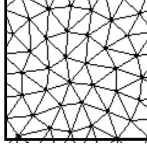
Numerical Methods in Geophysics:
Introduction



Why numerical methods?

simple geometries – analytical solutions
complex geometries – numerical solutions

Applications in geophysics

seismology
geodynamics
electromagnetism
... in all domains

History of computers

serial computers
vectorization
parallel computers
memory requirements

Numerical Methods in Geophysics: Introduction

Macroscopic and microscopic description

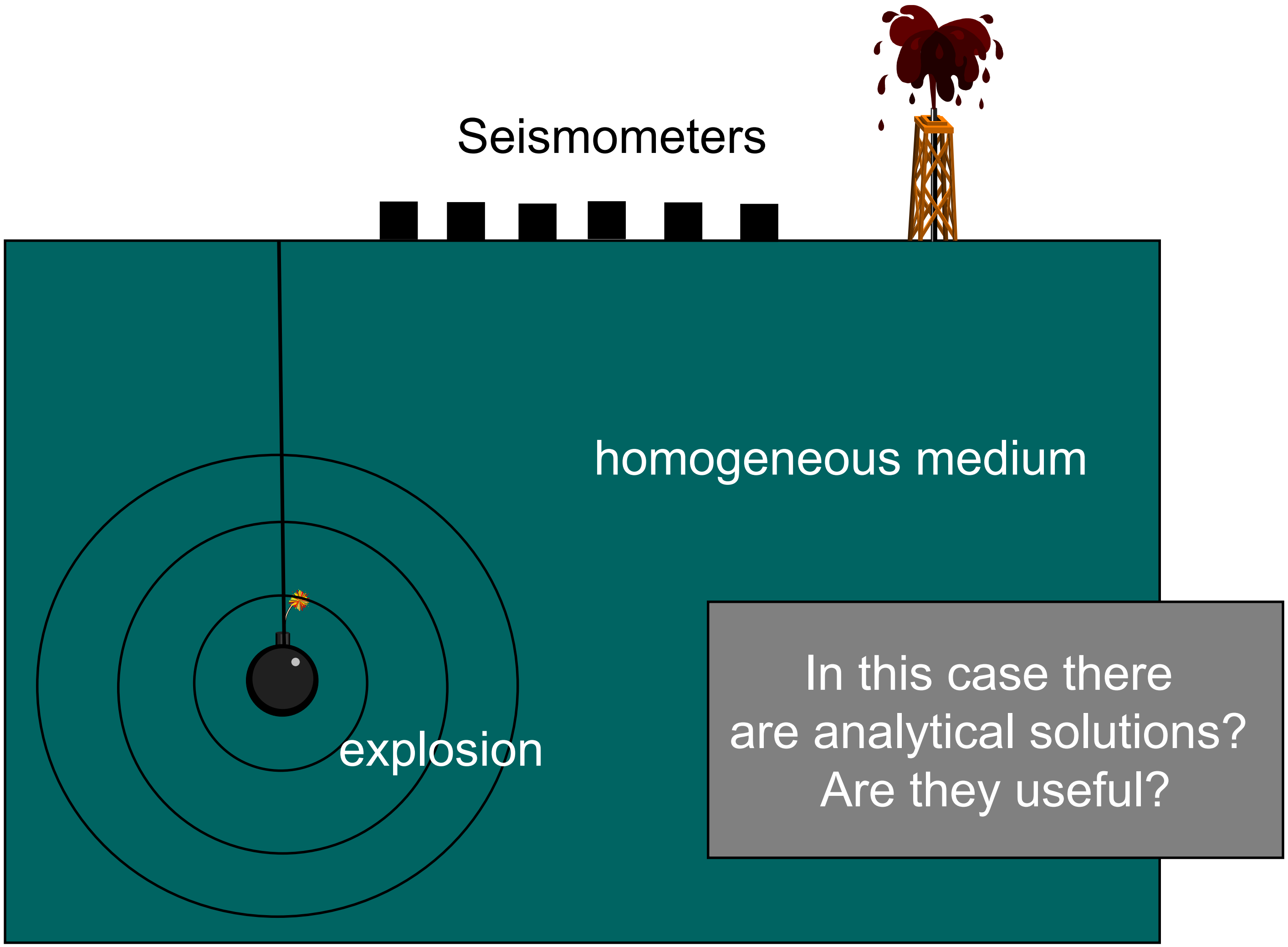
continuum mechanics
lattice gases
fluid mechanics
nonlinear processes

Partial differential equations in geophysics

conservation equations
constitutive laws
wave equation
diffusion equation
Navier-Stokes equation

Why numerical methods?

Example: seismic wave propagation



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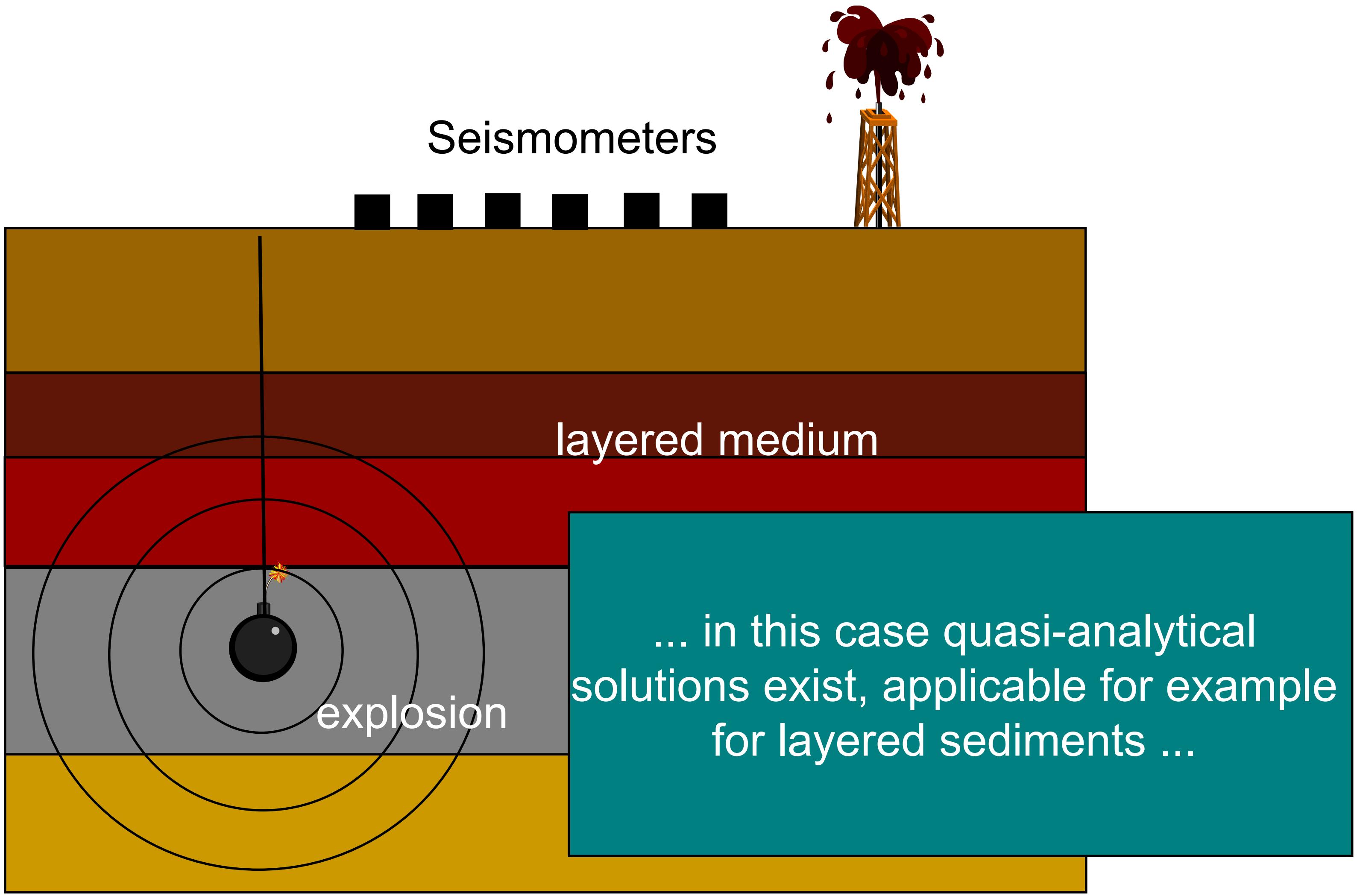
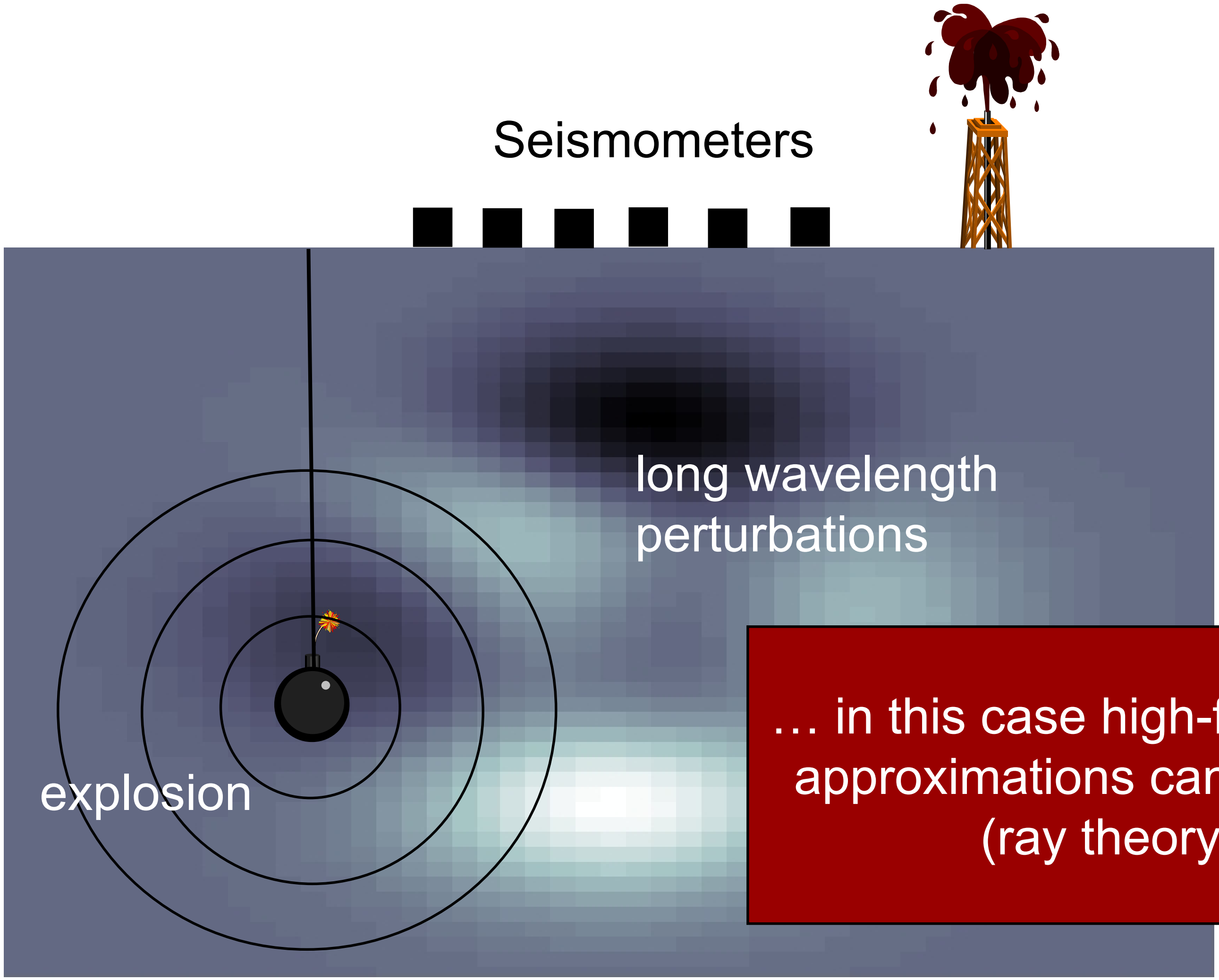
Example: seismic wave propagation



Why numerical methods?

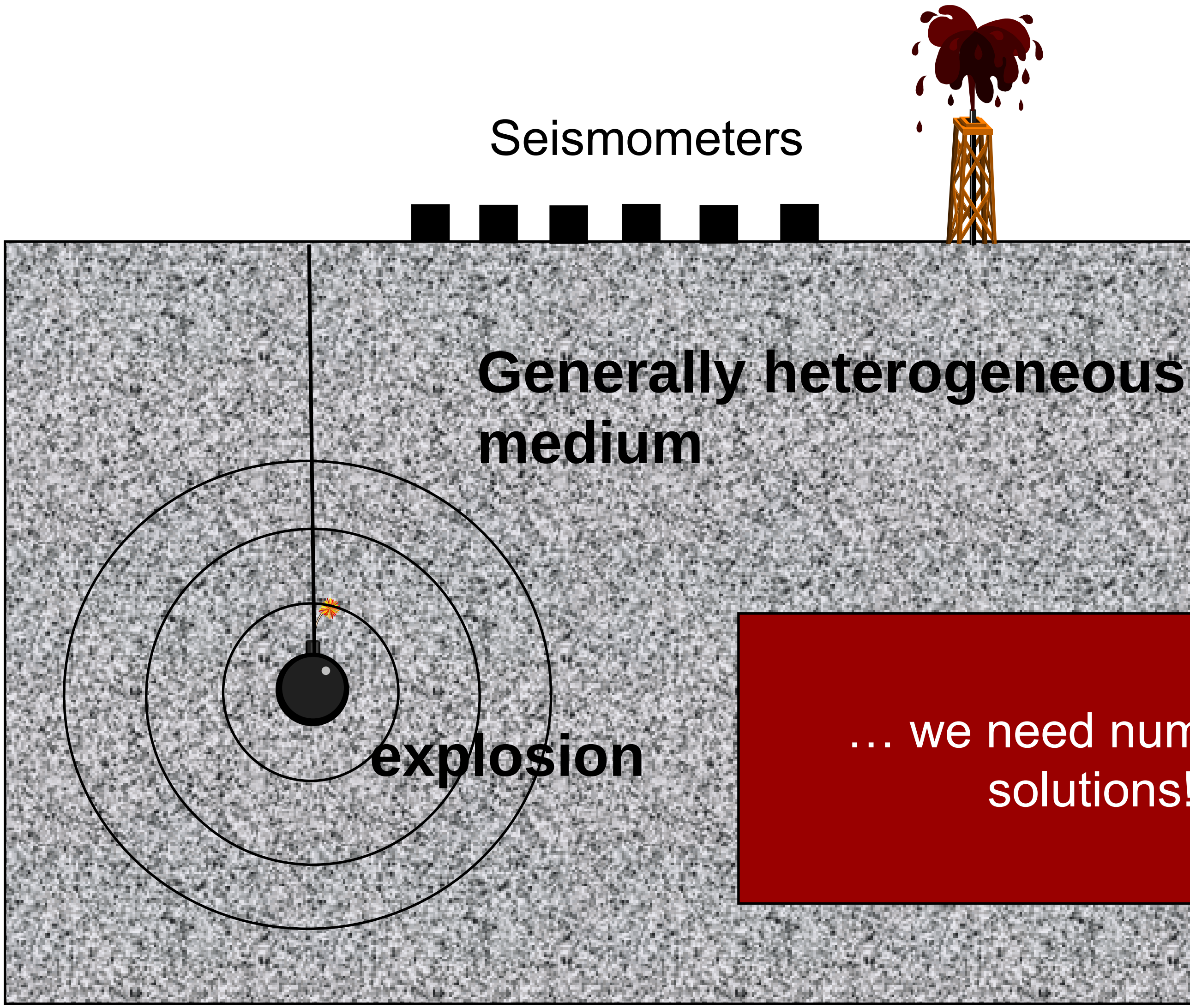
Example: seismic wave propagation



… in this case high-frequency approximations can be used (ray theory)

Why numerical methods

Example: seismic wave propagation



… we need numerical solutions!

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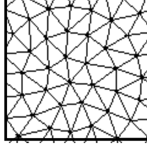

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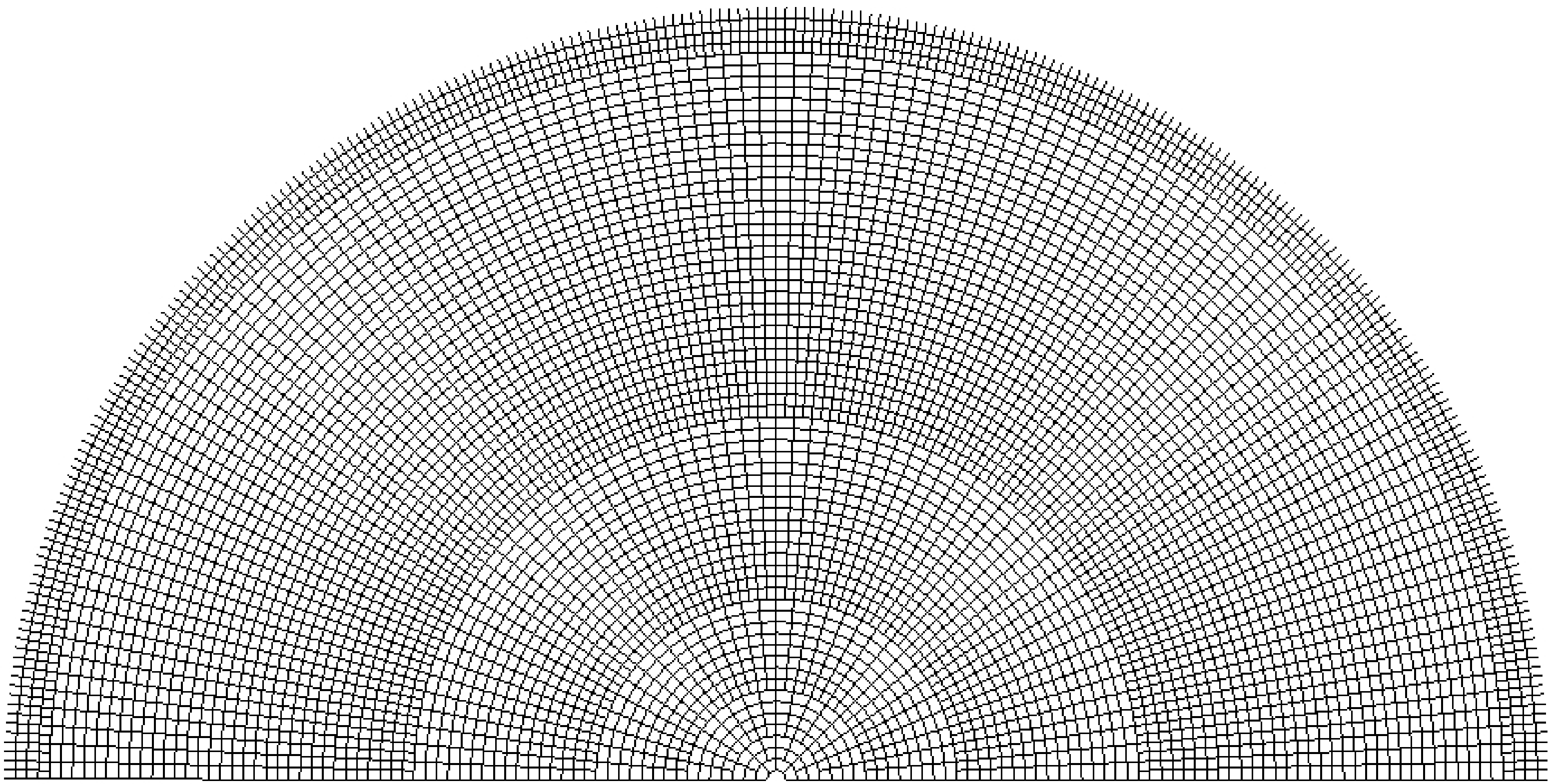
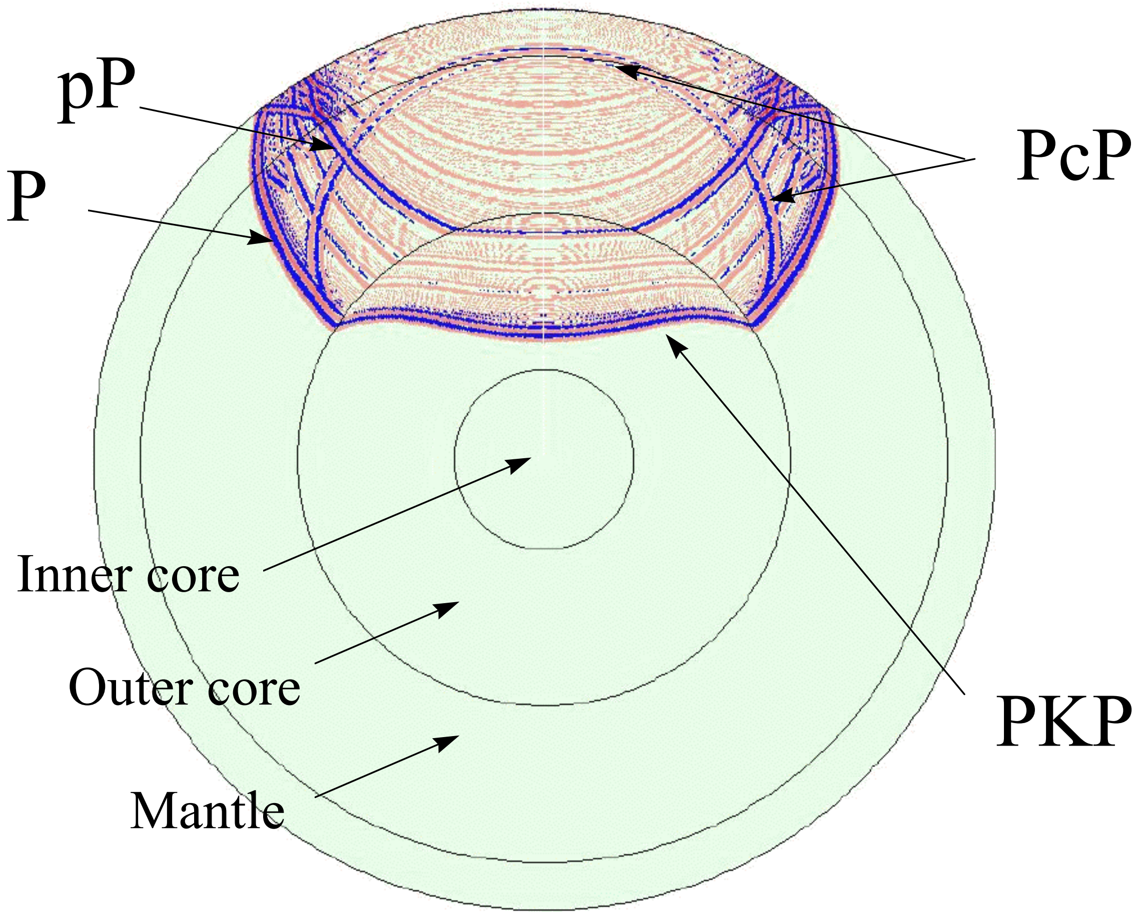
global seismology – spherical coordinates - axisymmetry



finite differences – multidomain method

global seismology – spherical coordinates - axisymmetry



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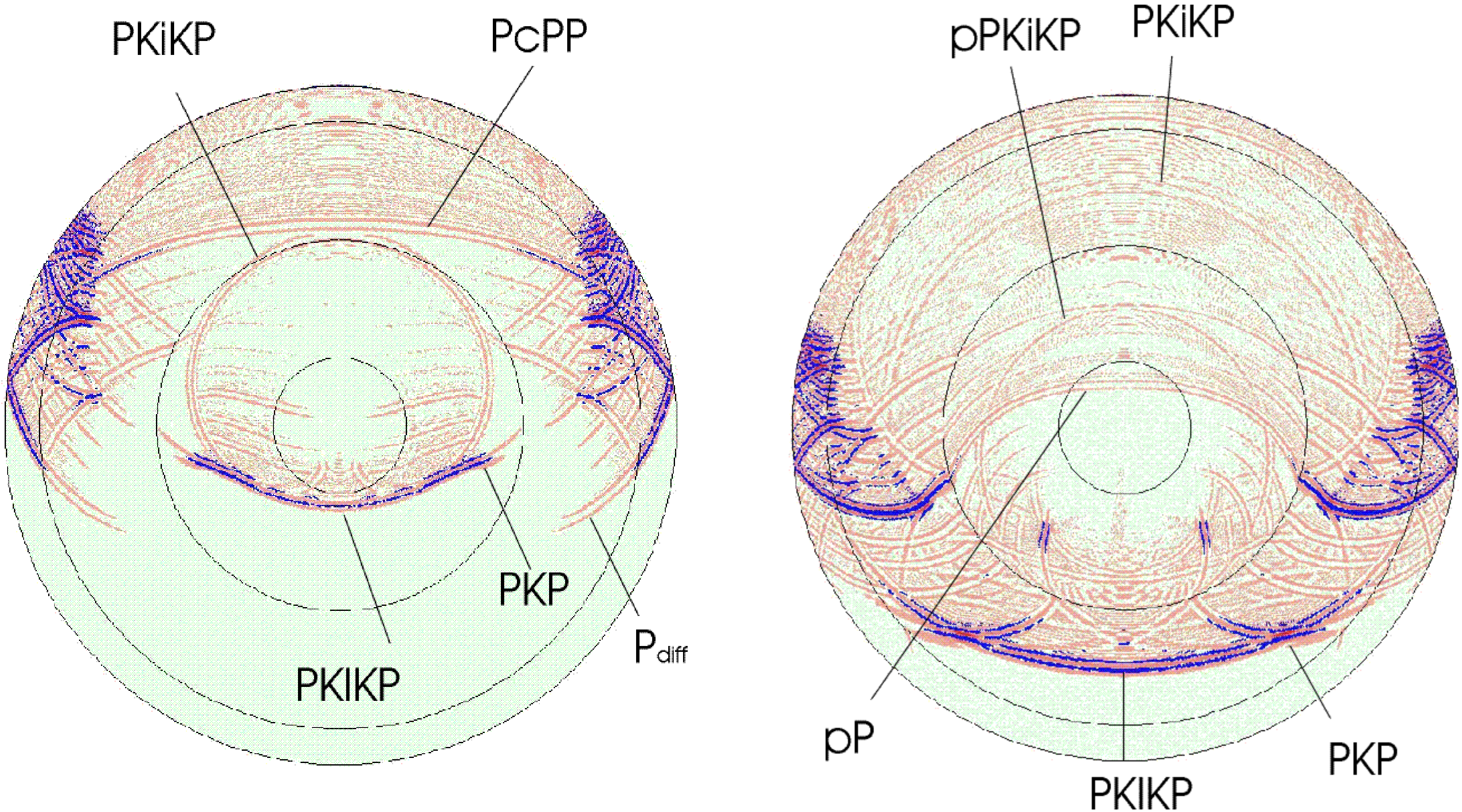
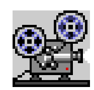


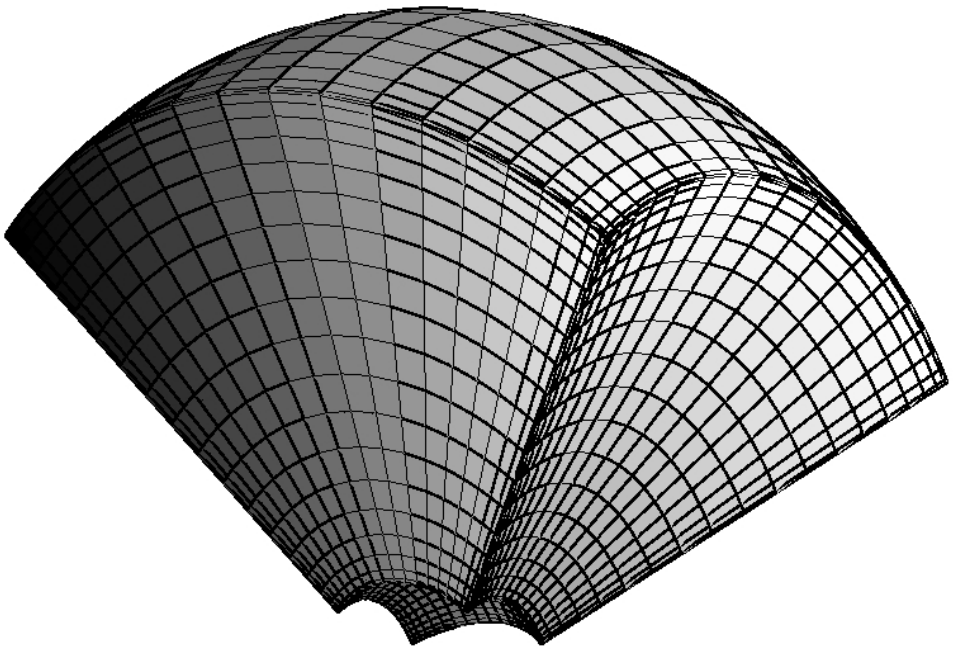
finite differences – multidomain method

P-waves in the Earth: the movie



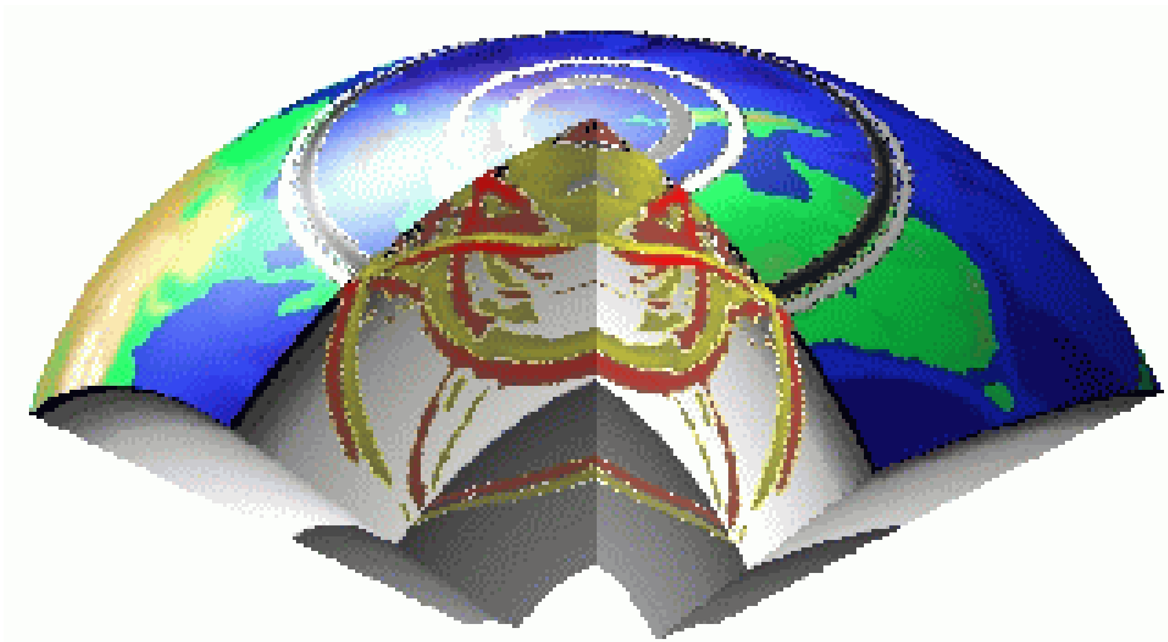
finite differences – multidomain method

global seismology



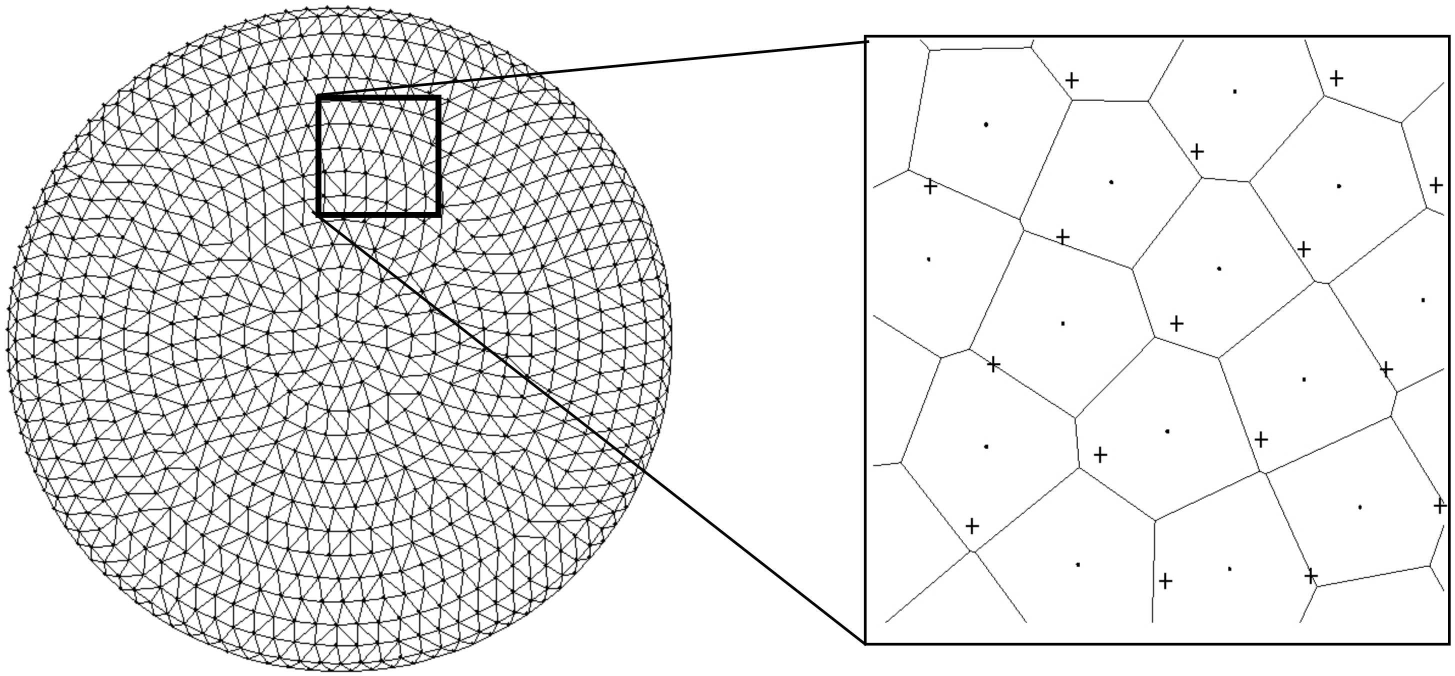
3-D grid for space-dependent parameters
Chebyshev collocation points denser near the boundaries
Better implementation of boundary conditions.

global seismology



Wave propagation in spherical sections.
Investigation of 3-D effects of subduction zones.
Grid size 200x200x200.
Grid distance 50km
Finite differences and pseudospectral methods

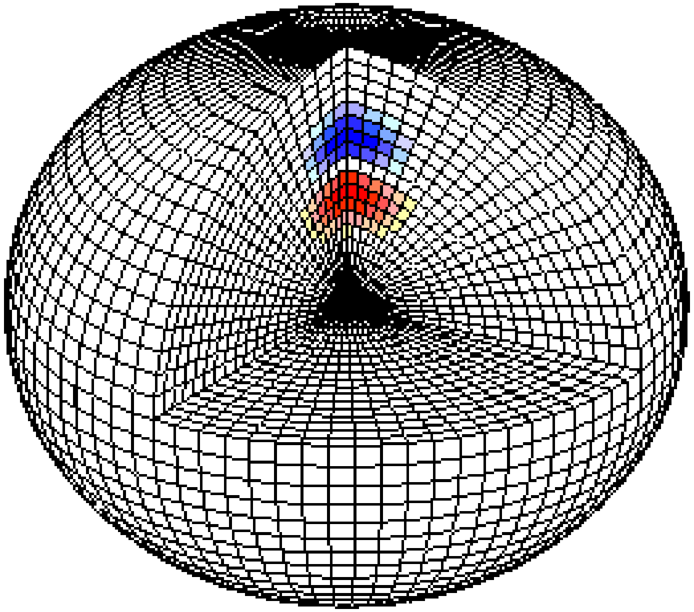
global seismology – spherical geometry – cartesian equations



Delauney triangulation – Voronoi cells
3-D – irregular grids – natural differences

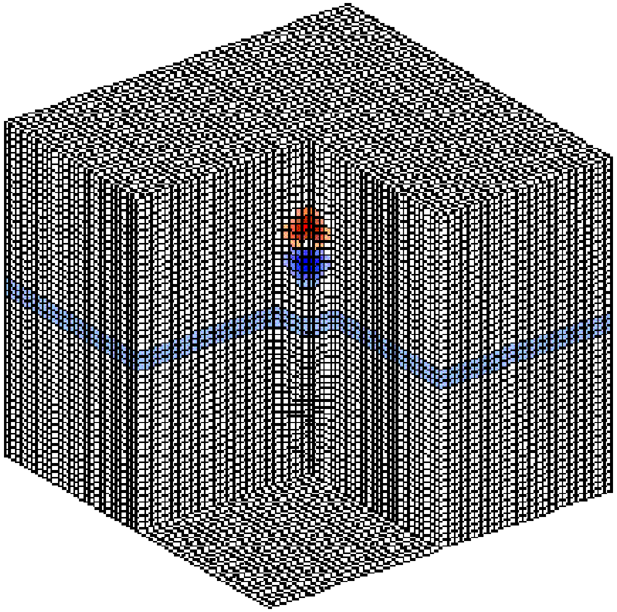
global seismology – spherical geometry – cartesian equations

3-D – irregular grid
natural differences
20000 grid points
test case

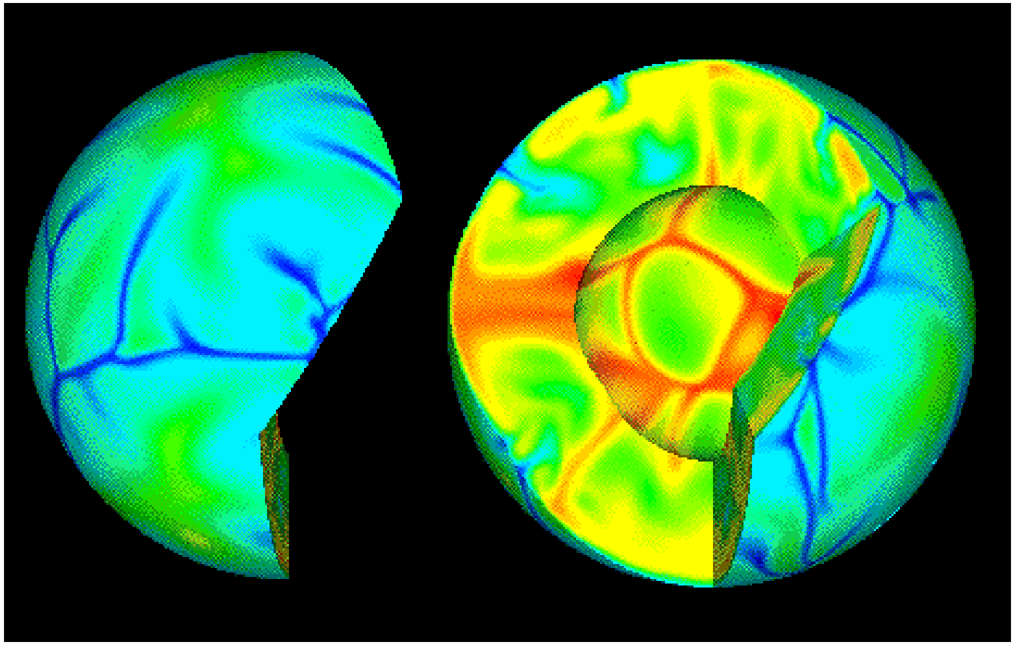


Exploration seismology

- wave propagation in media with thin curved layers
- finite difference method
- grid size 200x200x200
- orthogonal grid is stretched by analytical functions

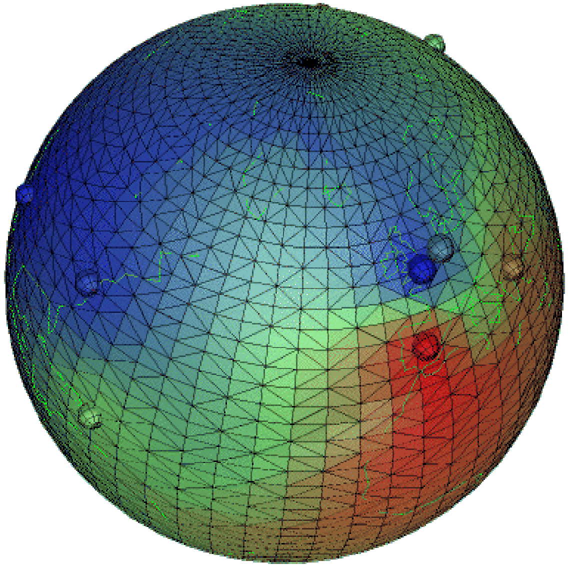


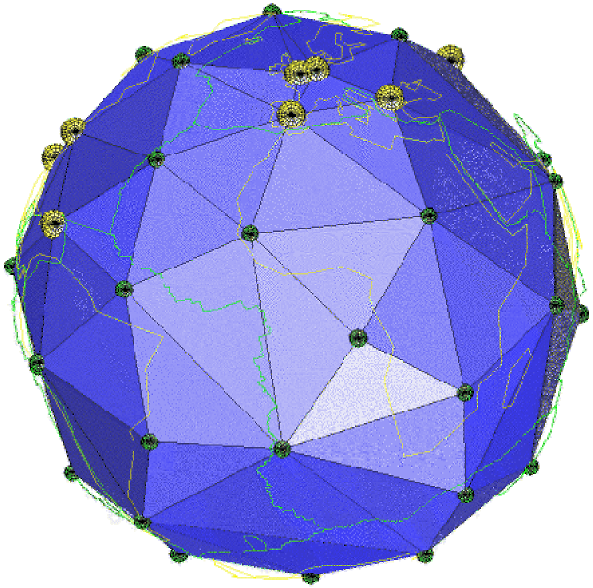
geodynamics – mantle convection



3-D finite-element modelling of mantle convection
in spherical geometry. 10 Million grid points.
Implementation on parallel hardware (P. Bunge, Munich).

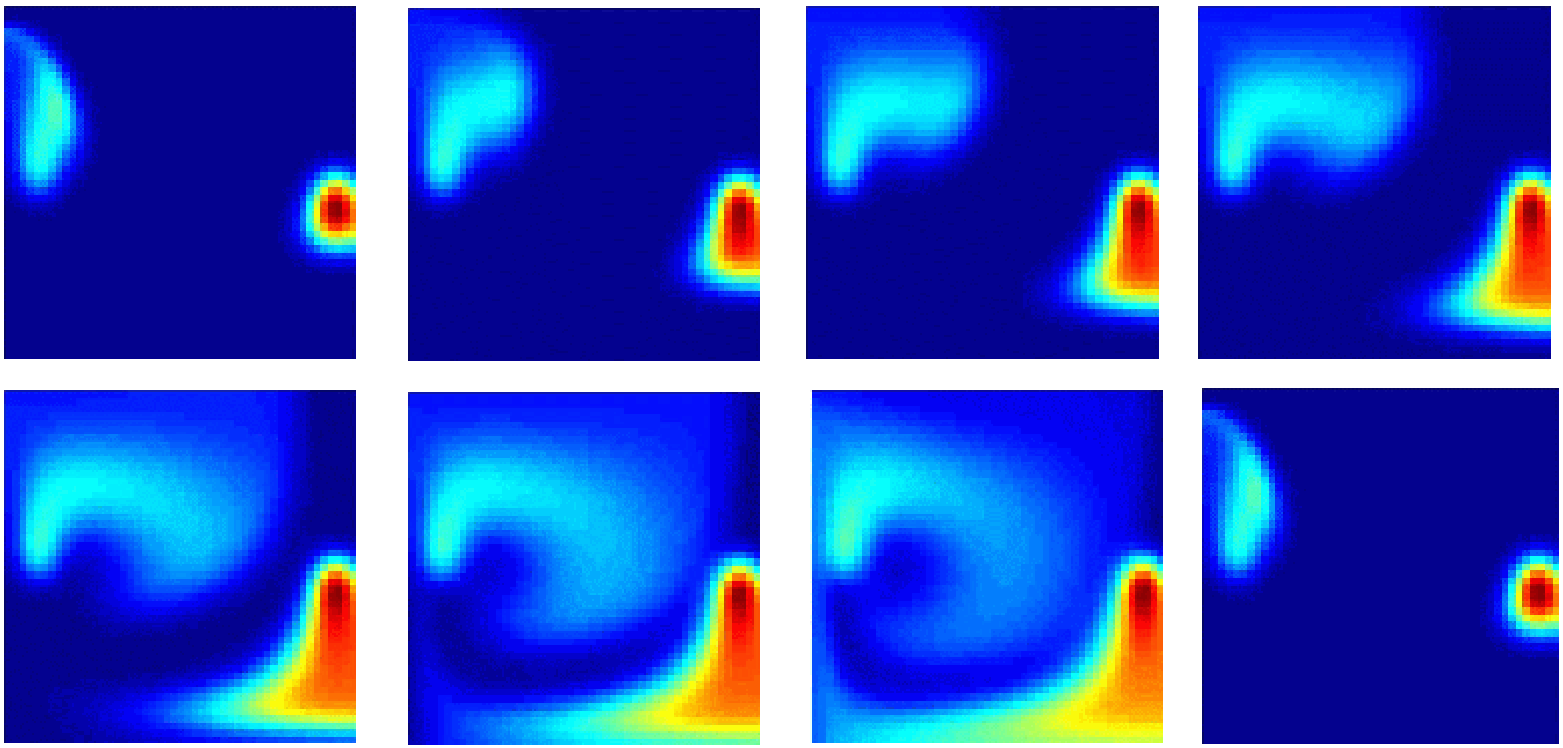
global electromagnetism – conductivity of the Earth's mantle





3-D finite-element modelling
(Schultz, Cambridge)

isotope mixing in the oceans
Stommel-gyre
input of isotopes near the boundaries (e.g. rivers)



diffusion – reaction – advection equation

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→ History of computers

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memory requirements

Example: seismic wave propagation, 2-D case

grid size: 1000x1000

number of grid points: 10^6

parameters/grid point: elastic parameters (3), displacement (2), stress (3) at 2 different times -> 16

Bytes/number: 8

required memory: $16 \times 8 \times 10^6 \times 1.3 \times 10^8$

You can do this on a standard PC!

130 Mbyte memory (RAM)

Memory Requirements

Example: seismic wave propagation, 3-D case

grid size:	1000x1000x1000
number of grid points:	10^9
parameters/grid point:	elastic parameters (3), displacement (3), stress (6) at 2 different times -> 24
Bytes/number:	8

This is a
GRAND CHALLENGE PROJECT
for supercomputers

required memory: $24 \times 8 \times 10^9 \times 1.9 \times 10^{11}$

190 Gbyte memory (RAM)

Memory requirements

... this would mean

...we could discretize our planet with volumes of the size

$$4/3\ \pi(6371\text{km})^3 / 10^9 \times 1000\text{km}^3$$

with an representative cube side length of 10km. Assuming that we can sample a wave with 20 points per wavelength we could achieve a dominant period T of

$$T = \lambda / c = 20\text{s}$$

for global wave propagation!

History

1960: 1 MFlops
1970: 10MFlops
1980: 100MFlops
1990: 1 GFlops
1998: 1 TFlops
2010: ?

What are parallel computations

Example: Hooke's Law
stress-strain relation

$$\sigma_{xx} = \lambda\,(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu\varepsilon_{xx}$$

$$\sigma_{ij}, \varepsilon_{ij} \Rightarrow f(x, y, z, t)$$

$$\lambda, \mu \Rightarrow f(x, y, z)$$

These equations hold at each point in time at all points in space

-> Parallelism

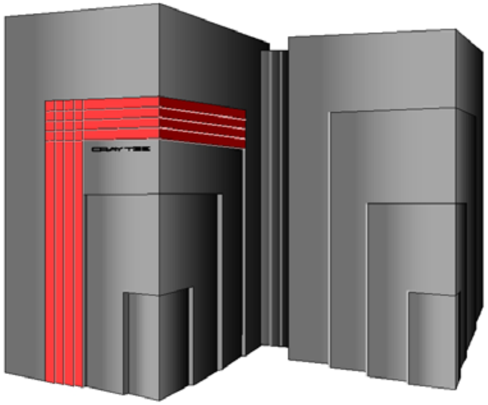
... in serial Fortran (F77) ...

at some time *t*

```
for i=1,nx
  for j=1,nz
  sxx(i,j)=lam(i,j)*(exx(i,j)+eyy(i,j)+ezz(i,j))+2*mu(i,j)*exx(i,j)
  enddo
enddo
```

add-multiplies are carried out sequentially

Parallelism



... in parallel Fortran (F90/95) ...

array syntax

```
sxx = lam*(exx+eyy+ezz) + 2*mu*exx
```

On parallel hardware each matrix is distributed on n processors. In our example no communication between processors is necessary. We expect, that the computation time reduces by a factor 1/n.

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Macroscopic and microscopic description

- continuum mechanics
- lattice gases
- fluid mechanics
- nonlinear processes

Partial differential equations in geophysics

- conservation equations
- constitutive laws
- wave equation
- diffusion equation
- Navier-Stokes equation

Macroscopic and Microscopic Description

Macroscopic description:

The universe is considered a continuum. Physical processes are described using partial differential equations. The described quantities (e.g. density, pressure, temperature) are really averaged over a certain volume.

Microscopic description:

If we decrease the scale length or we deal with strong discontinous phenomena we arrive at the discrete world (molecules, minerals, atoms, gas particles). If we are interested in phenomena at this scale we have to take into account the details of the interaction between particles.

Microscopic and Macroscopic Description - Examples

Macroscopic

- elastic wave equation
- Maxwell equations
- convection
- flow processes

Microscopic

- ruptures (e.g. earthquakes)
- waves in complex media
- tectonic processes
- gases
- flow in porous media

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conservation equations

$$\partial_t \rho + \partial_j (v_j \rho) = 0$$

mass

$$\partial_t (v_j \rho) + \partial_j (\rho\, v_i v_j - \sigma_{ij}) = f_i$$

momentum

$$f_i = s_i + g_i$$

gravitation (g)
und sources (s)

gravitation

$$g_i = -\partial_i \Phi$$

gravitational field

$$\Delta\Phi = -\rho\, 4\pi\, G$$

$$\Delta = (\partial_x^2 + \partial_y^2 + \partial_z^2)$$

gravitational potential
Poisson equation

still missing: forces in the medium

->stress-strain relation

stress and strain

$$\sigma_{ij} = \theta_{ij} + c_{ijkl}\partial_l u_k$$

prestress and incremental stress

$$\varepsilon_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j + \partial_i u_m \partial_j u_m)$$

nonlinear stress-strain relation

$$\varepsilon_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$$

… linearized ...

general viscoelastic solid

$$\sigma_{ij}(x,t) = \theta_{ij} + \int_0^\infty d\tau \Psi_{ijkl}(x,\tau)\partial_l u_k(x,t-\tau)$$

$$\Psi_{ijkl}(x,t) = c_{ijkl}\delta(t) + \nu_{ijkl}\delta_t(t)$$

relaxation functions

special case: $v \Longrightarrow 0$
small velocities

$$\partial_t(v_j\rho) + \partial_j(\rho\, v_i v_j - \sigma_{ij}) = f_i$$

$$v_i \rightarrow 0 \Rightarrow \rho v_i v_j \approx 0$$

We will only consider problems in the low-velocity regime.

special case: static density

$$\rho(x,t) = \rho_0(x) + \delta\rho(x,t) \approx \rho_0(x)$$

incompressible flow, wave phenomena

We will only consider problems with static density.

hyperbolic differential equations
e.g. the acoustic wave equation

$$\frac{1}{K}\partial_t^2 p - \partial_{x_i}\frac{1}{\rho}\partial_{x_i} p = -s$$

K compression
s source term

parabolic differential equations
e.g. diffusion equation

$$\partial_t T = D\partial_i^2 T$$

T temperature
D thermal diffusivity

elliptical differential equations
z.B. static elasticity

$$\partial_{x_i}^2 U(x) = F(x)$$

$$U = \partial_m u_m$$

$$F = \partial_m f_m / K$$

u displacement
f sources

Numerical Methods in Geophysics: Introduction

Summary:

Numerical method play an increasingly important role in all domains of geophysics.

The development of hardware architecture allows an efficient calculation of large scale problems through parallelisation.

Most of the dynamic processes in geophysics can be decribed with time-dependent partial differential equations.

The main problem will be to find ways to determine how best to solve these equations with numerical methods.