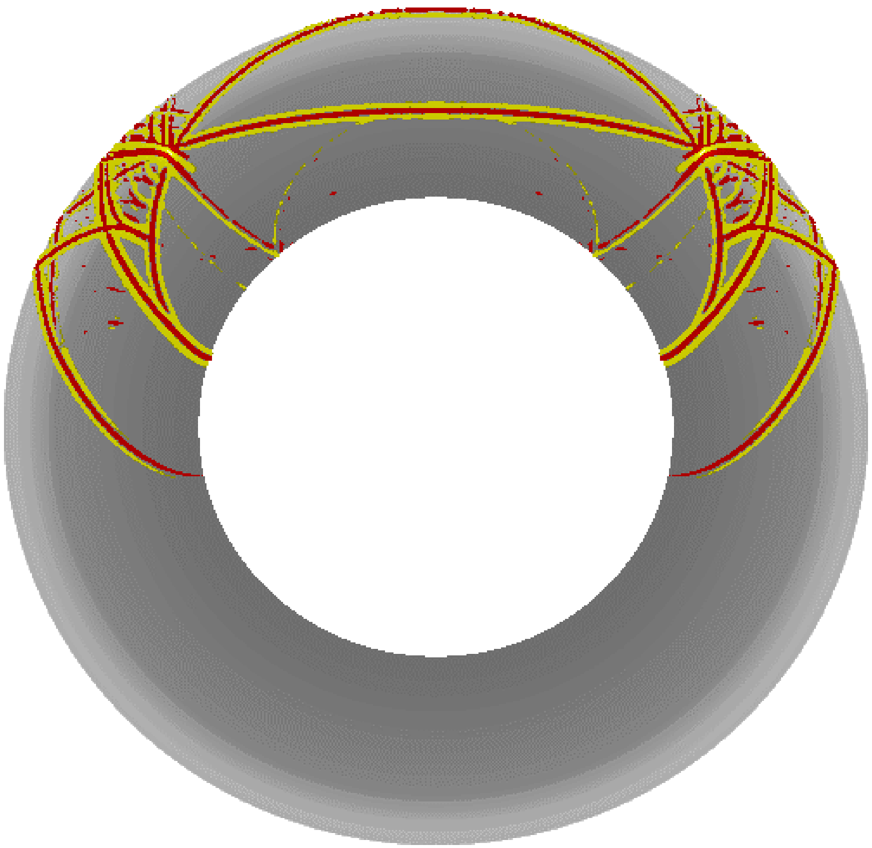
Implementing sources in axisymmetric methods

Axisymmetric grids - SH waves



SH waves - FD high order - Source at 600km depth
PREM model - dominant period 25 seconds

Axisymmetric grids - SH waves



SH waves - FD high order - Source at 600km depth
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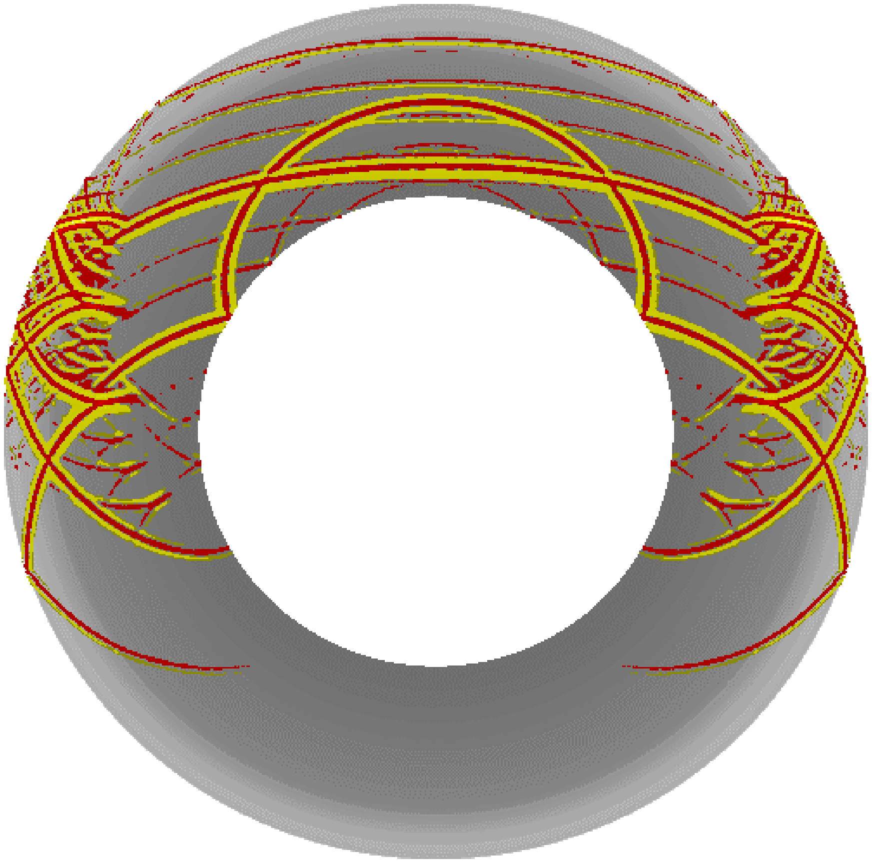
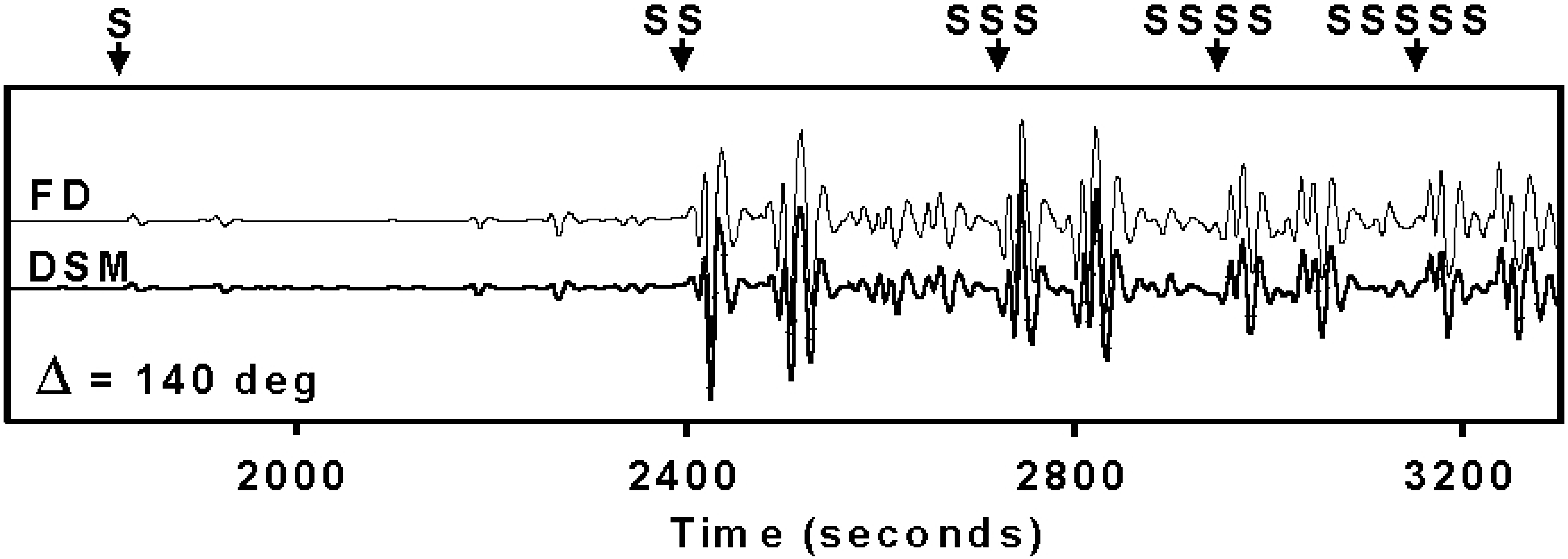
Axisymmetric grids - SH waves



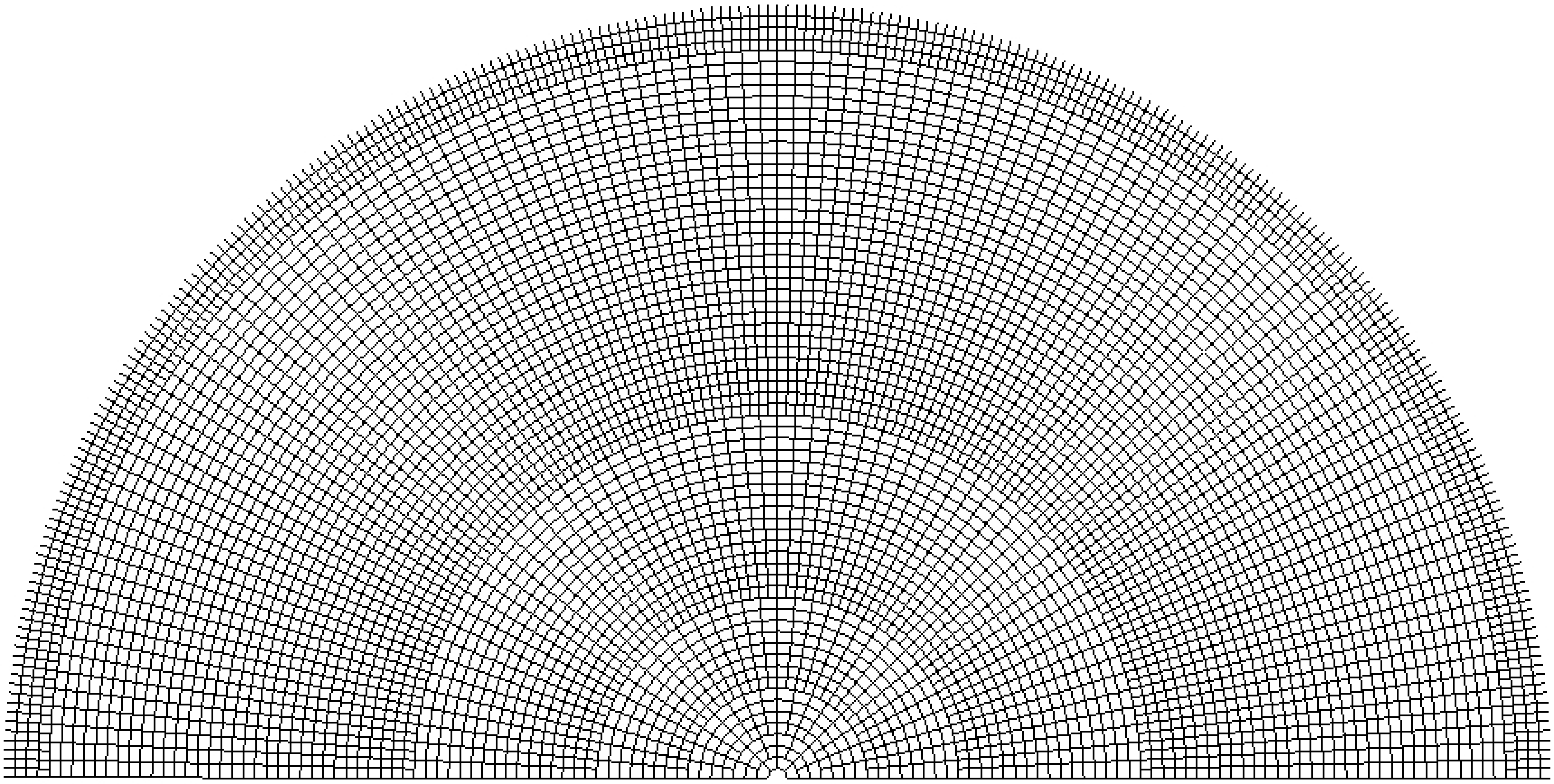
SH waves - FD high order - Source at 600km depth
PREM model - dominant period 25 seconds

Axisymmetric grids - seismograms



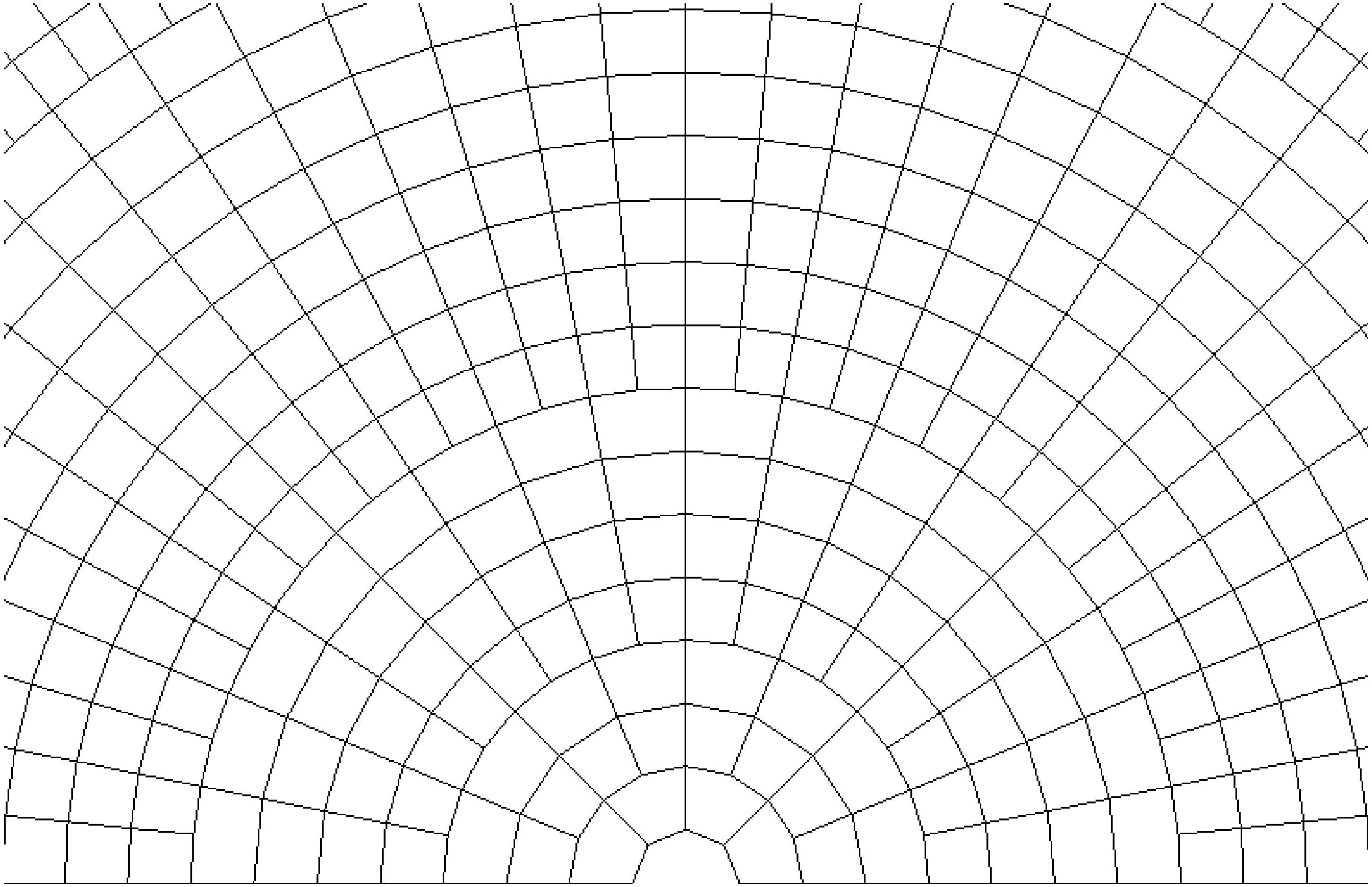
Comparison of *exact* method (DSM) with FD method.
Spherically symmetric model PREM.

Axisymmetric grids - whole Earth



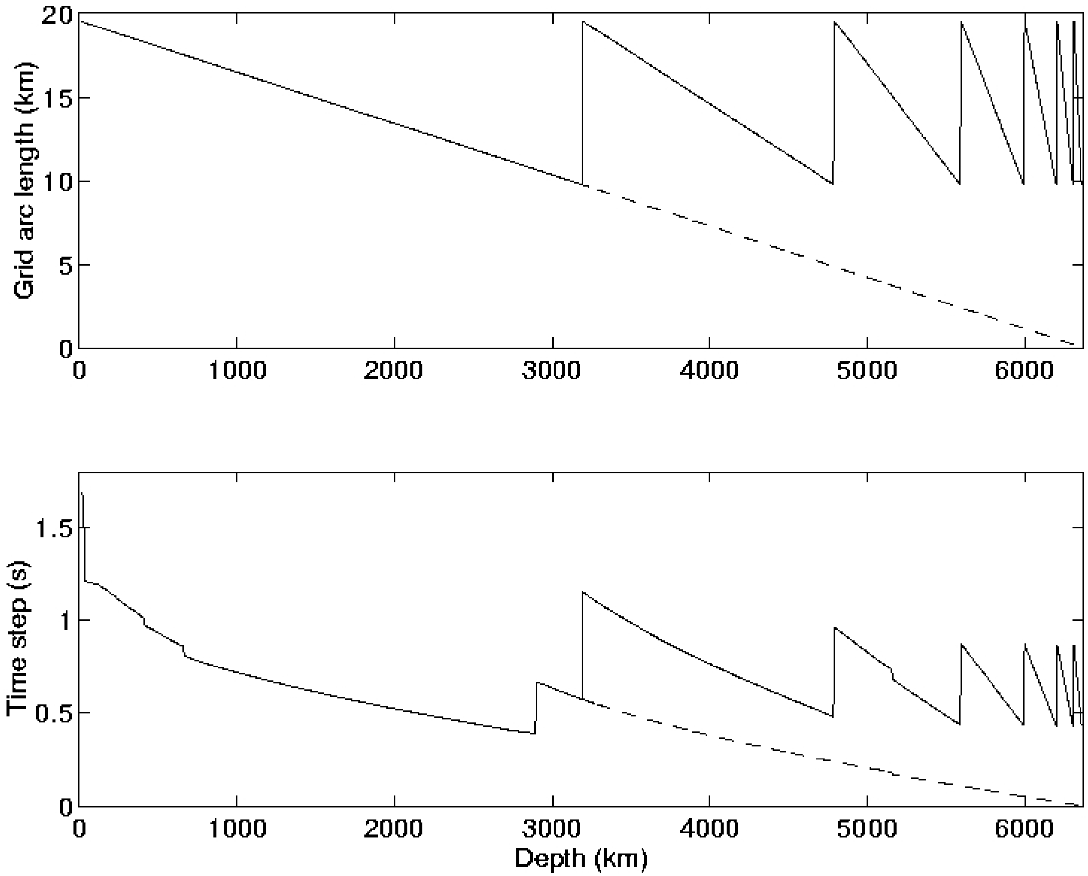
Single domain vs. multidomain

Axisymmetric grids - center



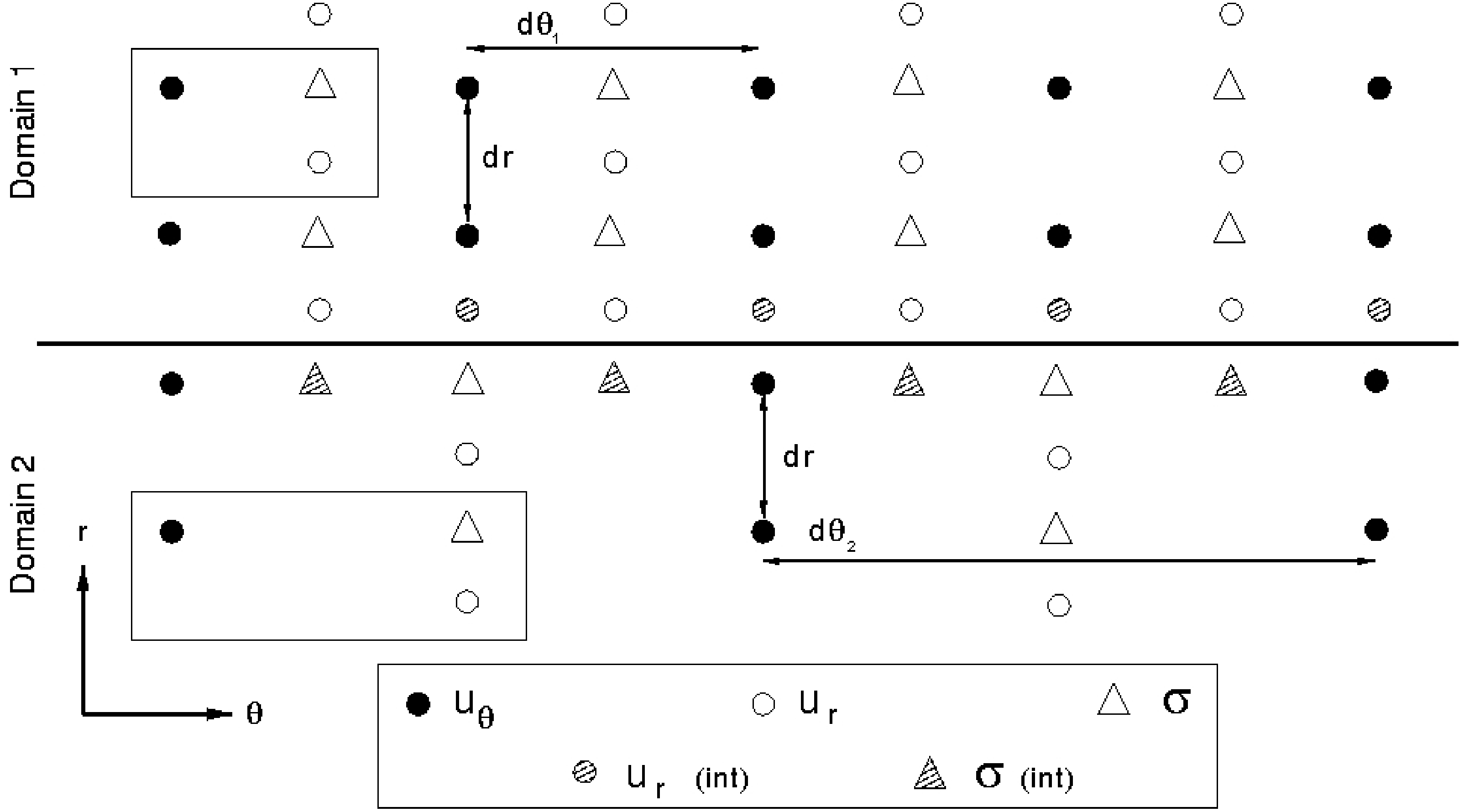
What should we do with the center?

Axisymmetric grids - stability



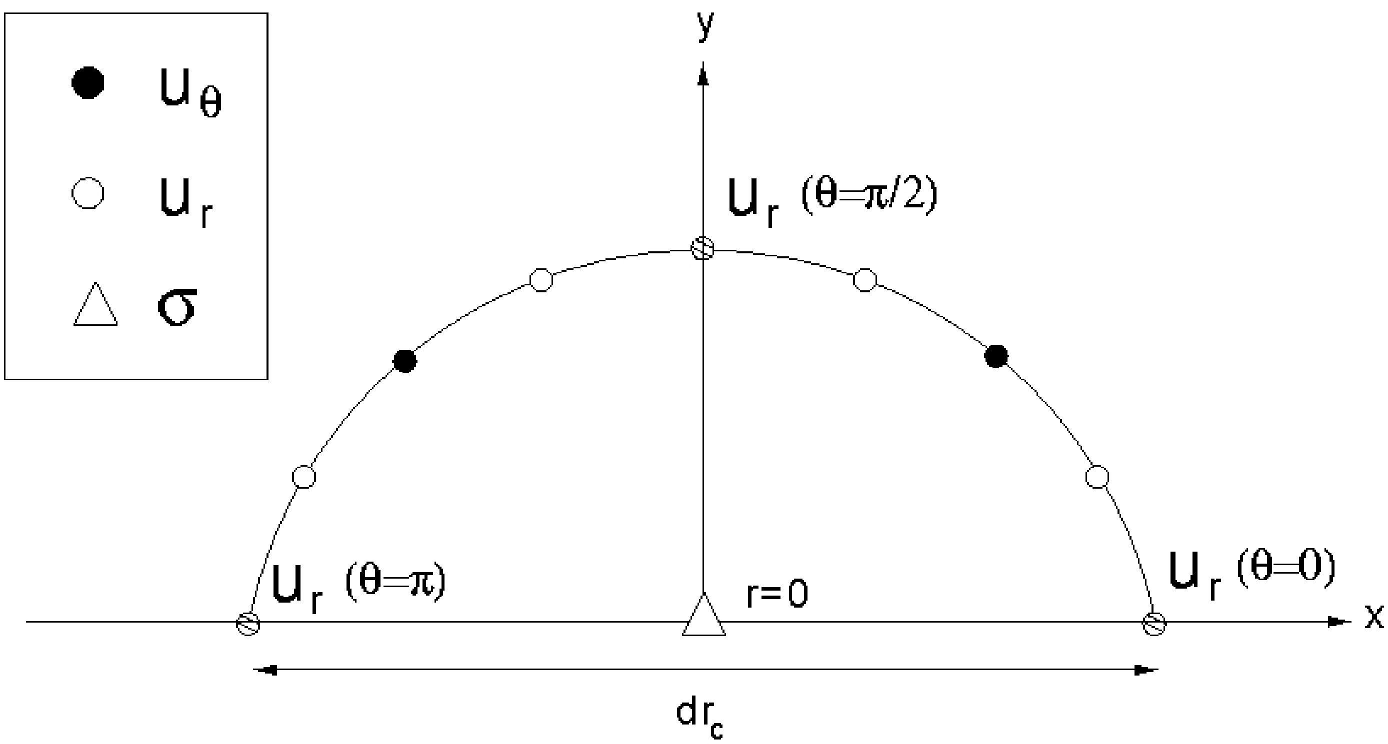
Top: lateral grid spacing as a function of depth multidomain and single domain (dashed). Bottom: Required time step for PREM for a stable calculation for multidomain and single domain (dashed).

Axisymmetric grids - staggering



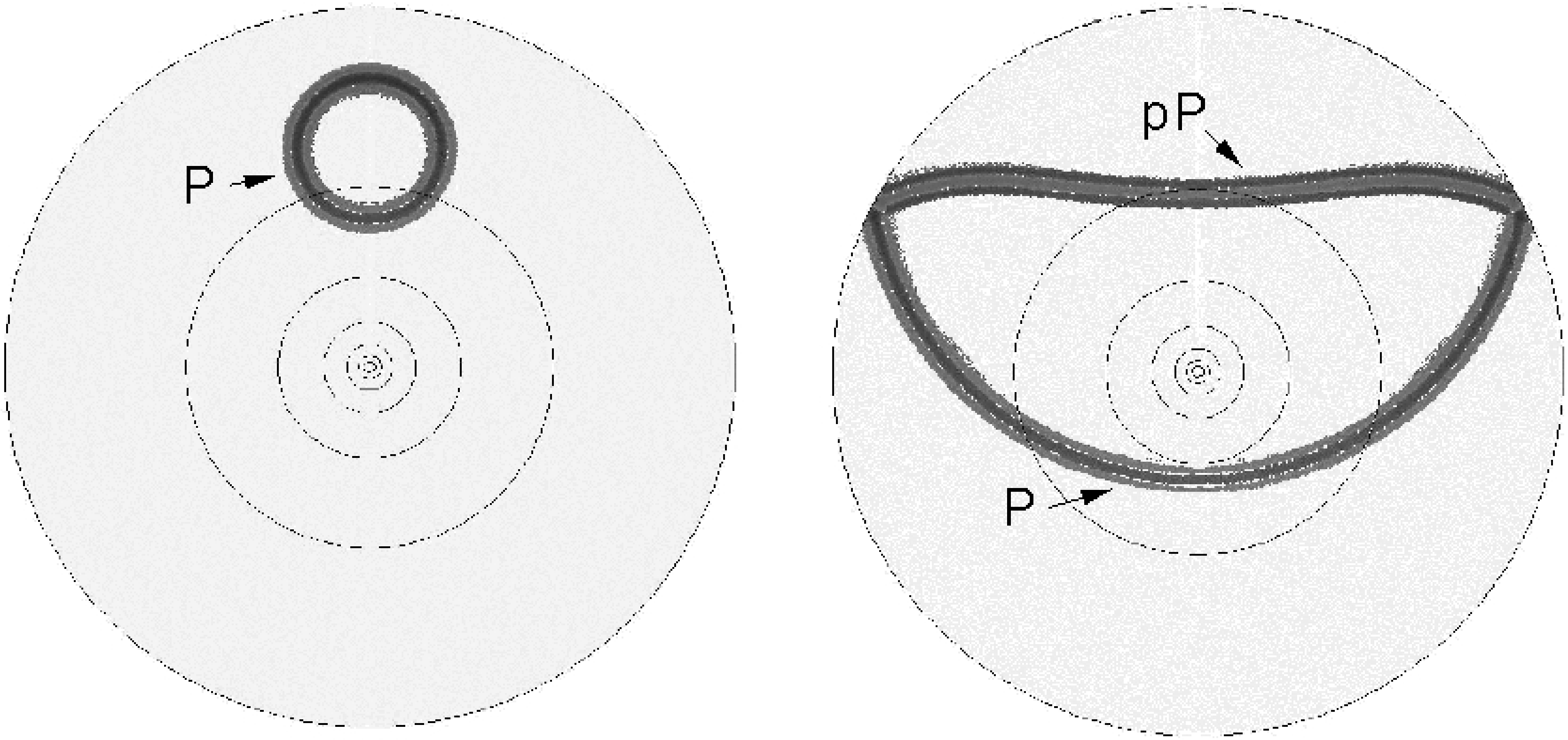
Staggering in the acoustic approximation. Multi-domain approach.
Domain connection through interpolation.

Axisymmetric grids - centre



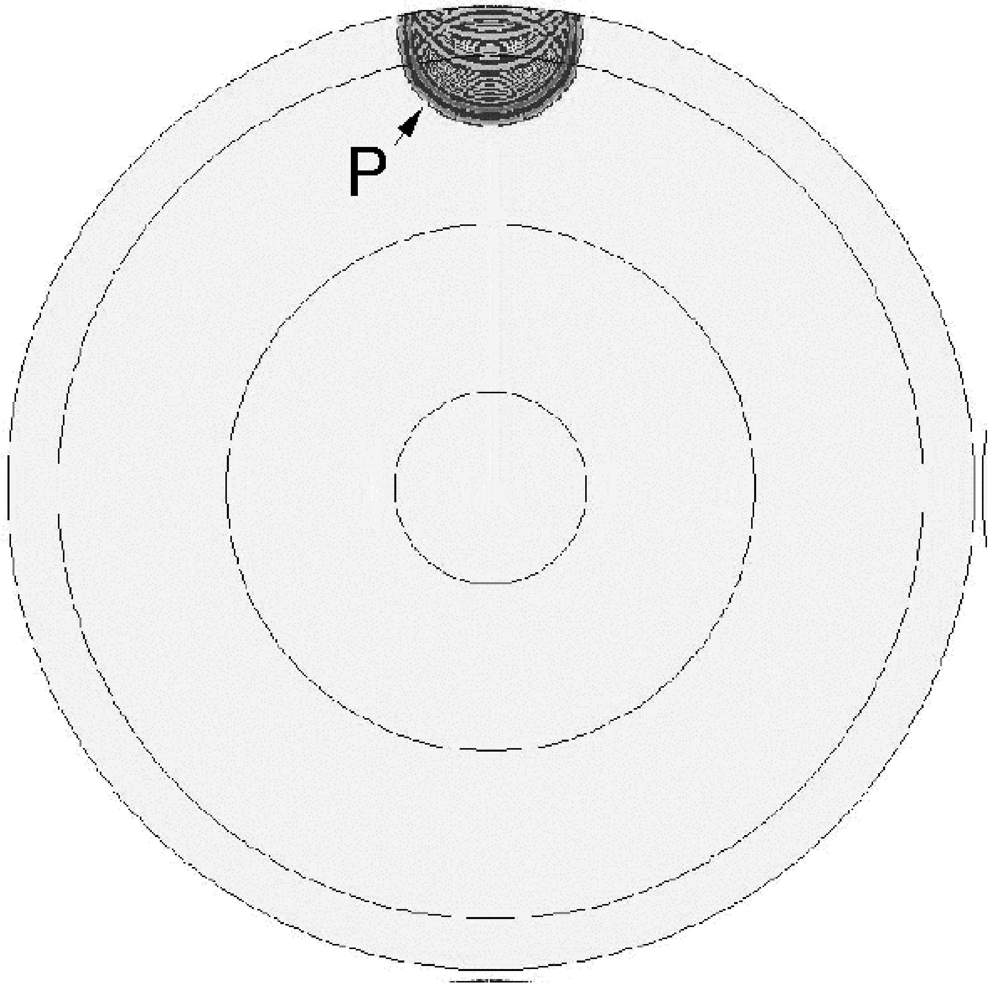
The stress at the center is calculated from the radial displacements at $r=dr_c/2$.

Acoustic case - homogeneous



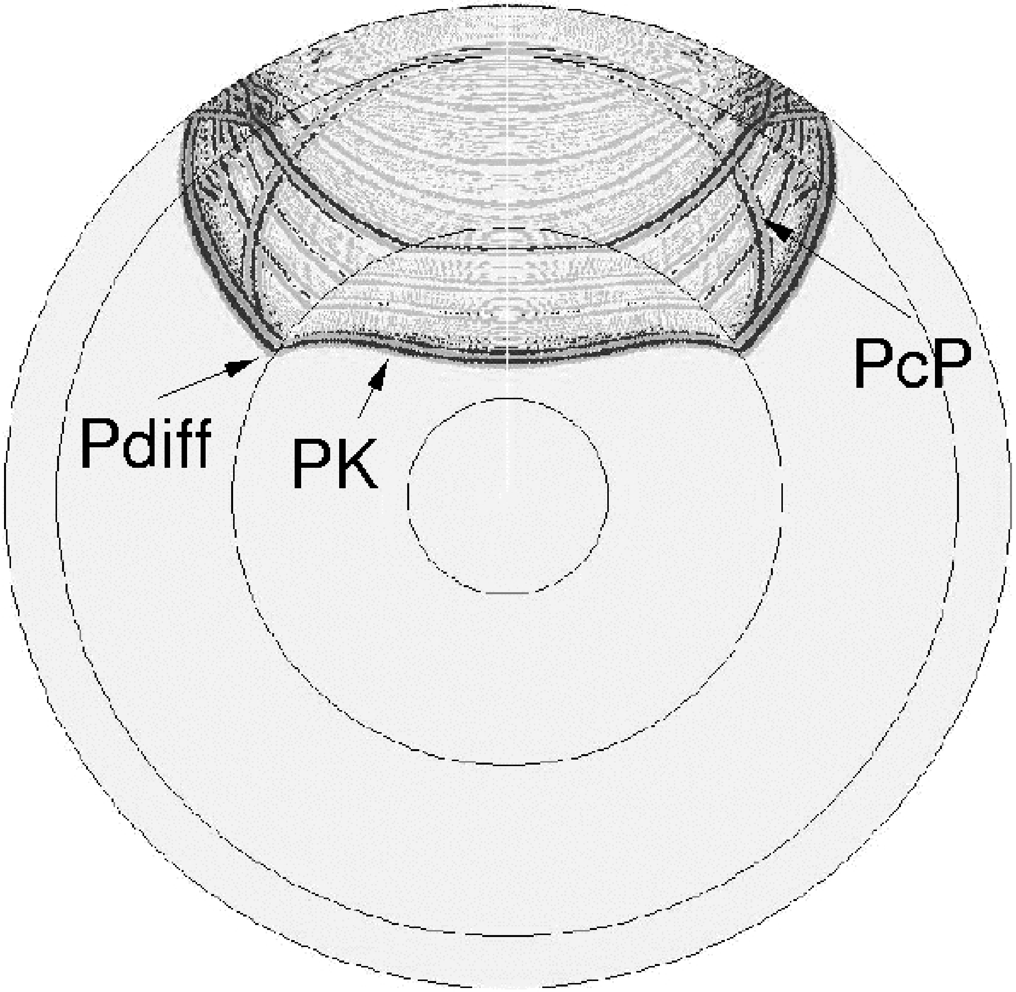
Acoustic waves in a homogeneous sphere. Multidomain approach.

Acoustic case - PREM model



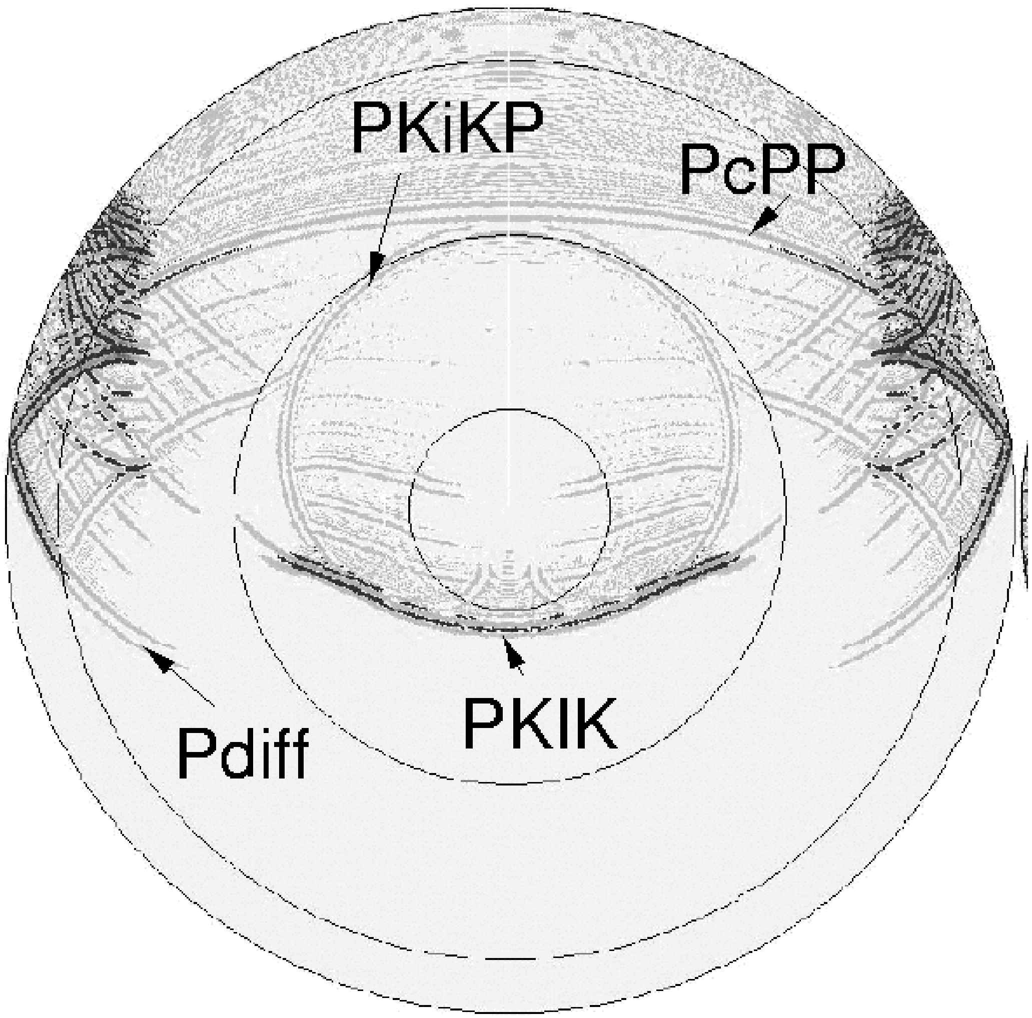
Acoustic waves in a sphere. Multidomain approach. PREM model.

Acoustic case - PREM model



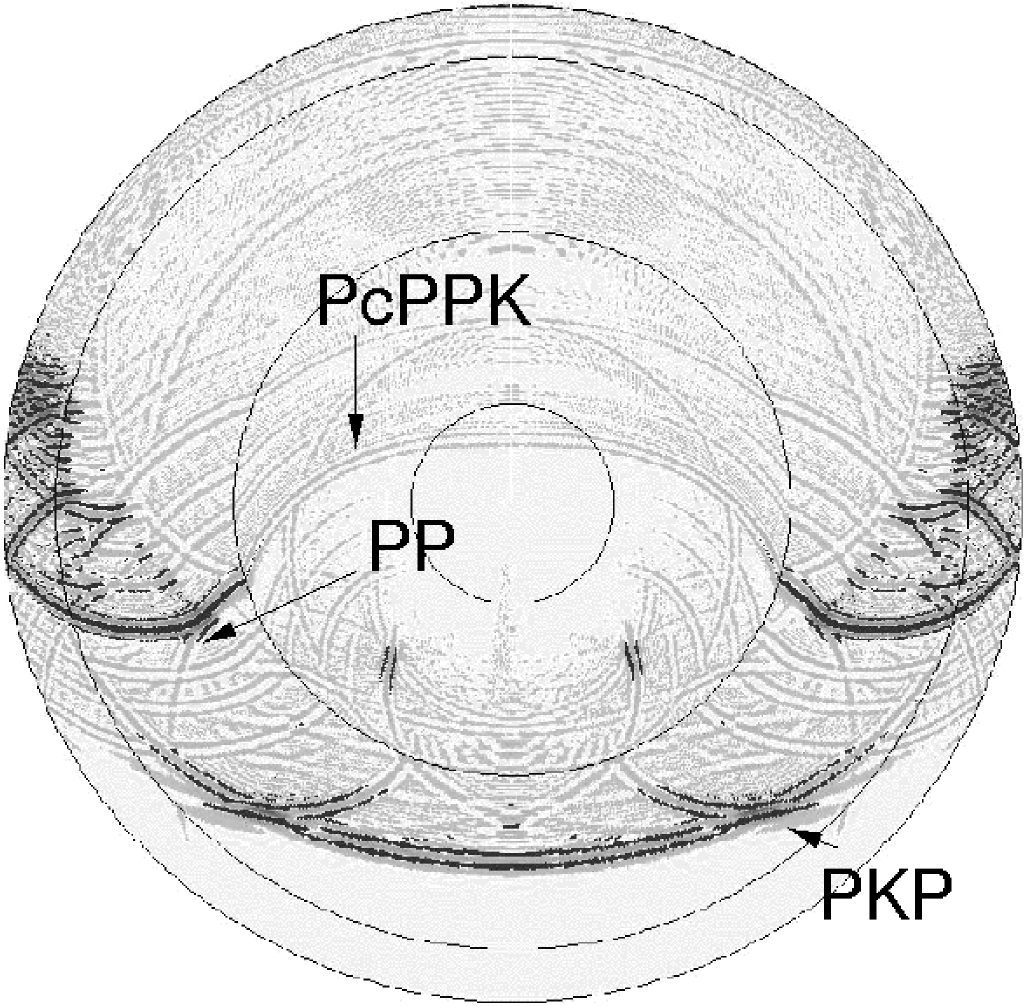
Acoustic waves in a sphere. Multidomain approach. PREM model.

Acoustic case - PREM model



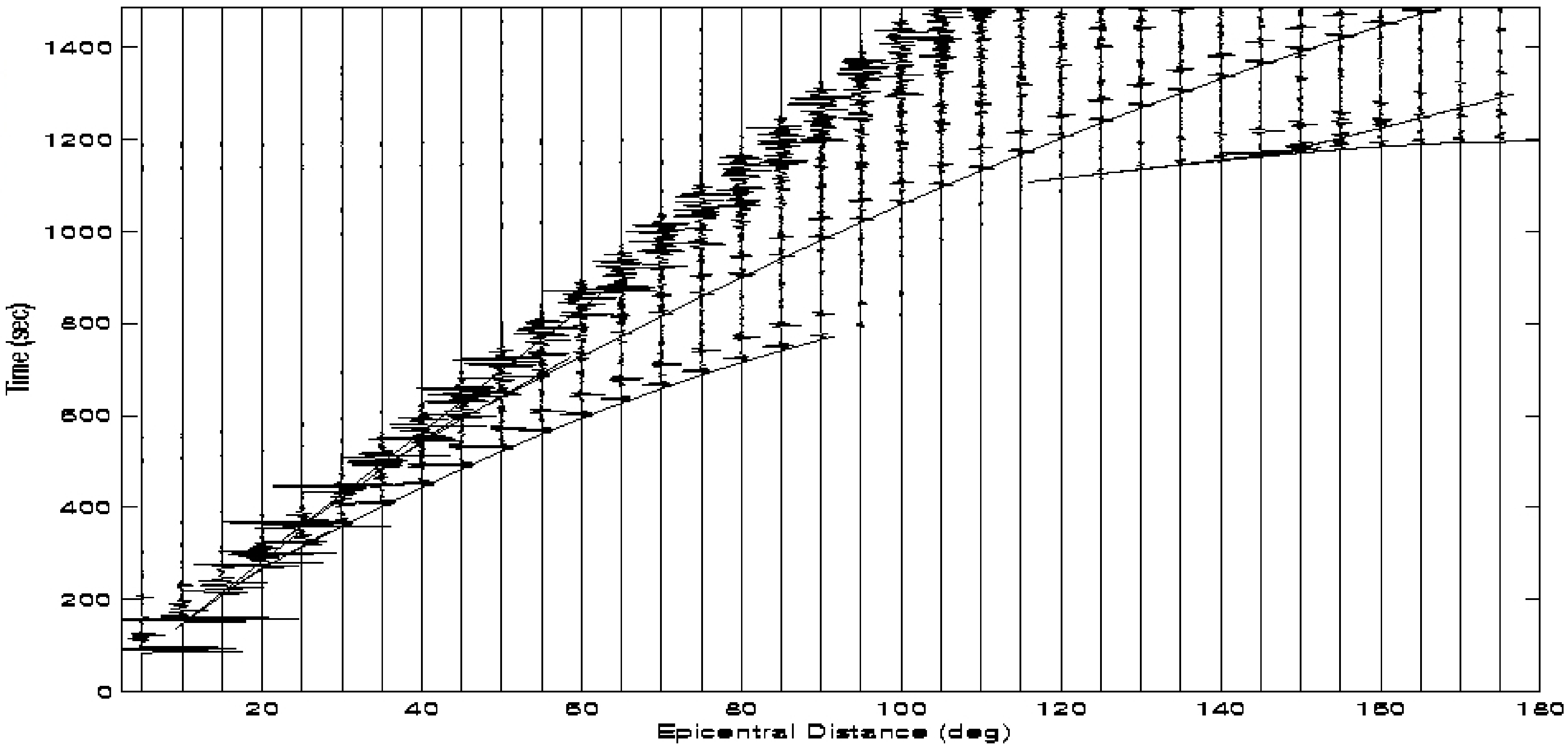
Acoustic waves in a sphere. Multidomain approach. PREM model.

Acoustic case - PREM model



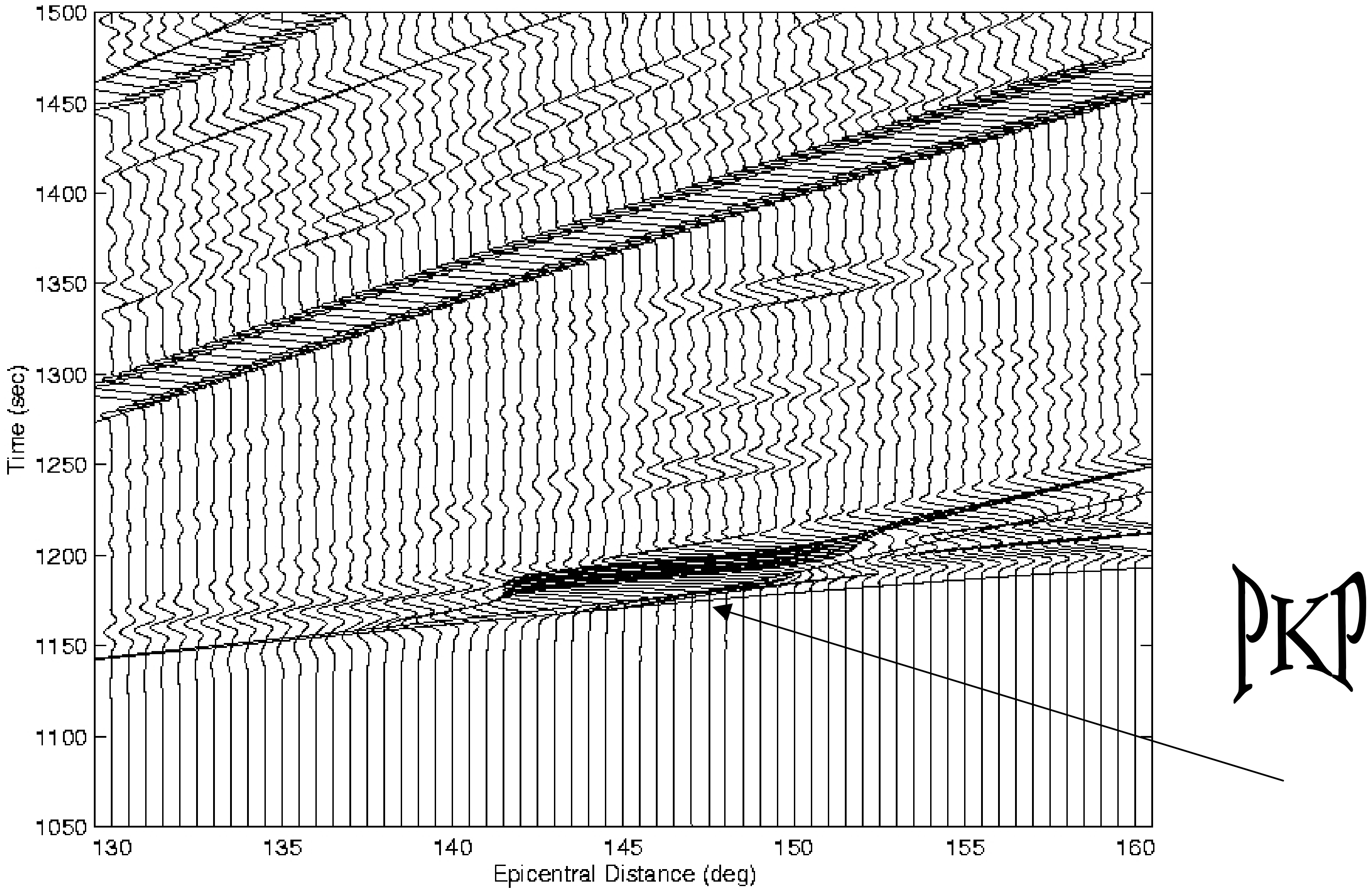
Acoustic waves in a sphere. Multidomain approach. PREM model.

Acoustic case - seismograms



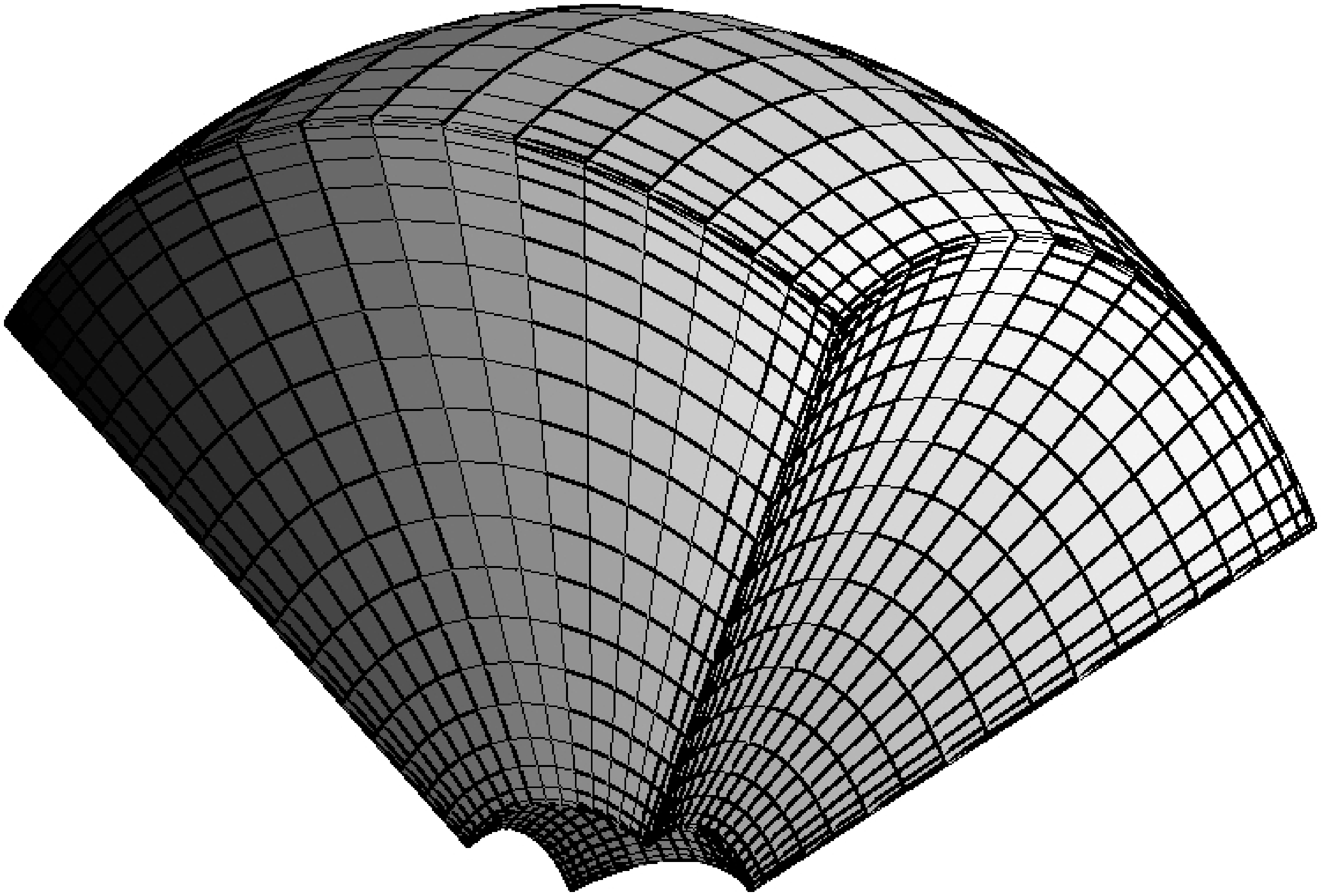
Seismograms for source at 200km depth. Dominant period 10s.

Acoustic case - seismograms



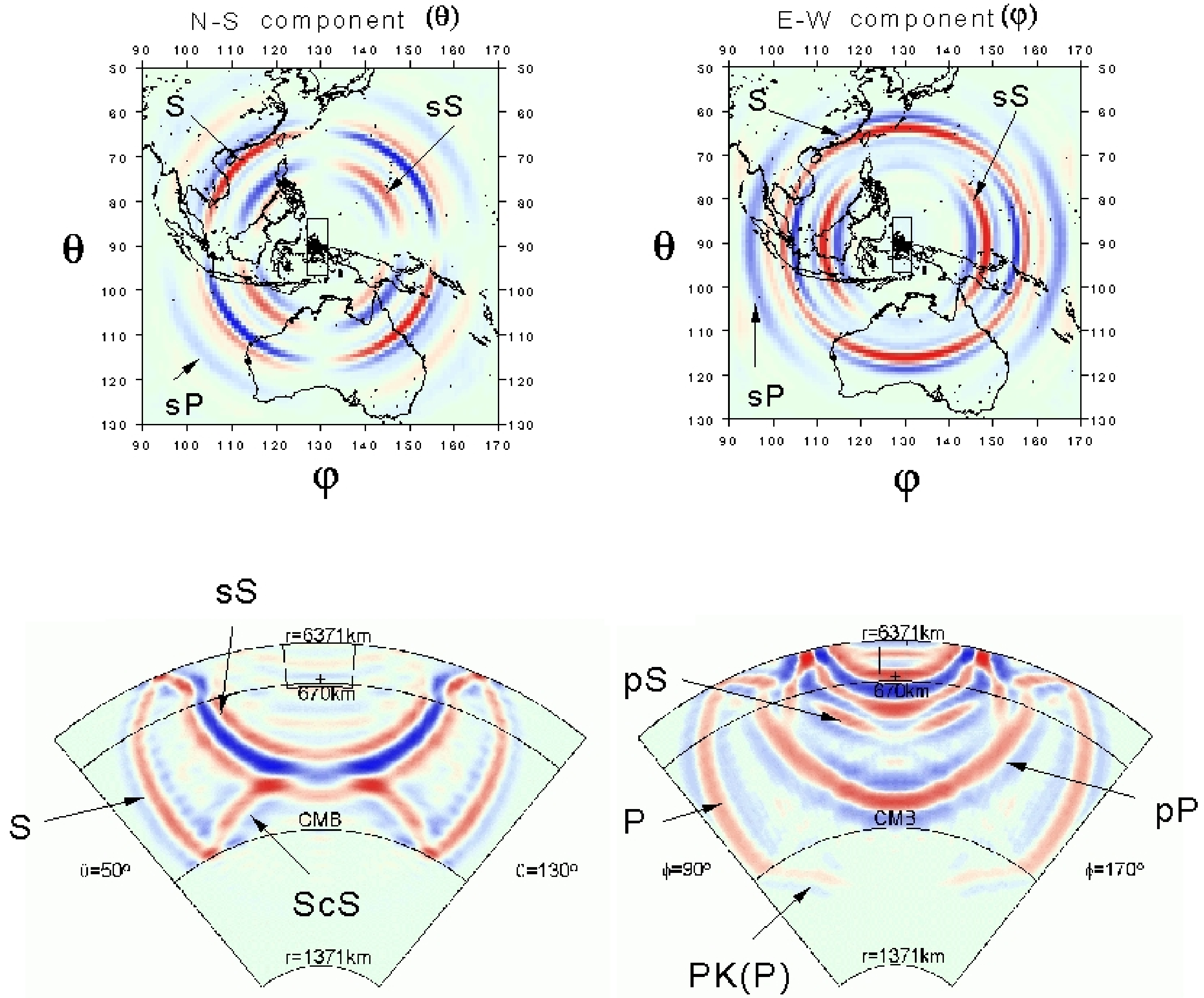
Seismograms for source at 200km depth. Dominant period 10s. PKP phase.

Spherical sections

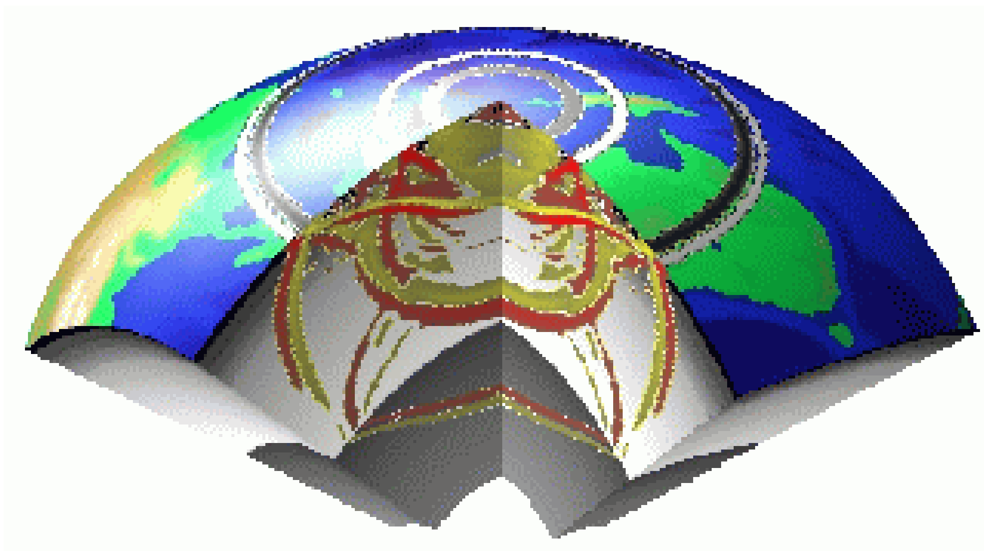


Chebyshev grid - spherical section

Spherical sections - snaps

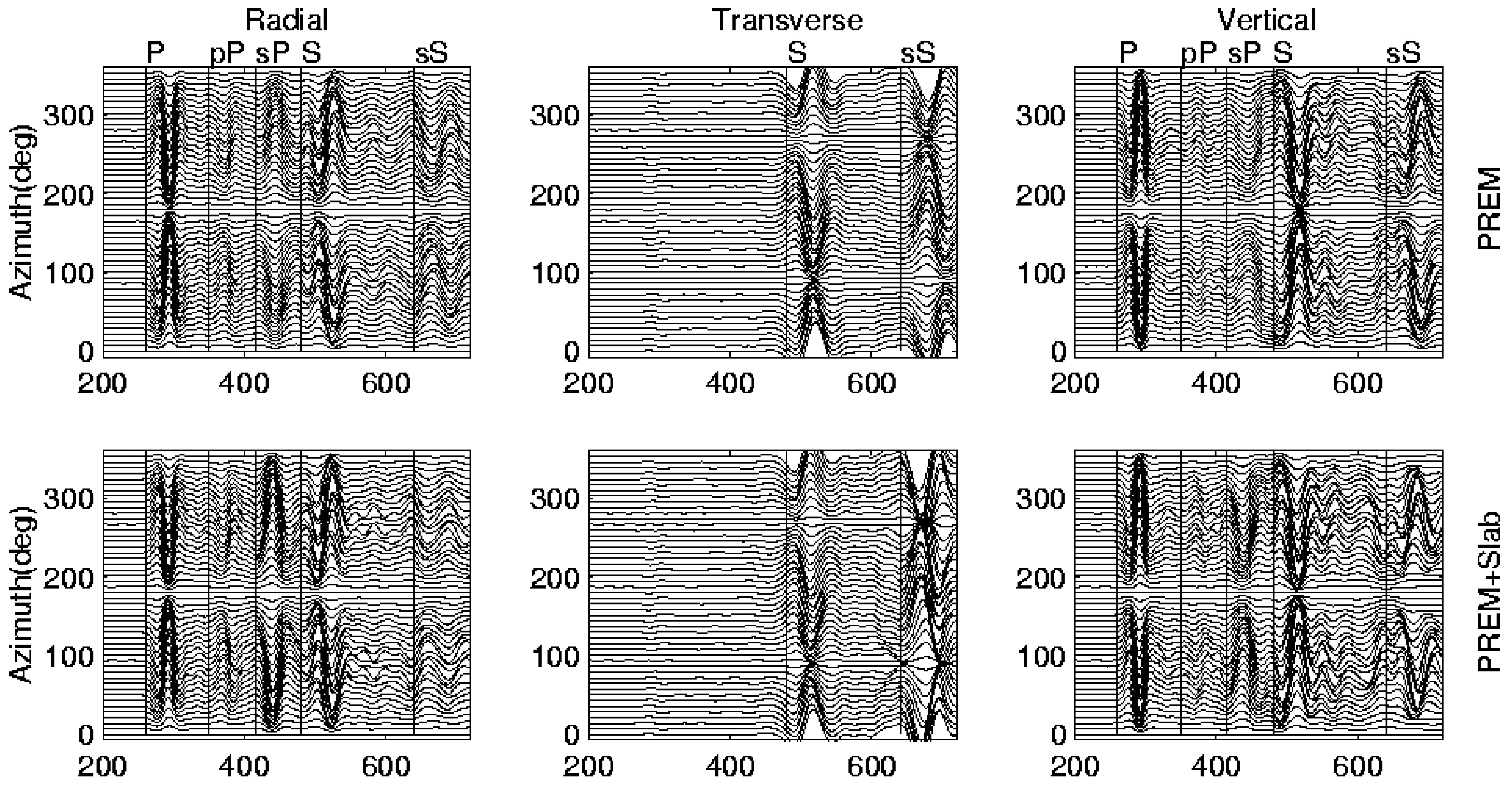


Spherical sections - snaps



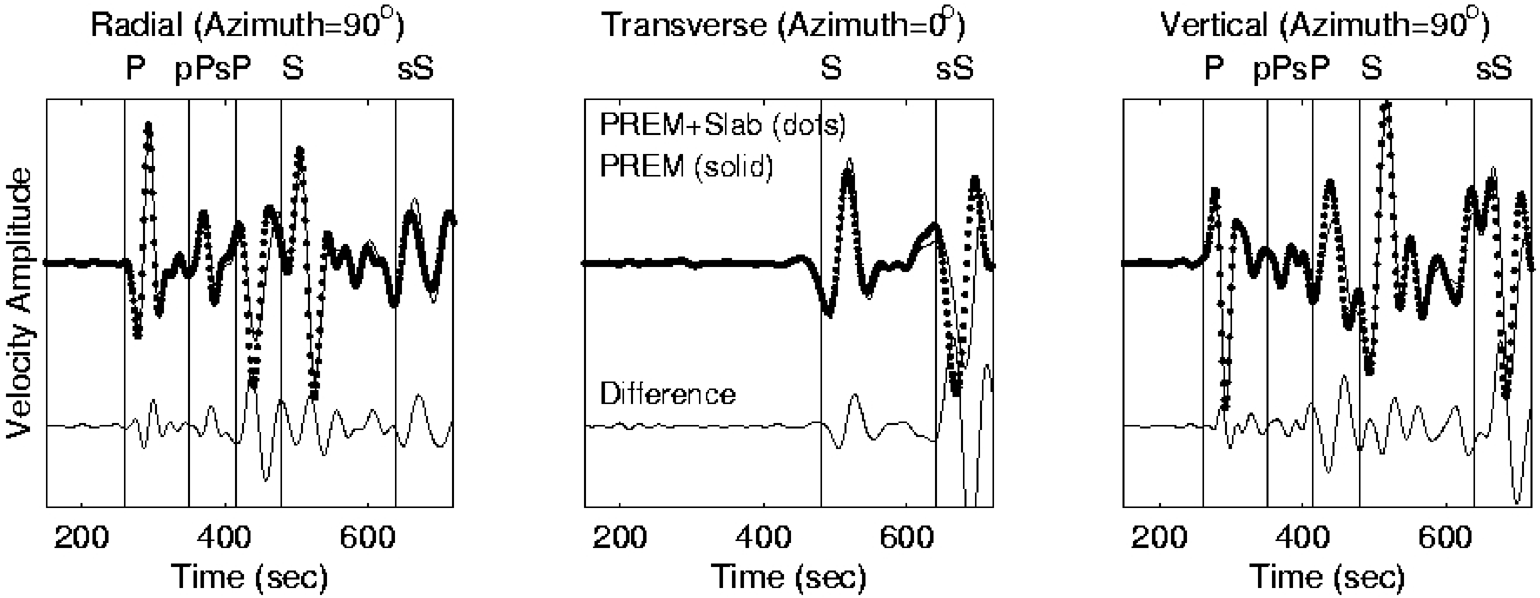
... seen this one before ?

Spherical sections - seismograms



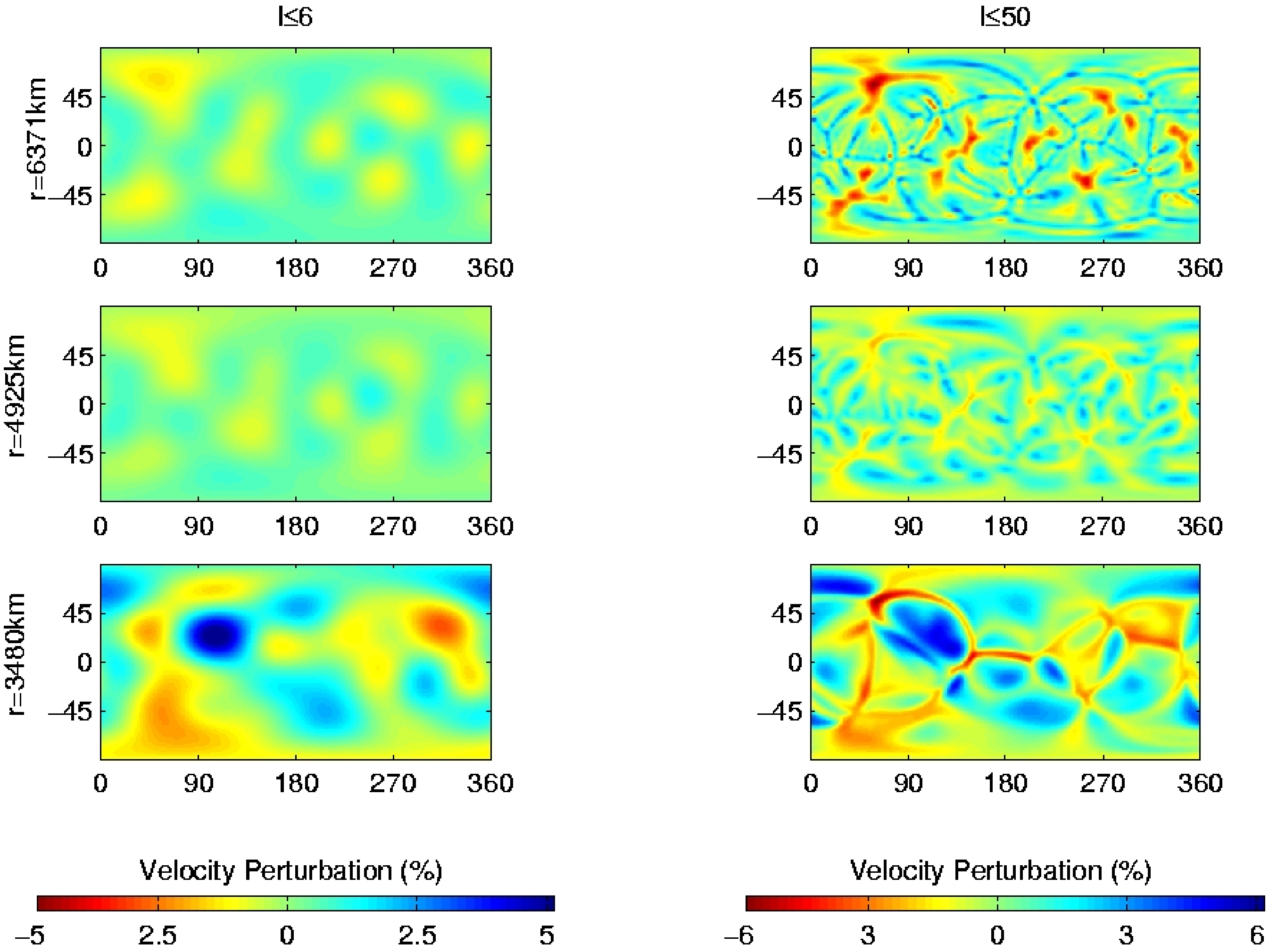
Azimuthal effects of a slab.

Spherical sections - seismograms



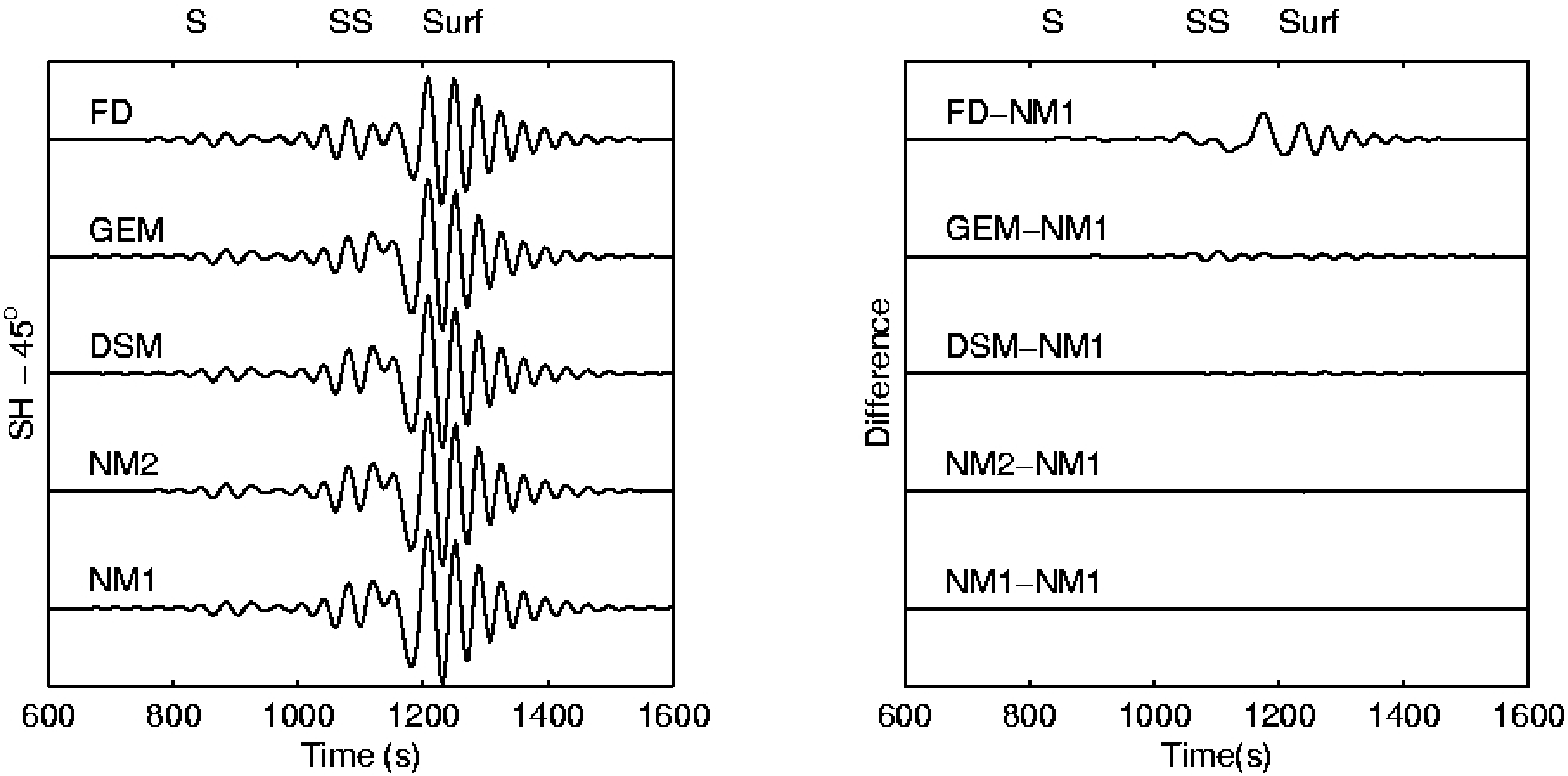
Azimuthal effects of a slab. Single seismogram.

The COSY Project



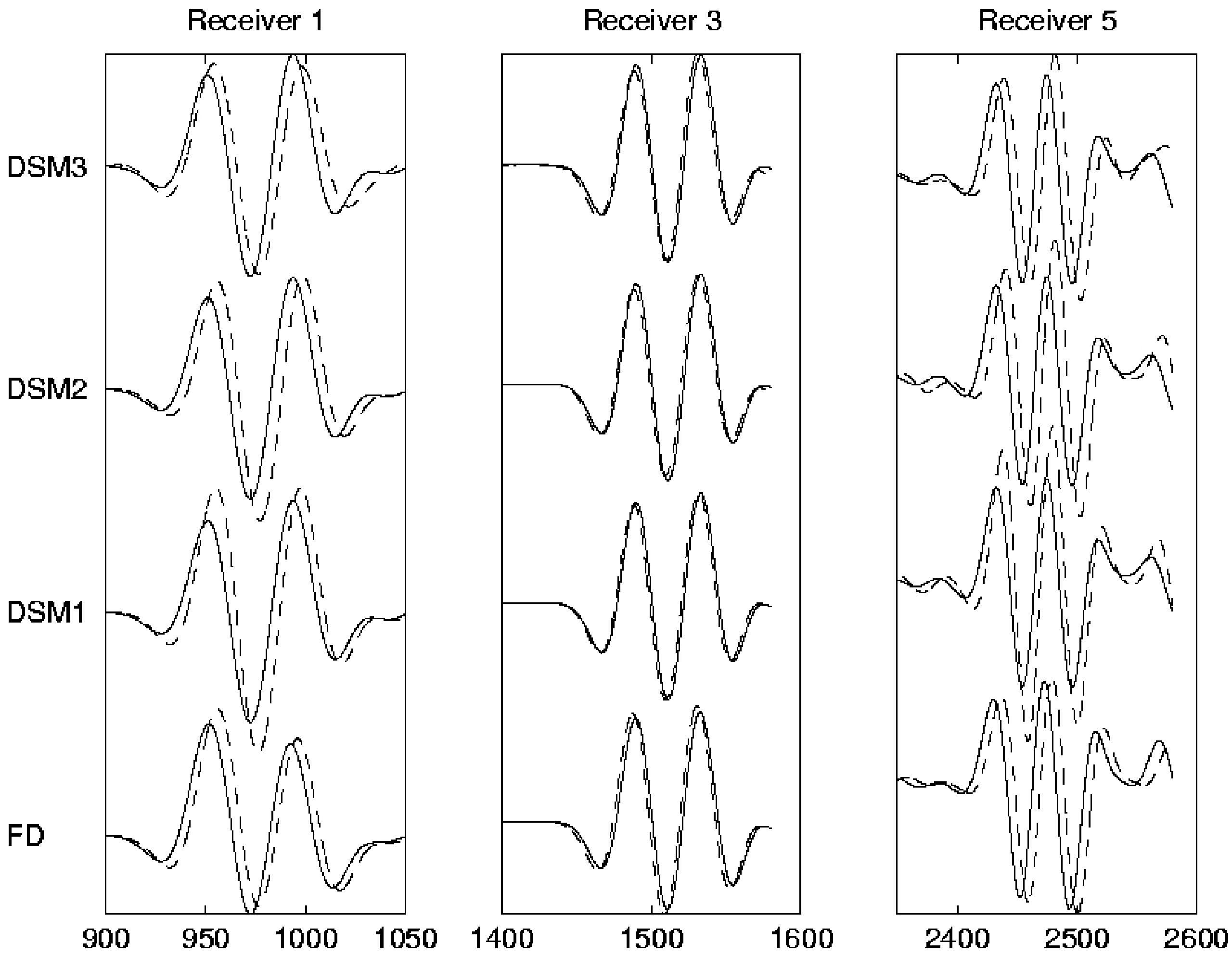
3D model pertubations, COSY Project

The COSY Project



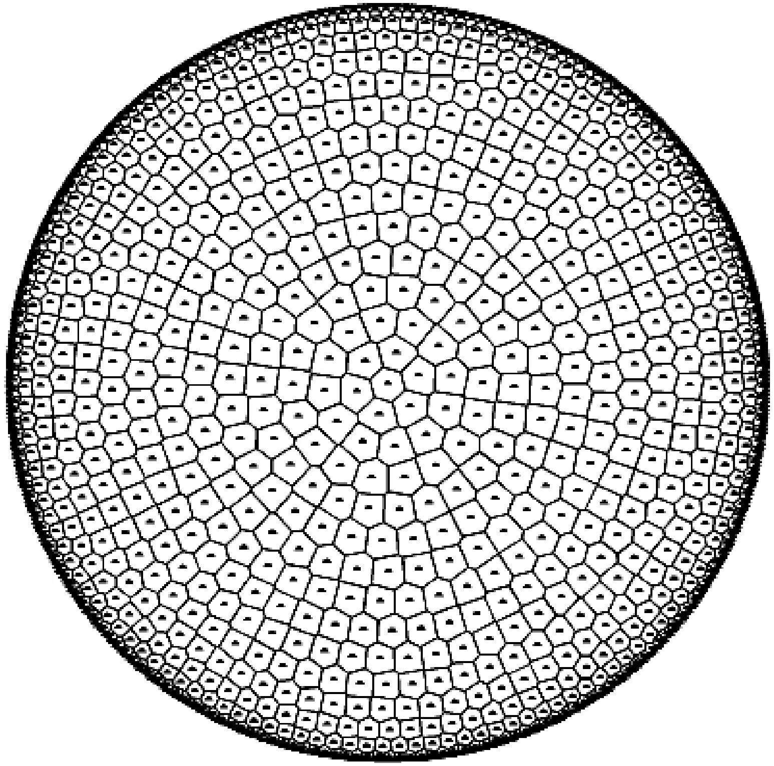
SH seismograms from different methods. PREM model.

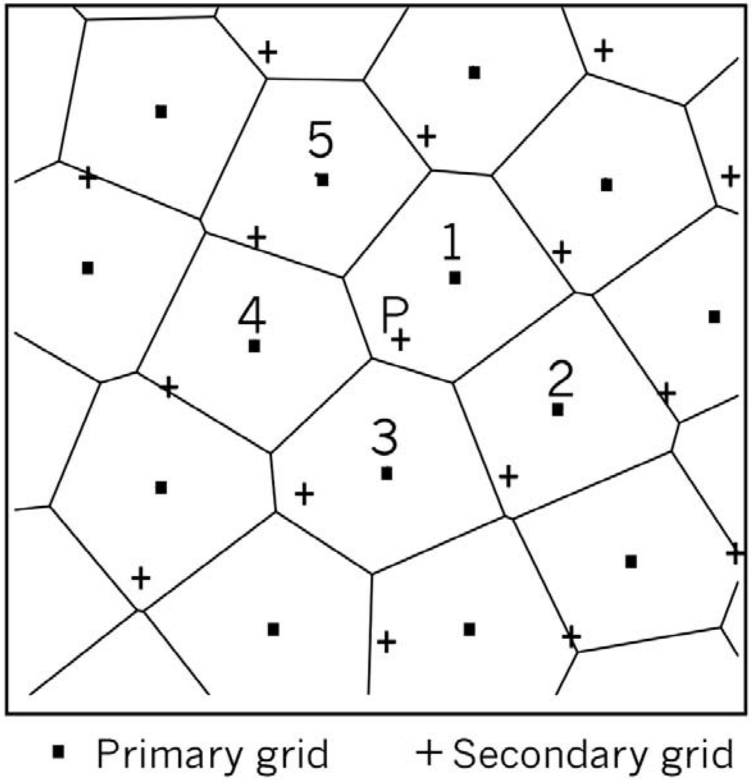
The COSY Project



3-D effects with different techniques.

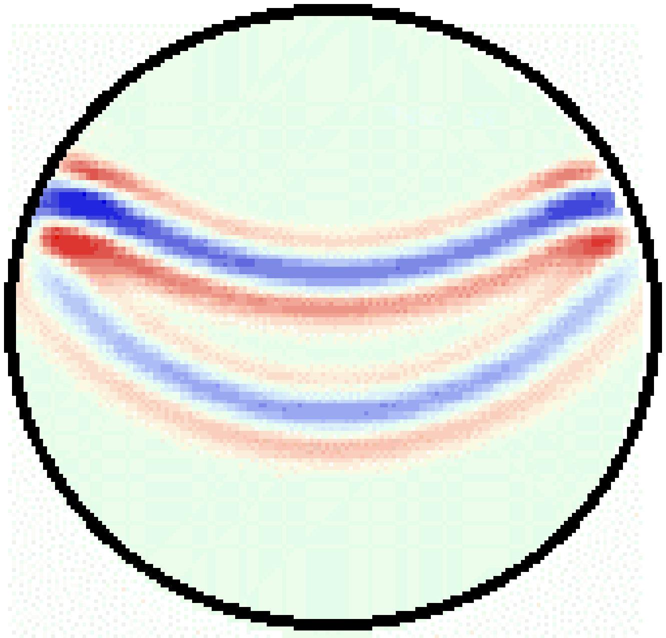
The whole Earth - Future

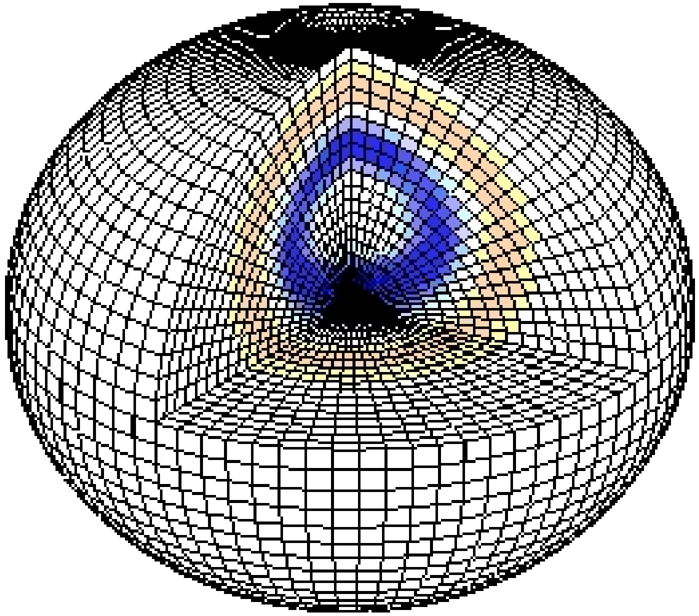




Voronoi cells - natural neighbours

The whole Earth - Future



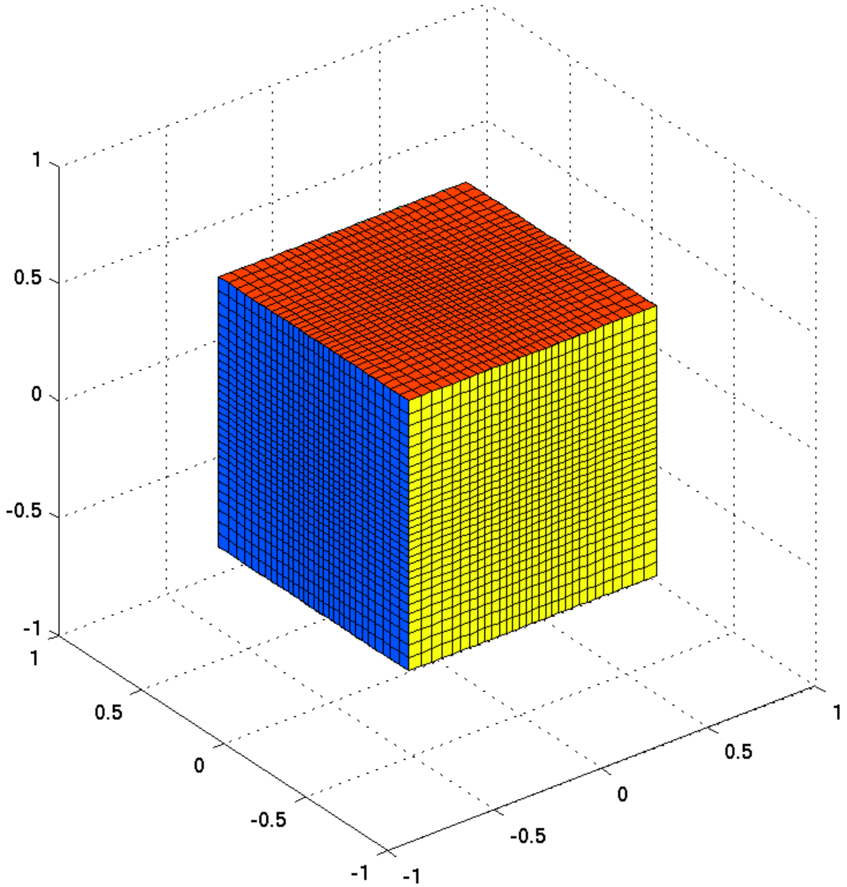


Waves in a

Cylinder

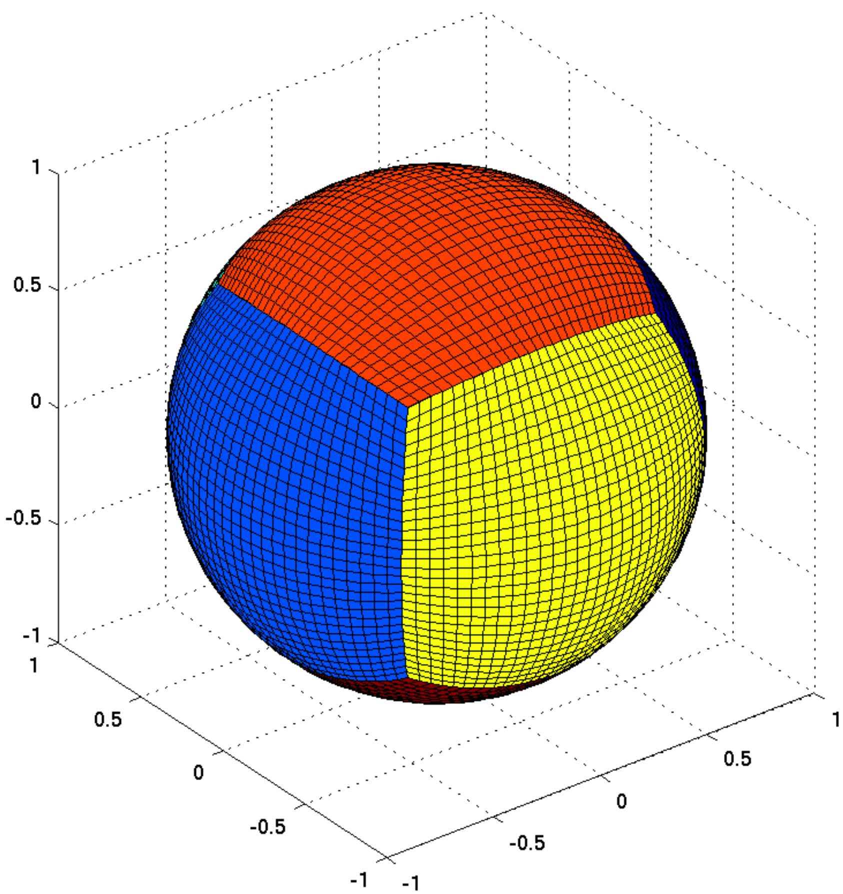
Sphere

Cubed Sphere



- gnomic projection of a cube to the surface of a sphere
- 6 equivalent chunks
- 6 different coordinate transformations
- 4 “coordinates”: chunk, ξ, η, r
- spherical shell mesh

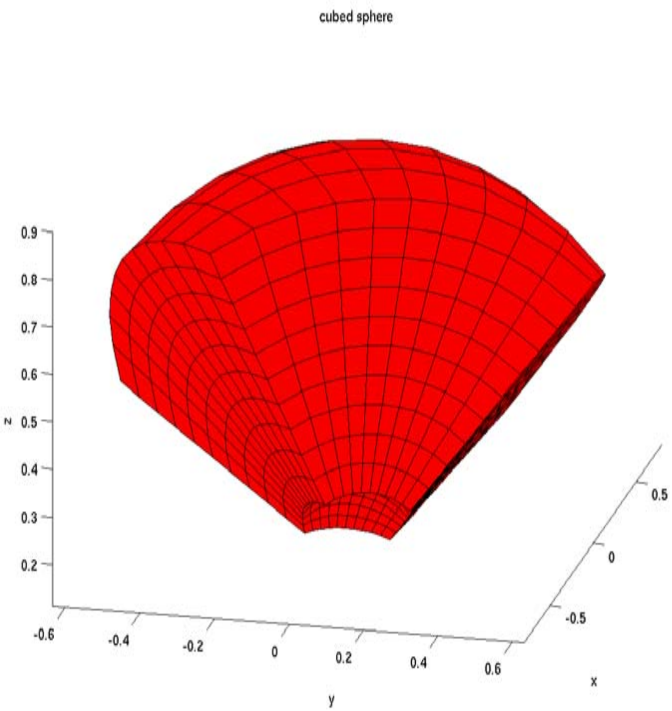
Cubed Sphere

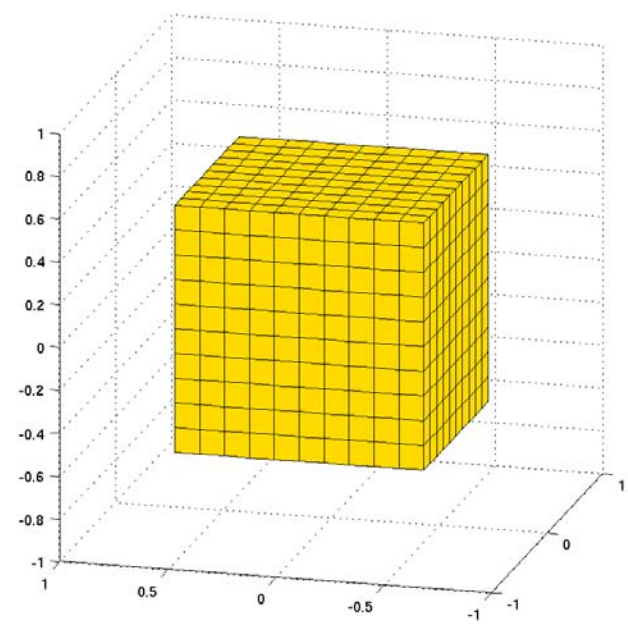


- gnomic projection of a cube to the surface of a sphere
- 6 equivalent chunks
- 6 different coordinate transformations
- 4 “coordinates”: chunk, ξ, η, r

coordinate transformation

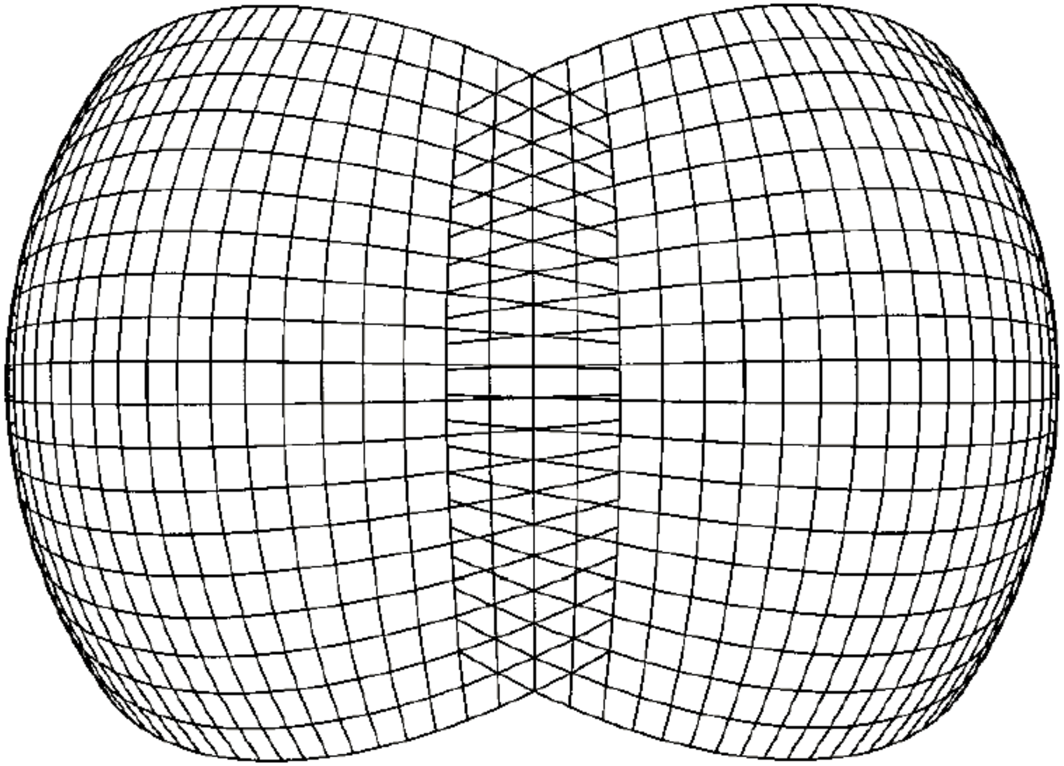
Cubed Sphere <=> Cartesian



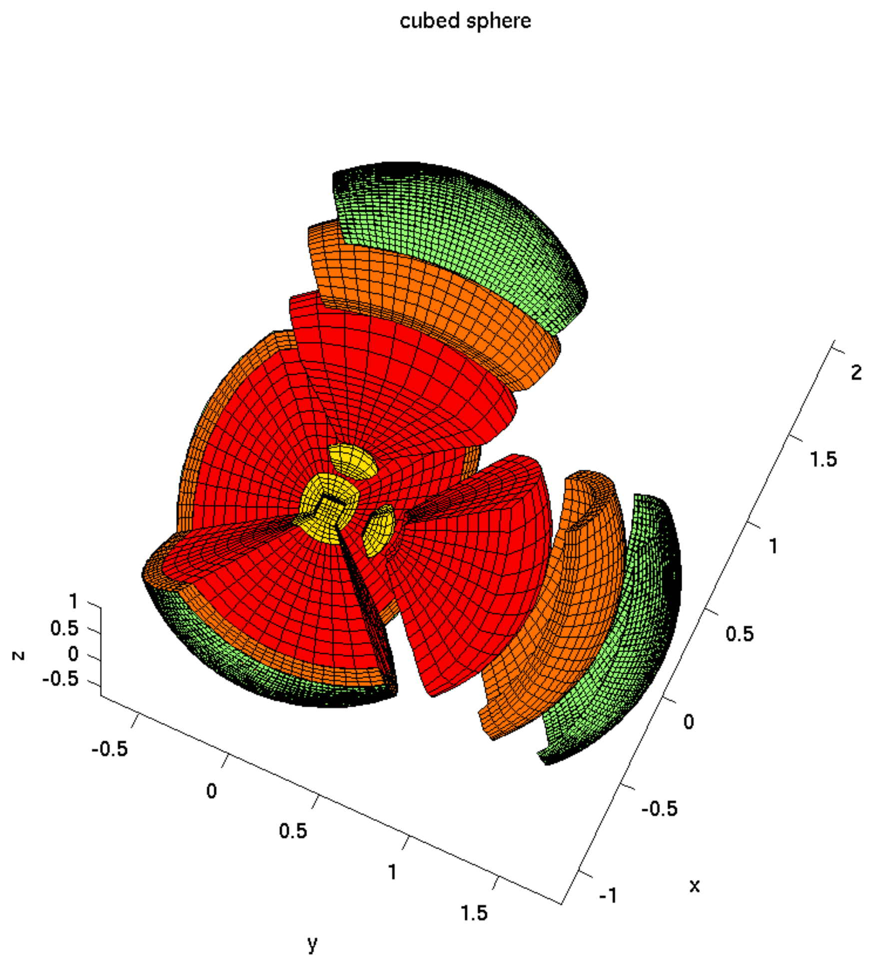


Chunk overlap

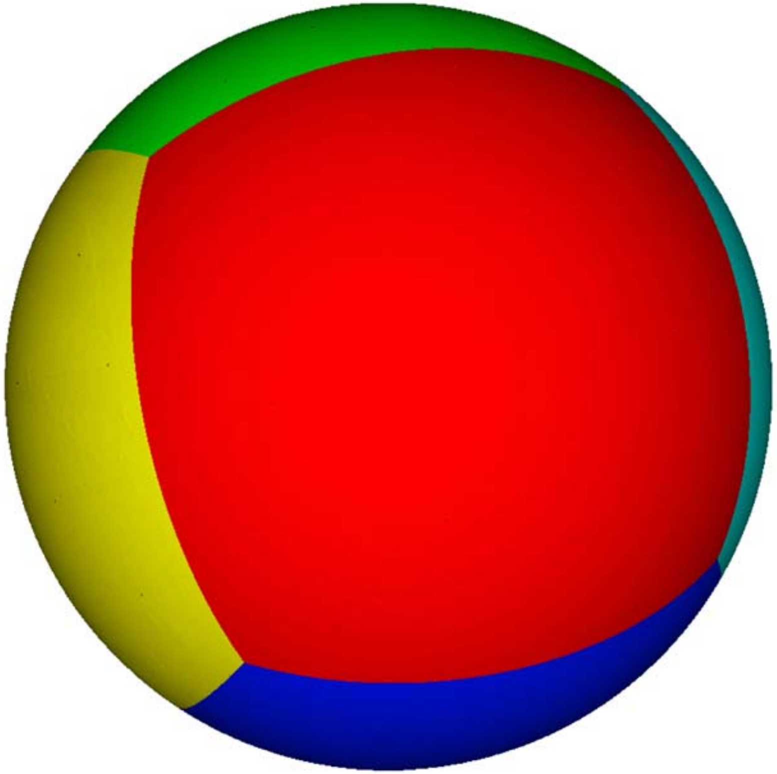
fitting chunks together



Solid Sphere extension

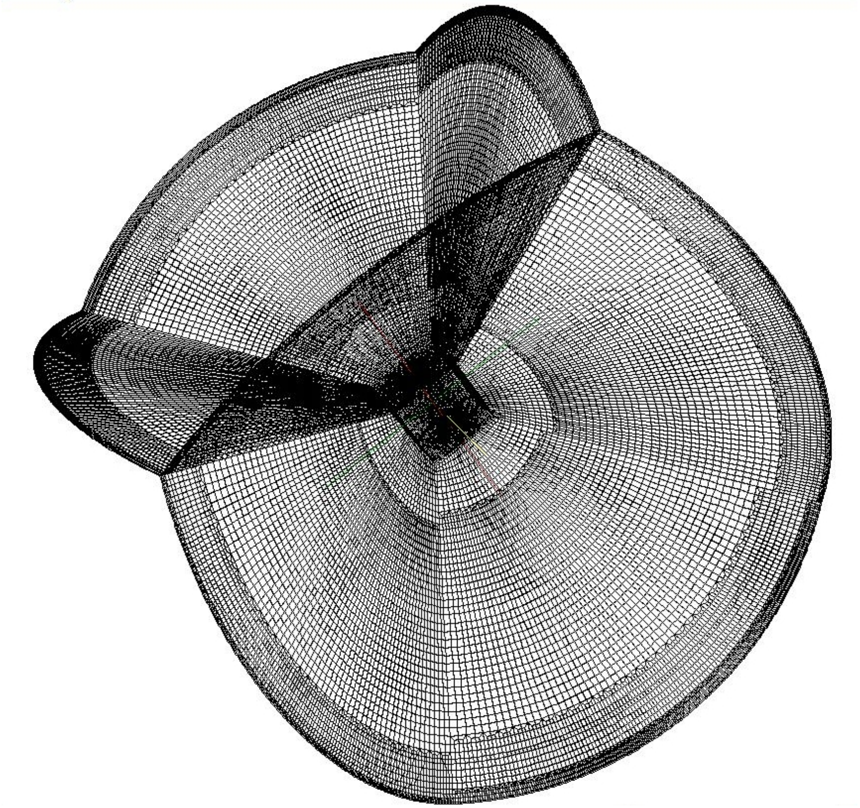


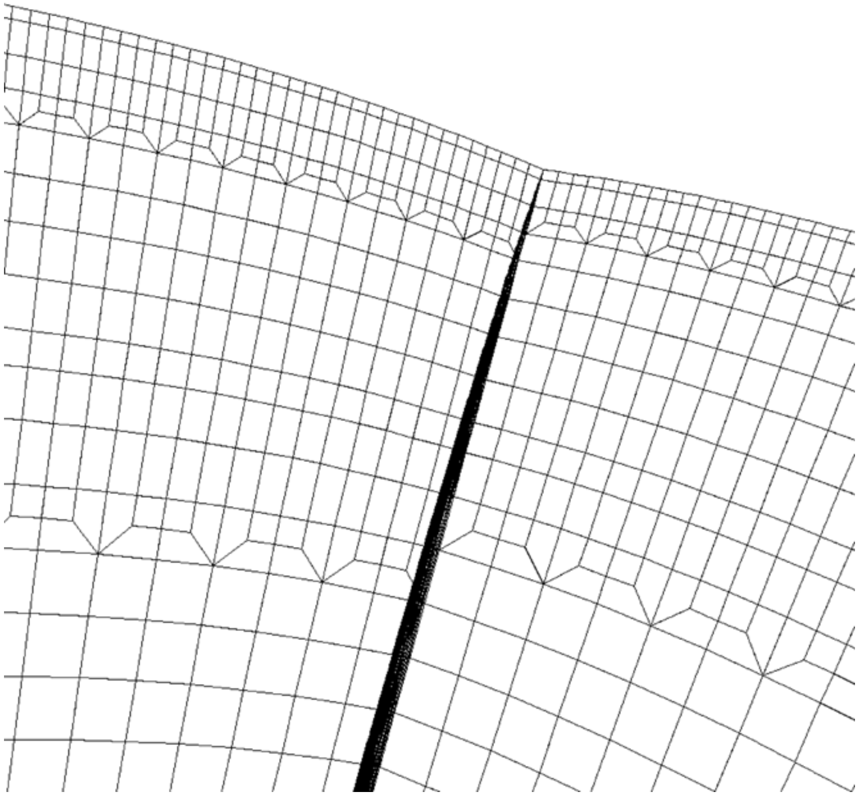
Cubed Sphere



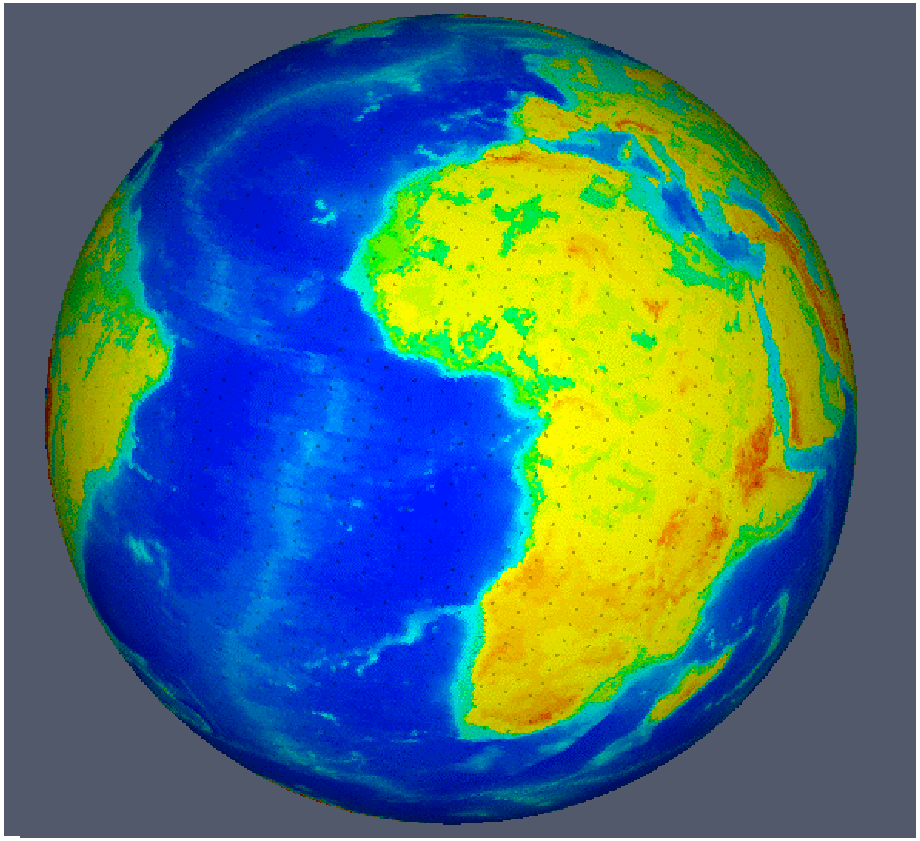
Chunk Partitioning

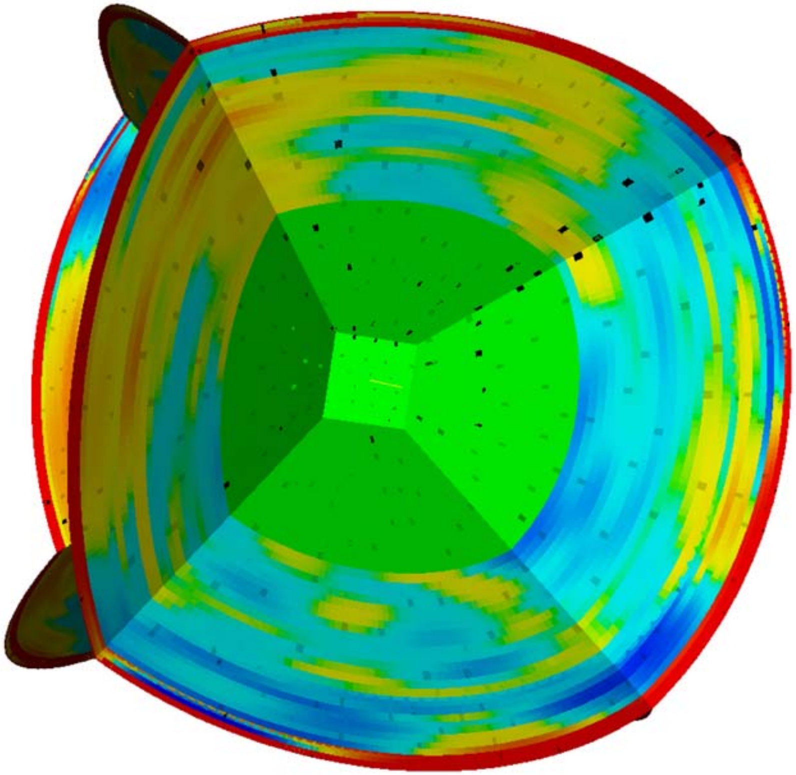
Special Issues of the Spherical Code



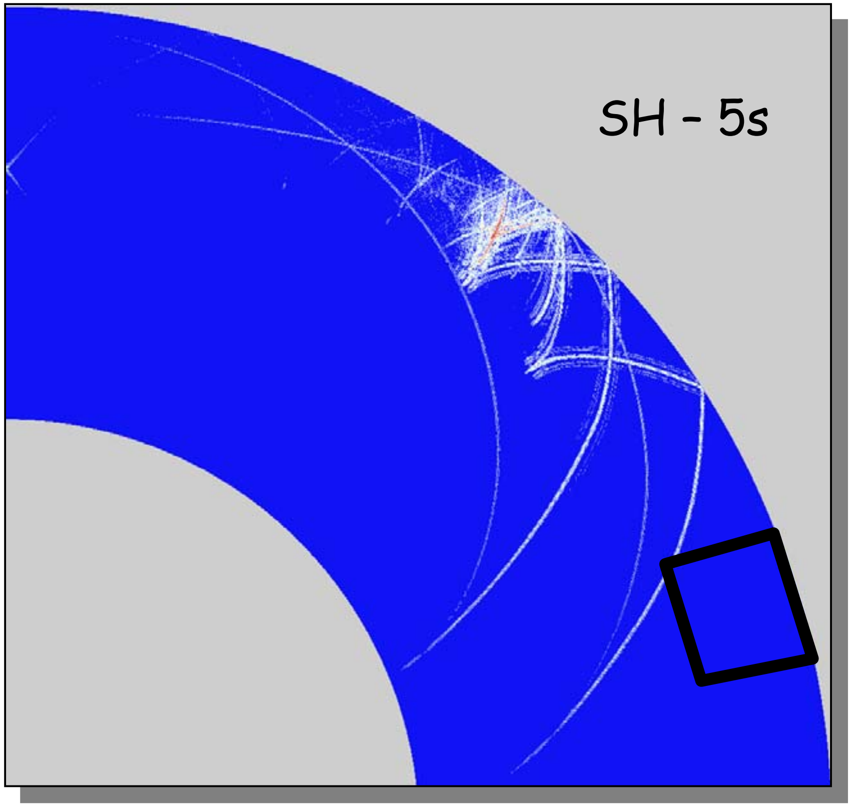


Status Quo





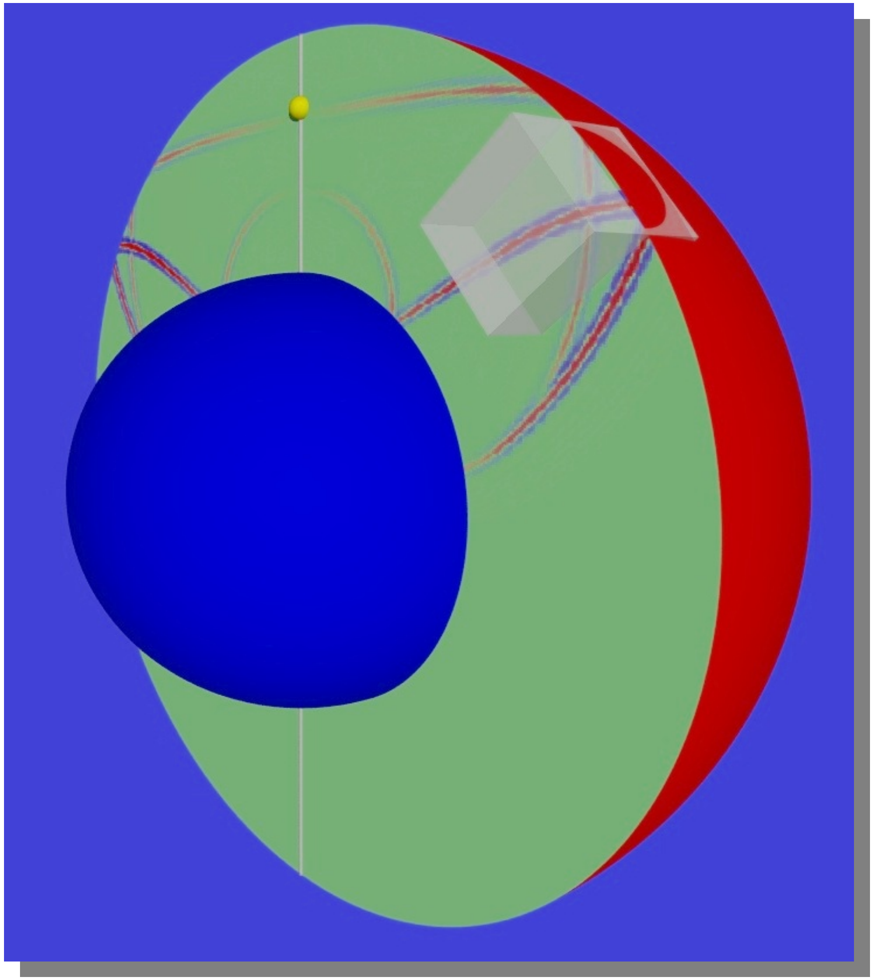
Combine axisymmetric approach with spherical section



Can we have a teleseismic wavefield flow into a local 3D box?

Global seismology

hybrid approach



- Combining axisymmetric approach with 3D spherical section
- Allows modelling higher frequencies
- Localized 3D structure (e.g. plumes, subduction zones)
- Phenomenological studies

Global seismology

scattering from a plume