Handling Unstructured Grids

- Triangulation of unstructured grids
- Interpolation of a function defined on unstructured grids
- Differentiate a function on unst. grids
- Relevant for:

  - geographical information
  - plotting problems
  - solving partial differential equations

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Voronoi Cells

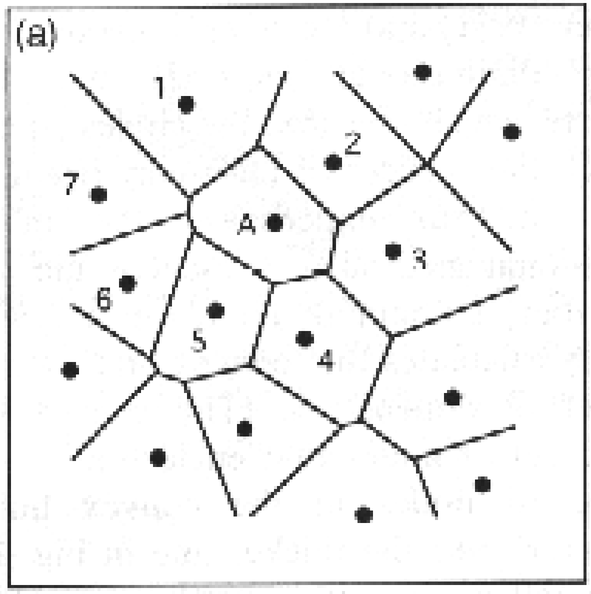
The Voronoi diagrams of an unstructured set of nodes divides the plane into a set of regions, one for each node, such that any point in a particular region is closer to that regions node than to any other.

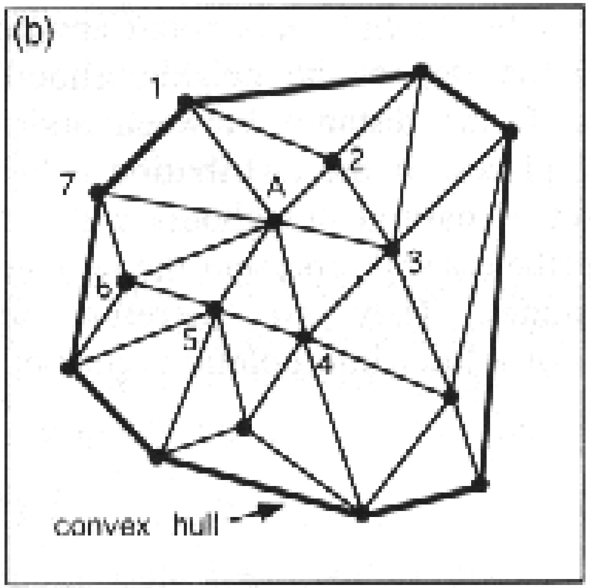
... one of the most fundamental and important geometrical constructs determined by an irregular set of points ...

Delaunay Triangulation

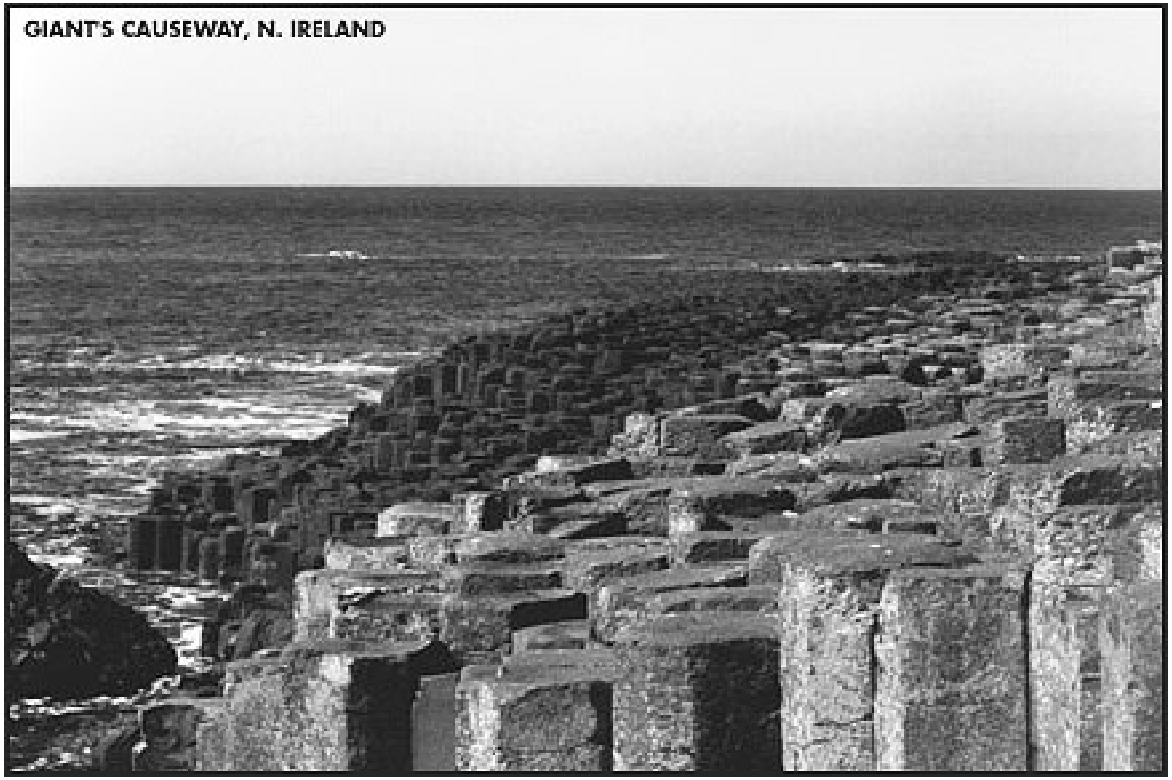
The Delaunay triangles are formed by connecting the nodes whose Voronoi cells have a common boundary.

Voronoi and Delaunay

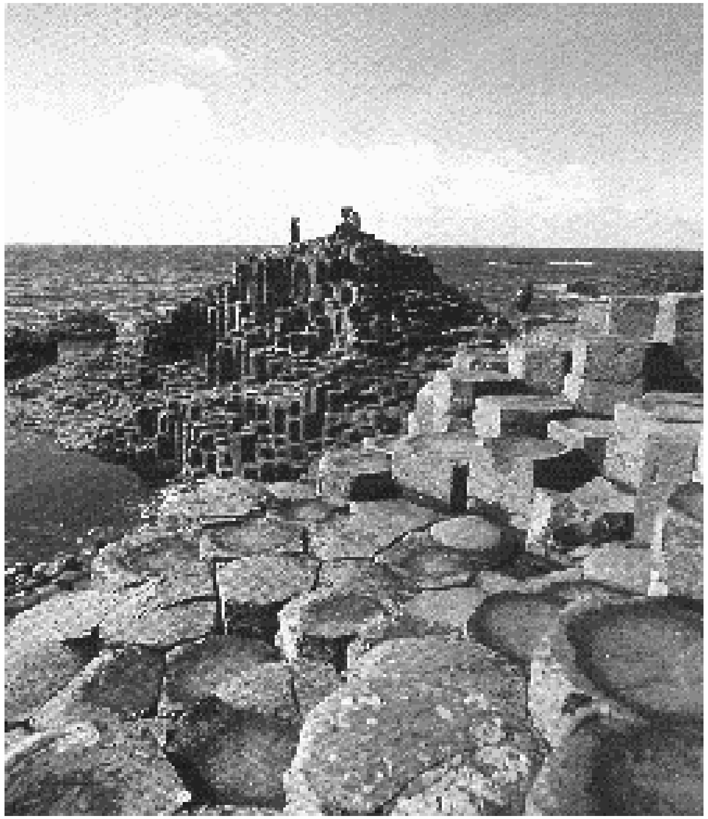


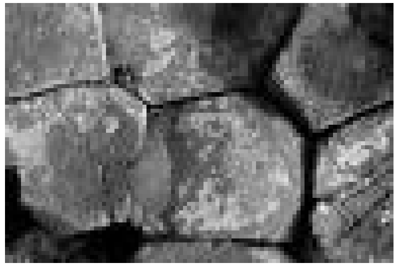


Voronoi Cells in Nature

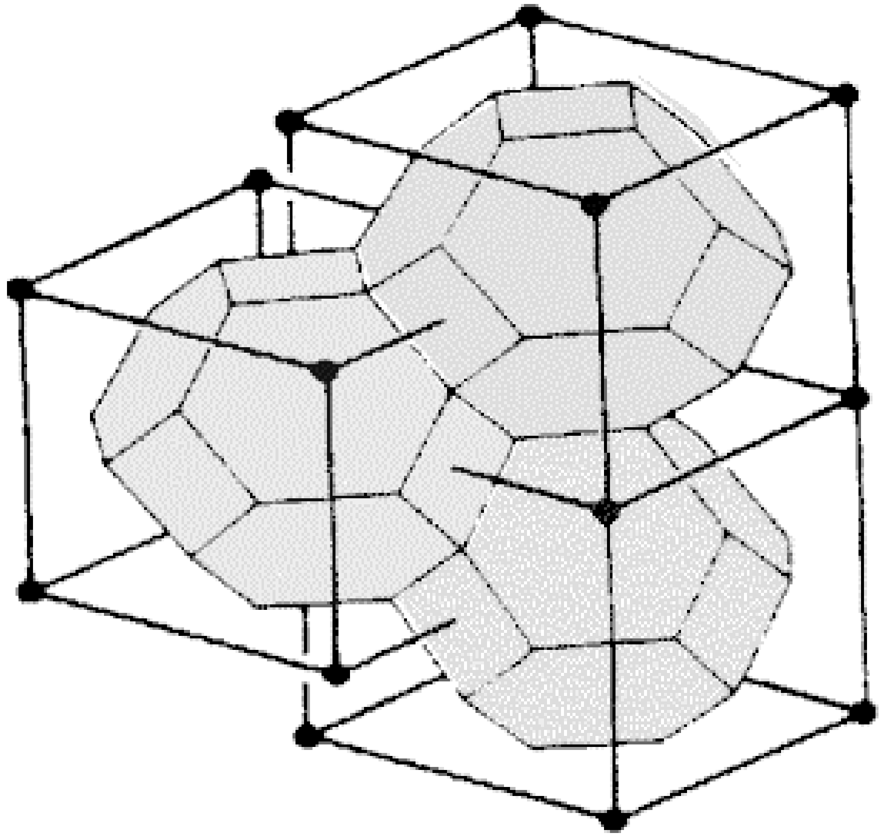


Voronoi Cells in Nature





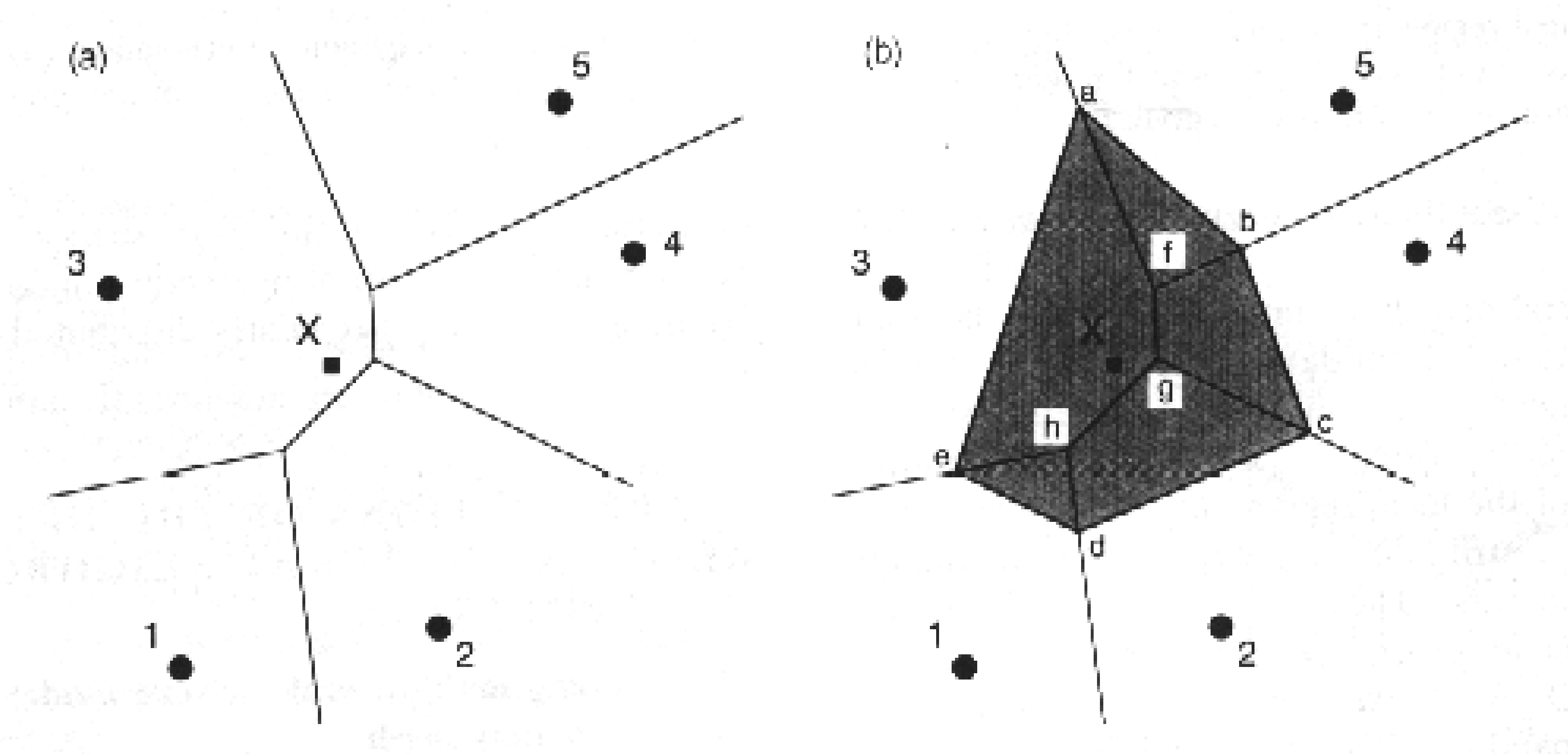
Voronoi in Physics



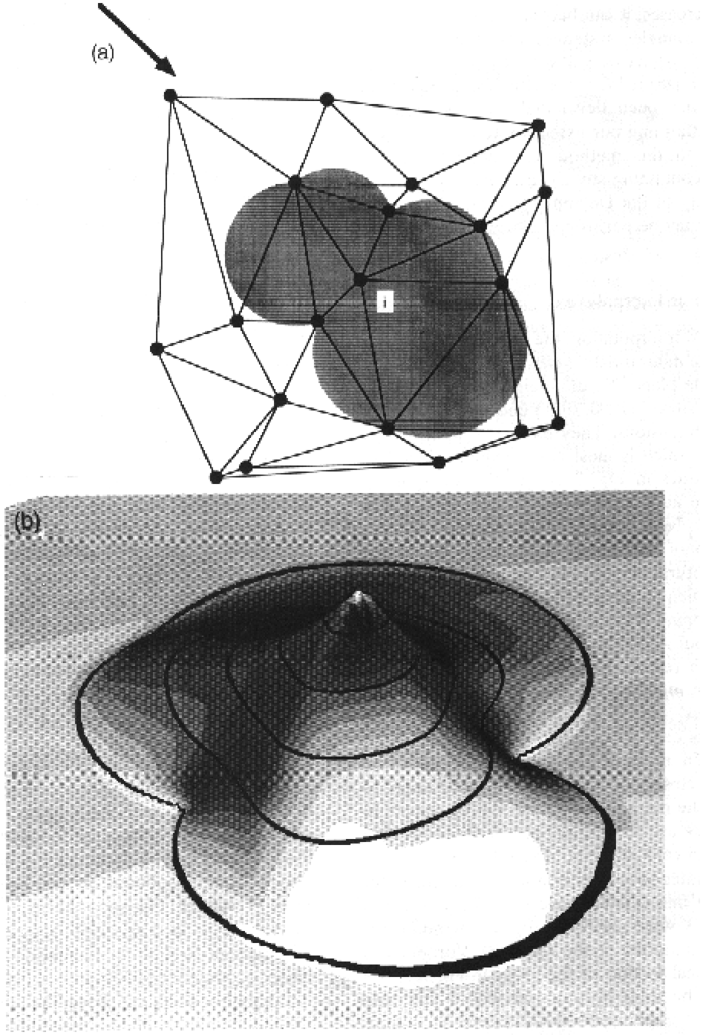
Natural Neighbours

- Basis for local interpolation
- Linear interpolation using triangles
- Distance weighting
- Natural neighbour interpolation
- Differential weights
- Examples

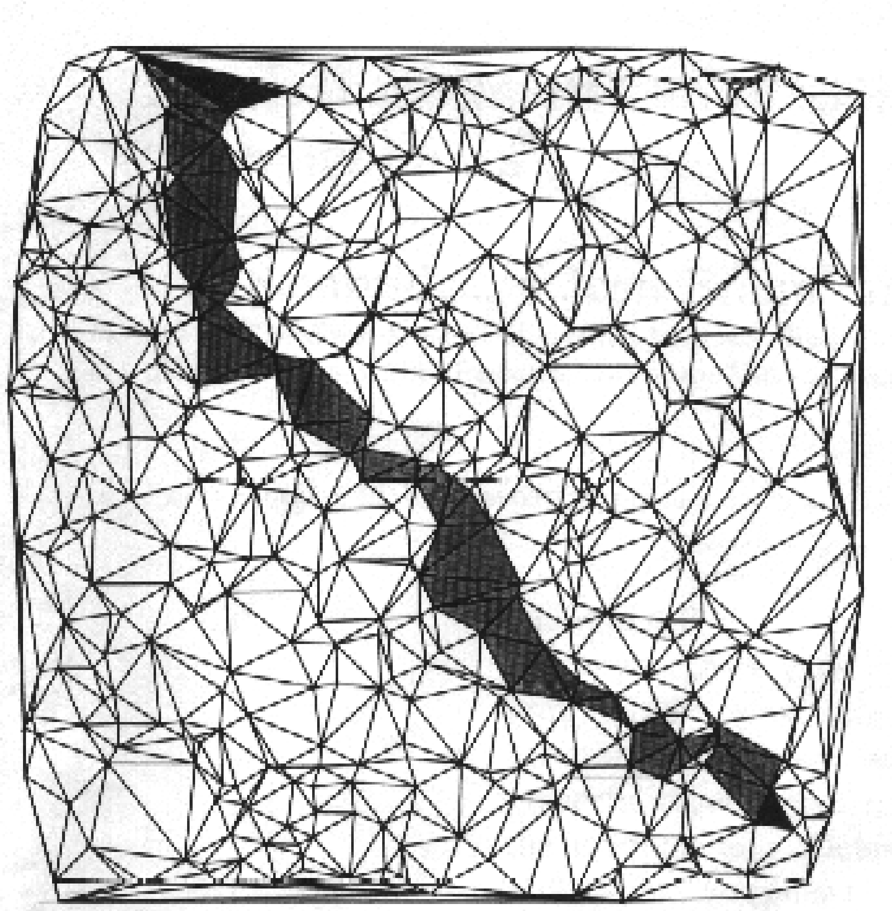
Voronoi: Overlapping Regions



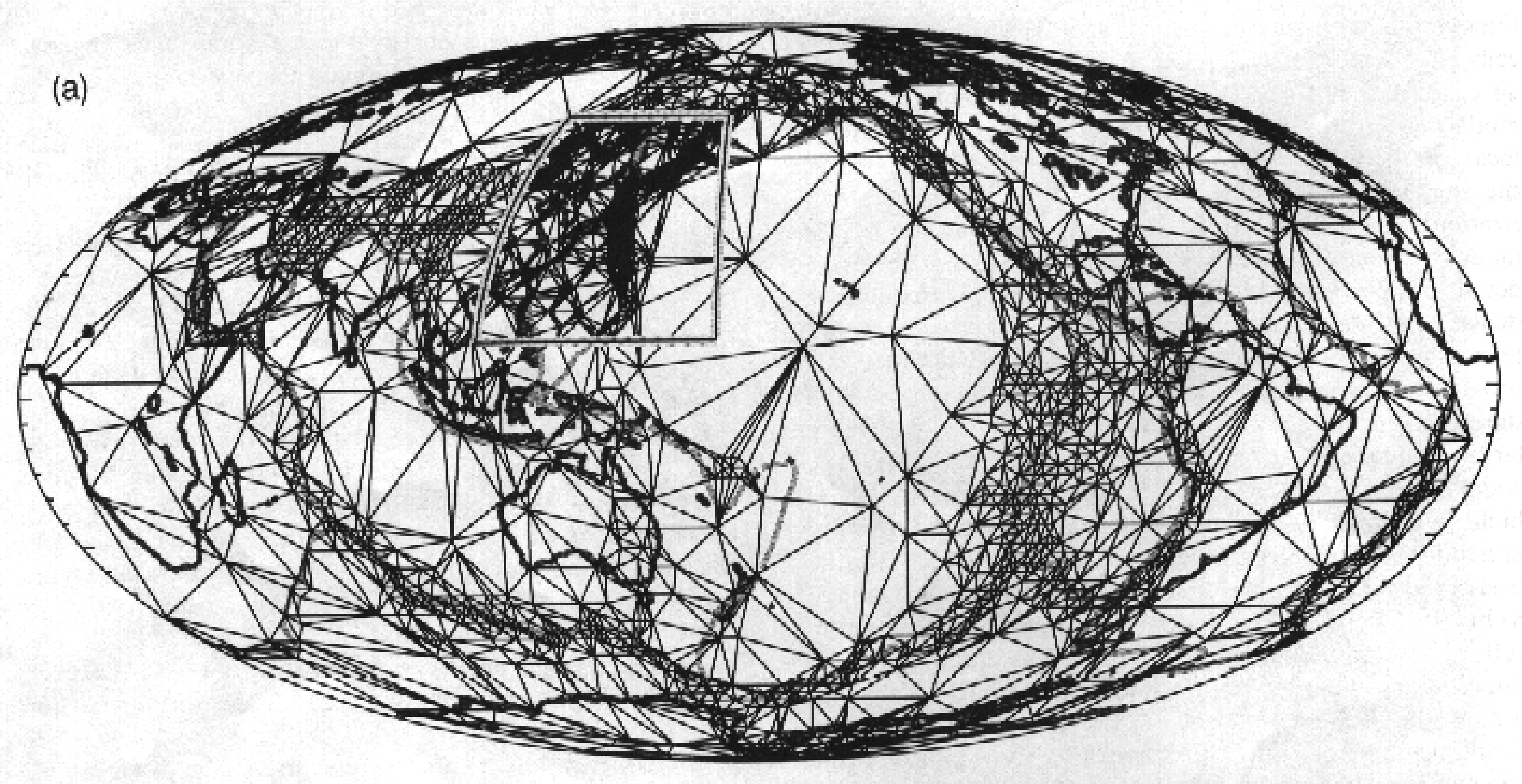
Influence region: Basis Functions



Finding a Triangle



Applications: Global Data Sets



Applications: Regional Data Sets

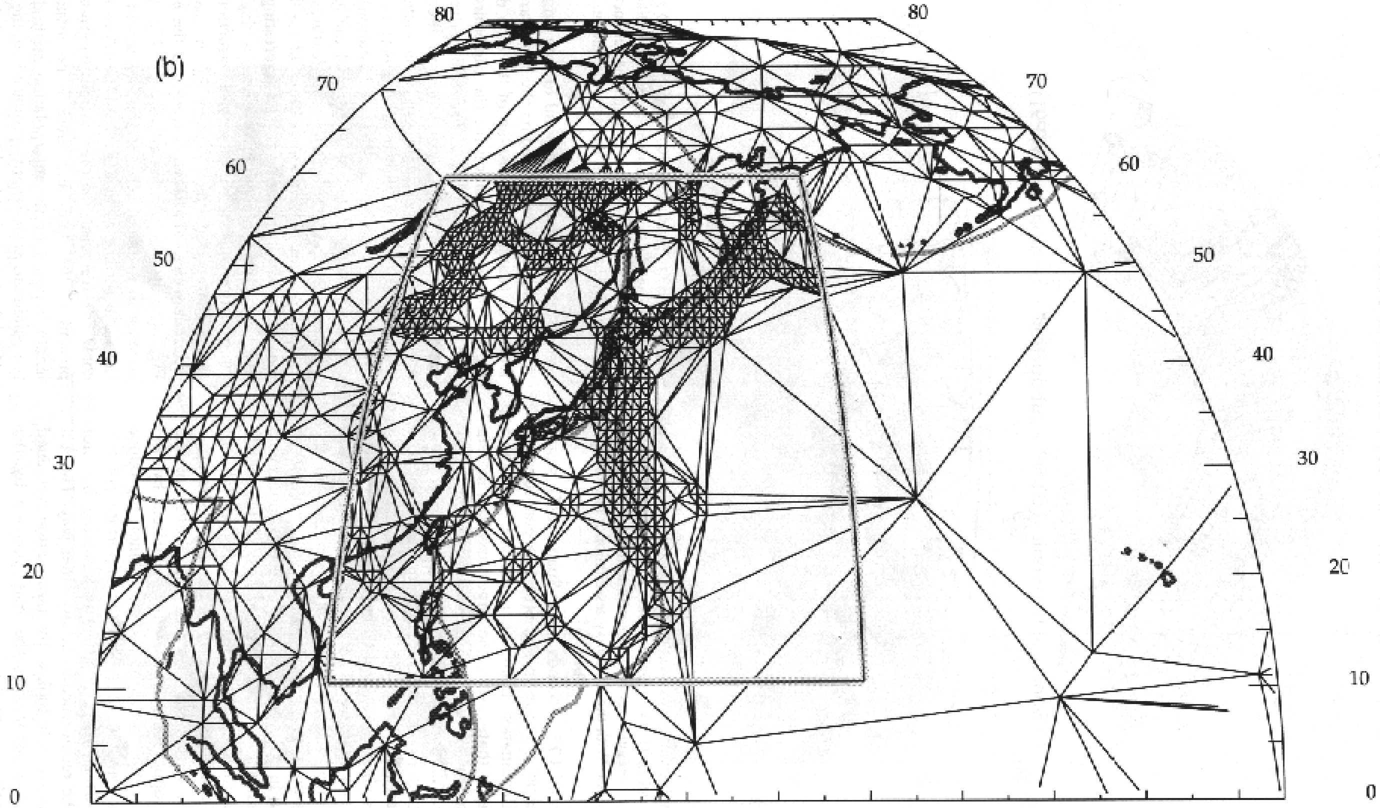
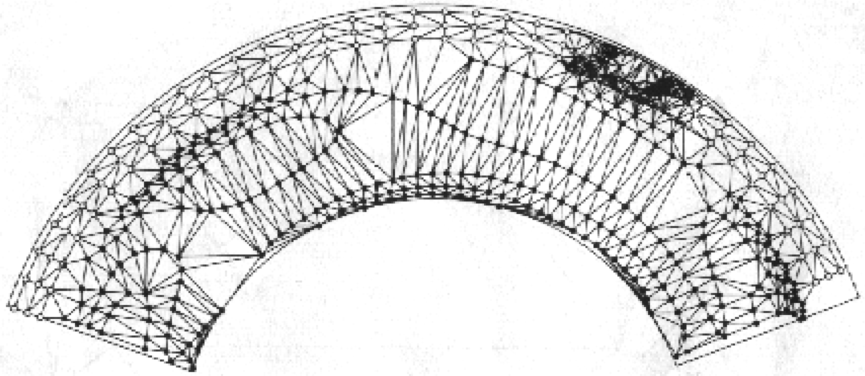
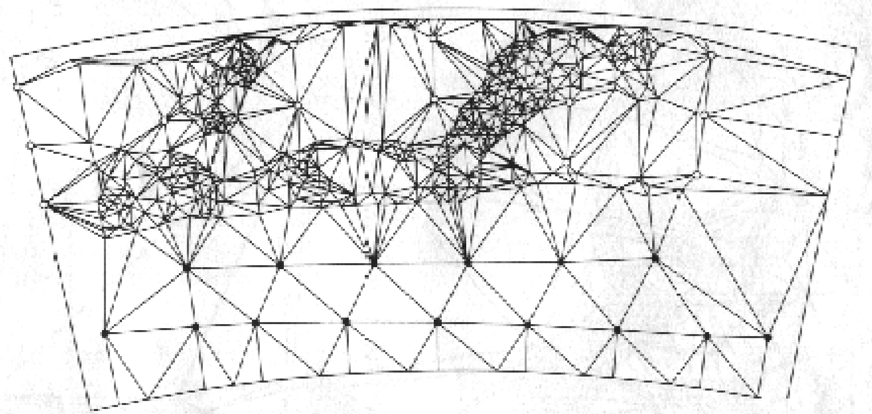


Figure 4. (*Continued.*)

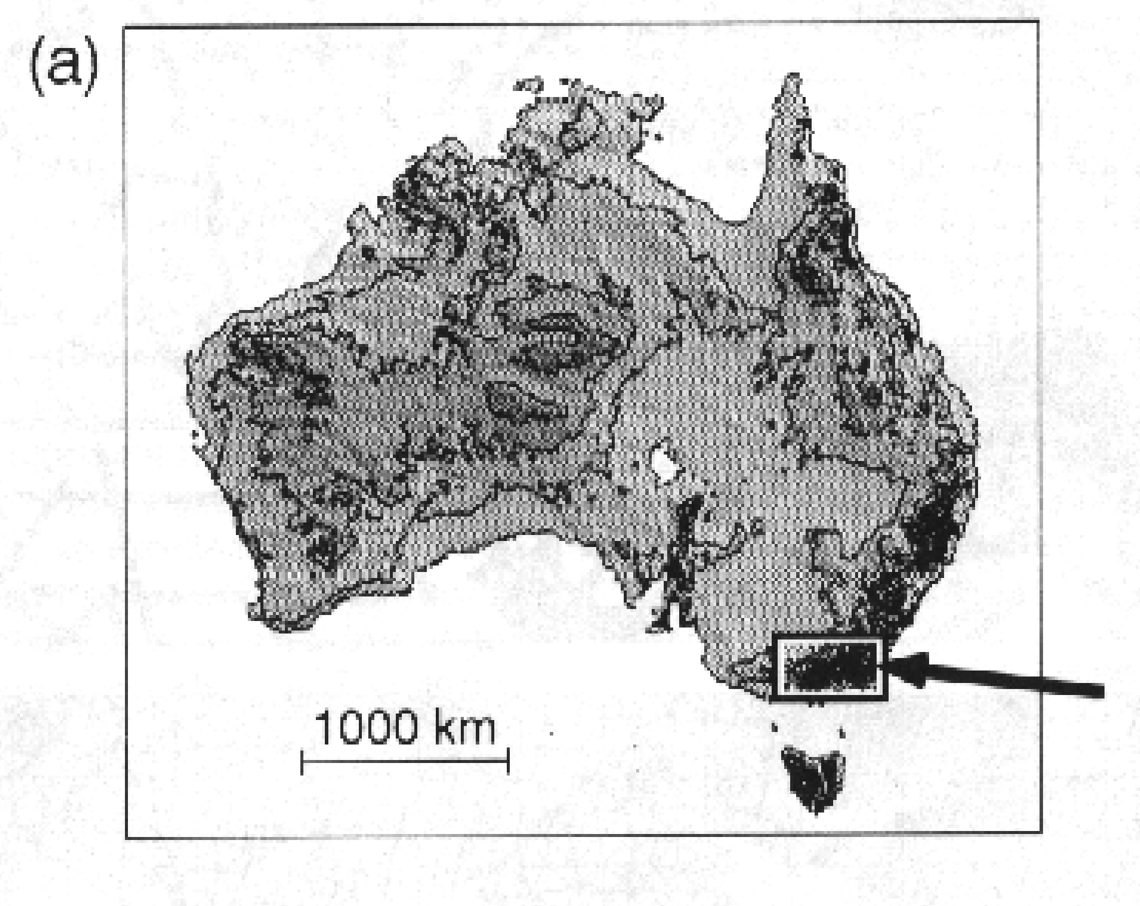
Applications: Regional Data Sets



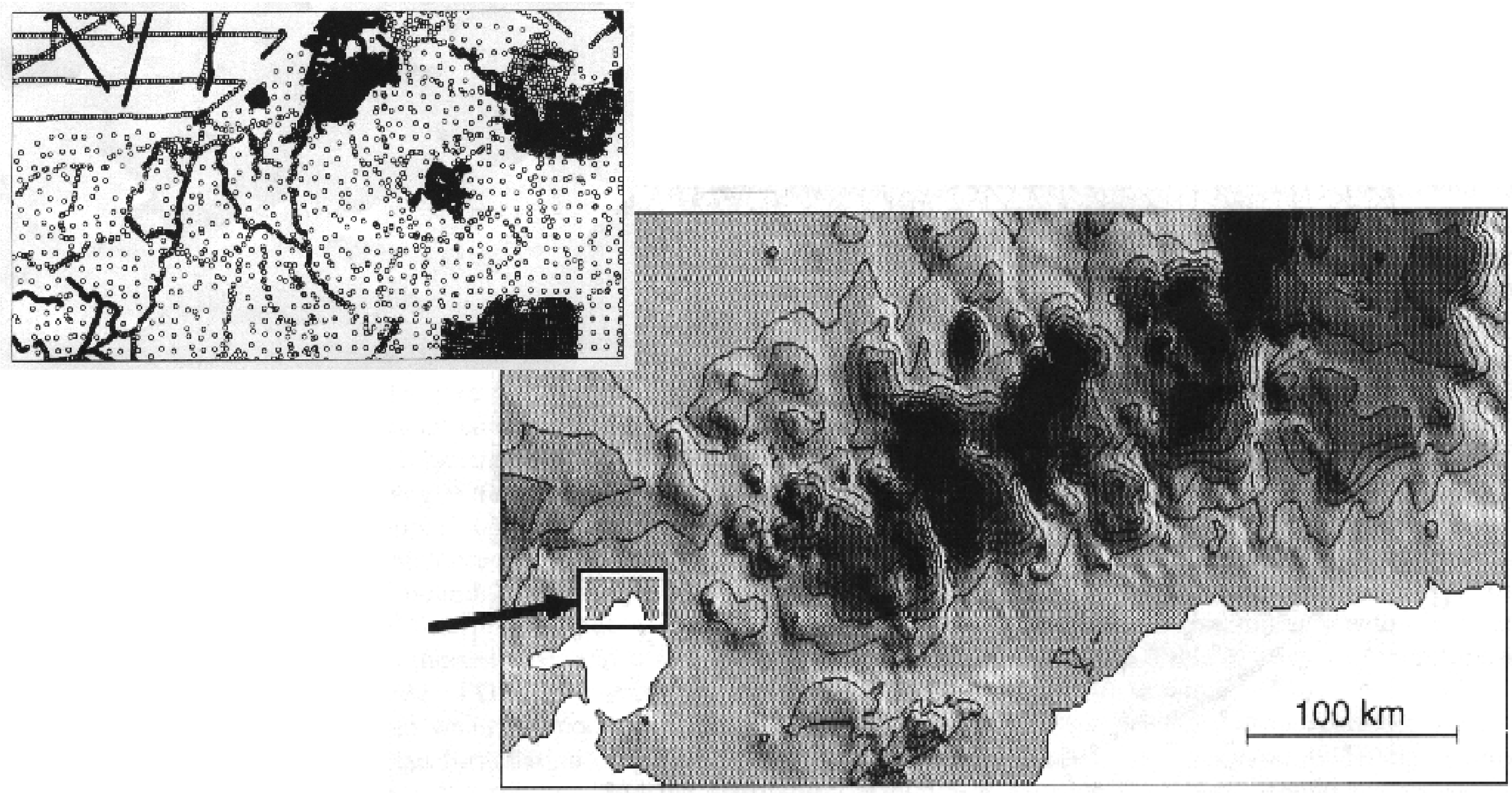
(b) Spherical harmonic (L02.56) + subduction zone models (NWP91)



Applications: Interpolation



Applications: Interpolation



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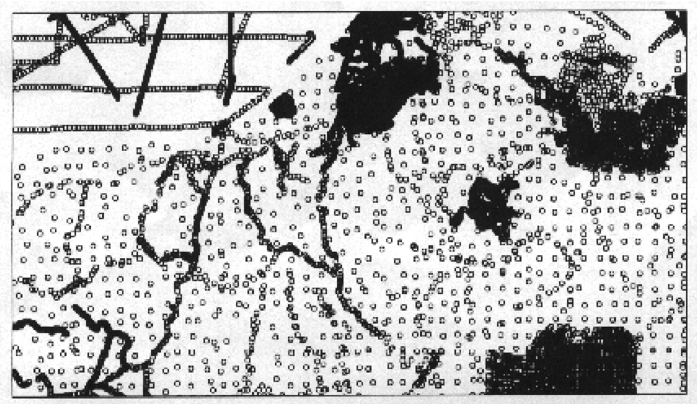
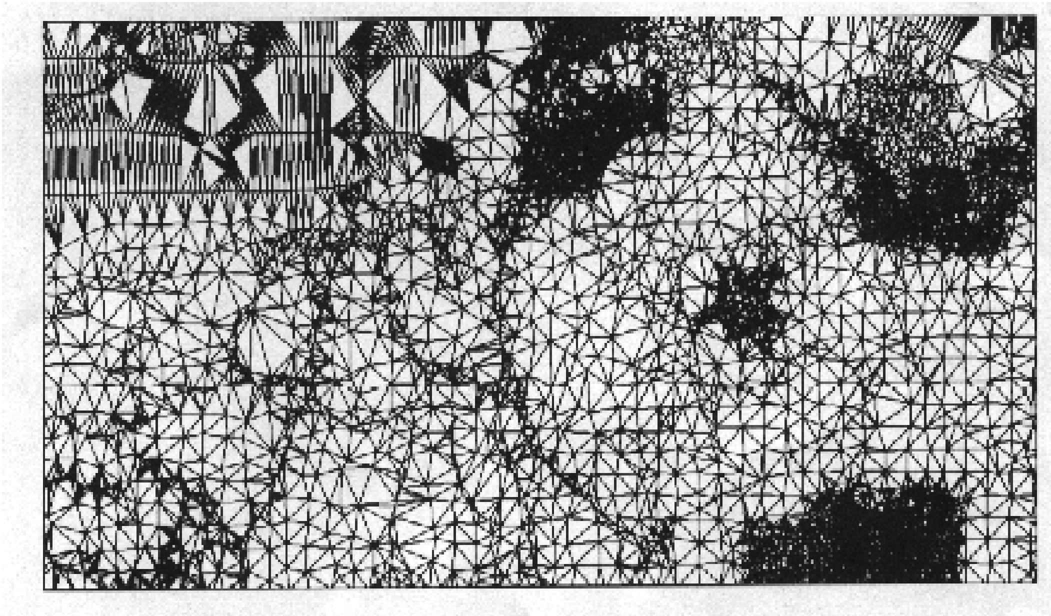
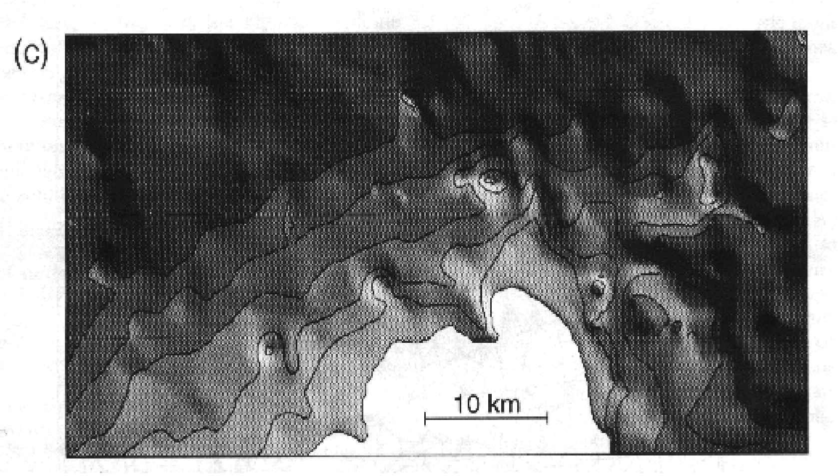
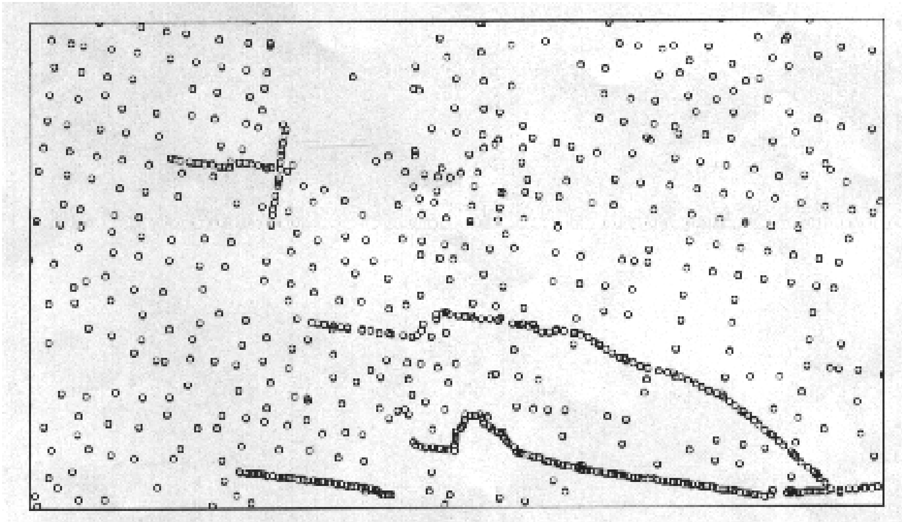
Applications: Interpolation

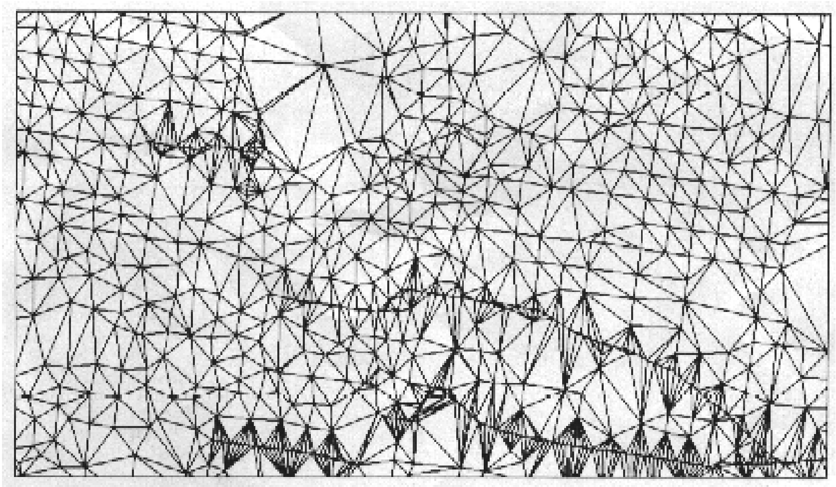




Applications: Interpolation



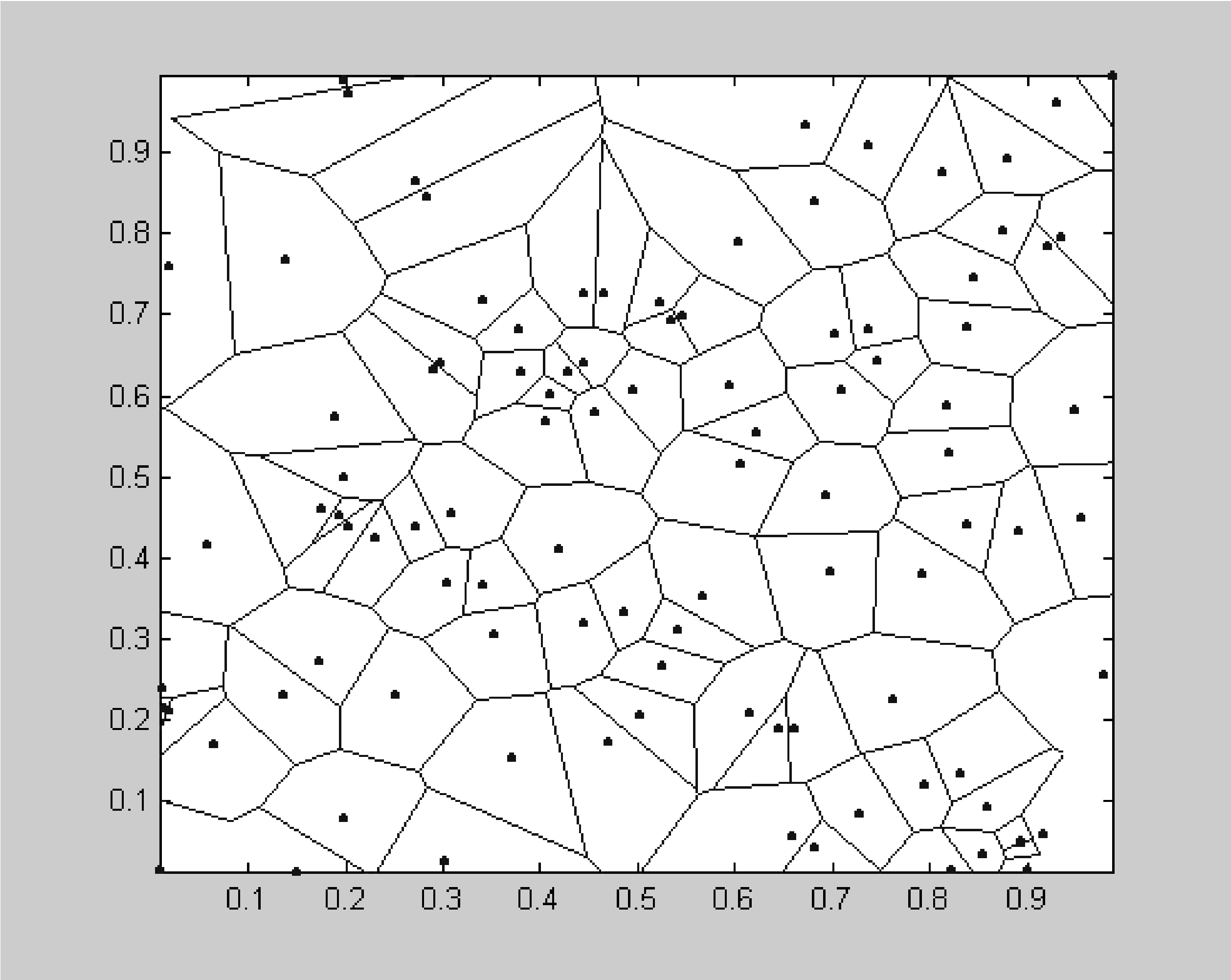




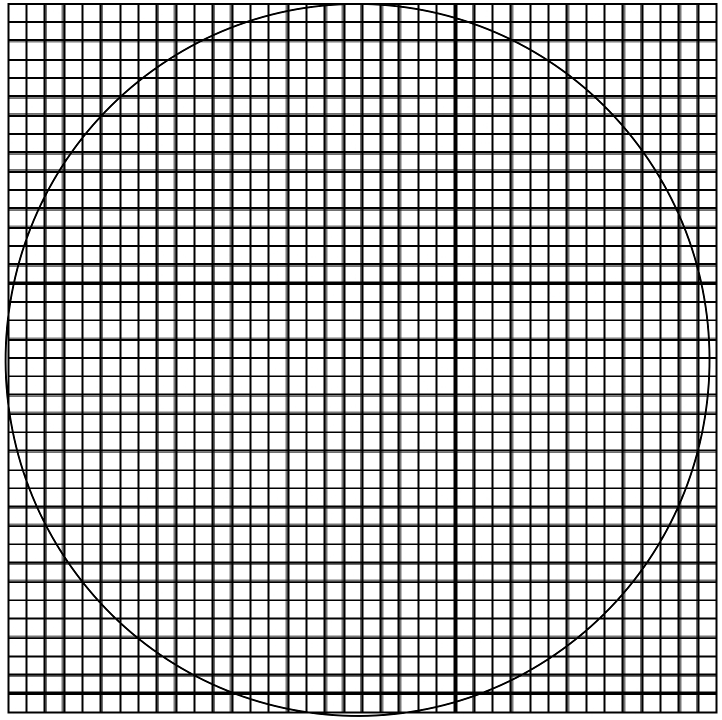
Irregular Grid Methods

- FD, PS Method not applicable to whole sphere
- FE difficult to parallelise
- Singularities in spherical system
- Can we extend FD to irregular grids?
- Can we work in Cartesian formulation
  but on a spherical grid?

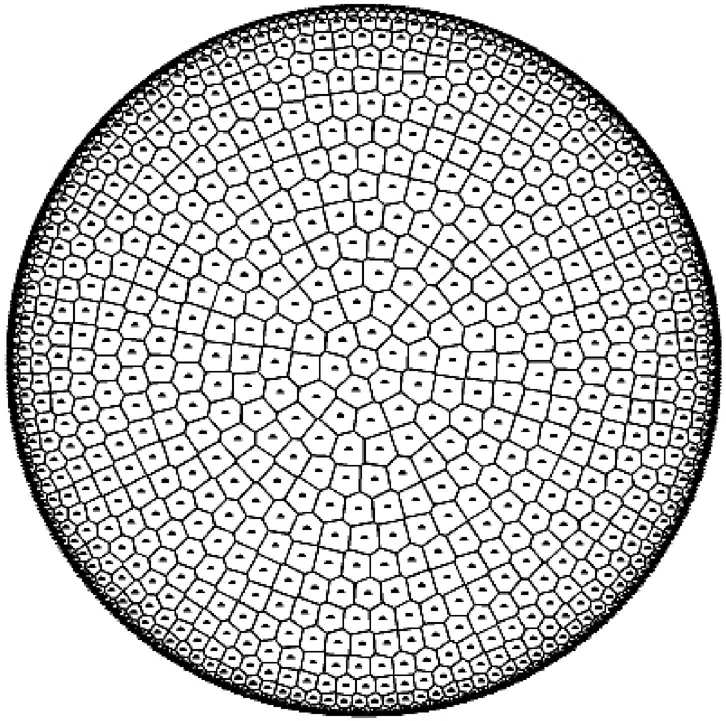
Voronoi cells



Irregular Grid Methods



Regular Grid



Voronoi cells

... remember ...

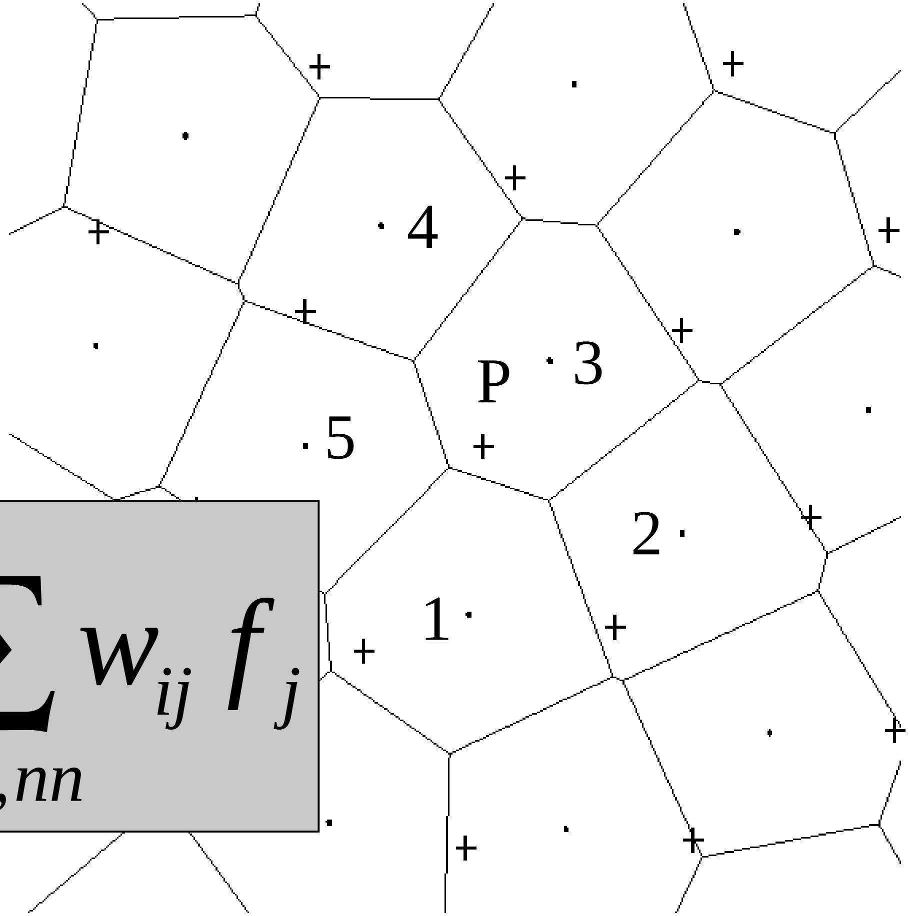
$$\rho\partial_t^2 u_i = \partial_j(\sigma_{ij} + M_{ij})$$

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}$$

(in order of appearance)

ρ	mass density
u_i	displacement
σ_{ij}	stress
M_{ij}	Moment
c_{ijkl}	elasticity
ε_{kl}	deformation

Waves on irregular grids



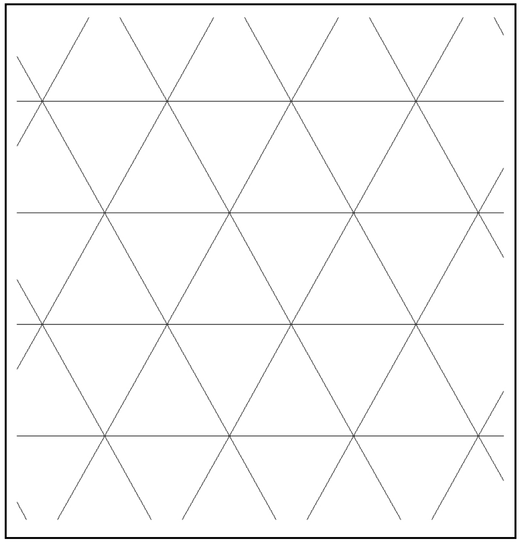
\+ Primary grid (velocities)

. Secondary grid (stresses)

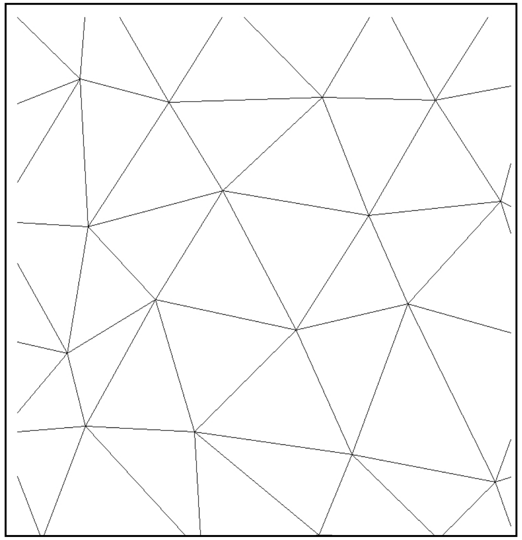
Point P with 5 natural neighbours

$$\partial f_P^i = \sum_{i=1,nn} w_{ij} f_j$$

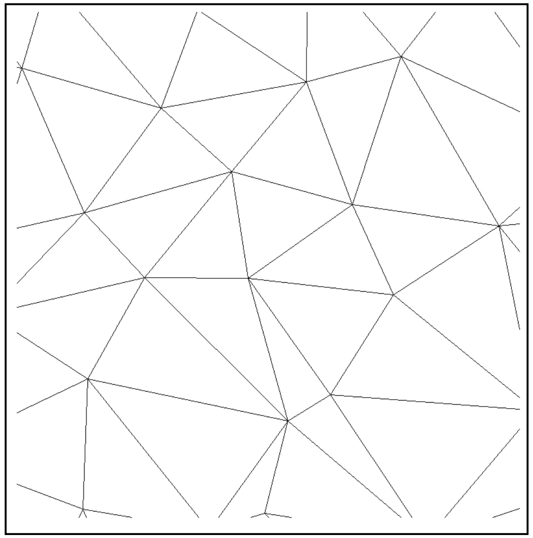
Grid quality



q_{mean}=1.0



q_{mean}=0.9

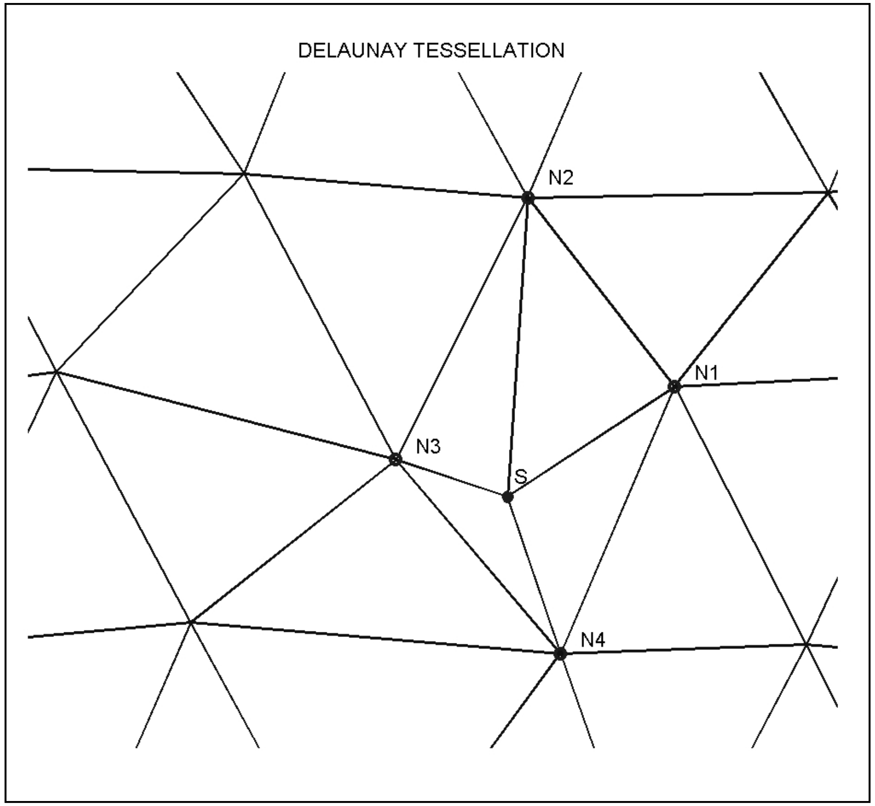


q_{mean}=0.8

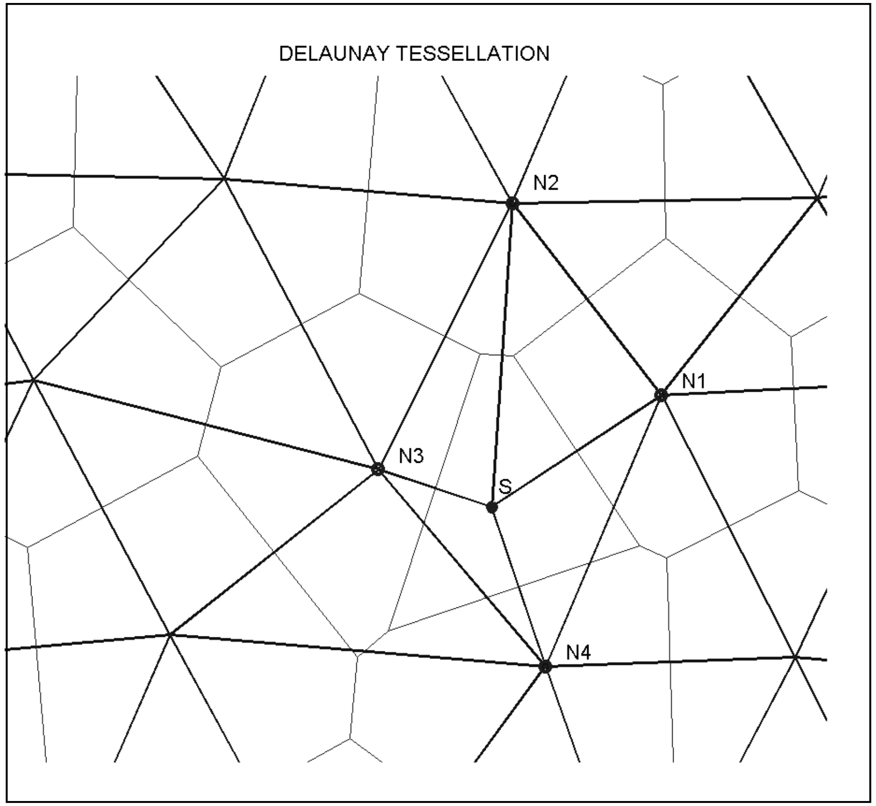
$$q = \frac{4\sqrt{3}A}{a^2 + b^2 + c^2}$$

Triangle quality q

Method 1: Natural Neighbour Coordinates



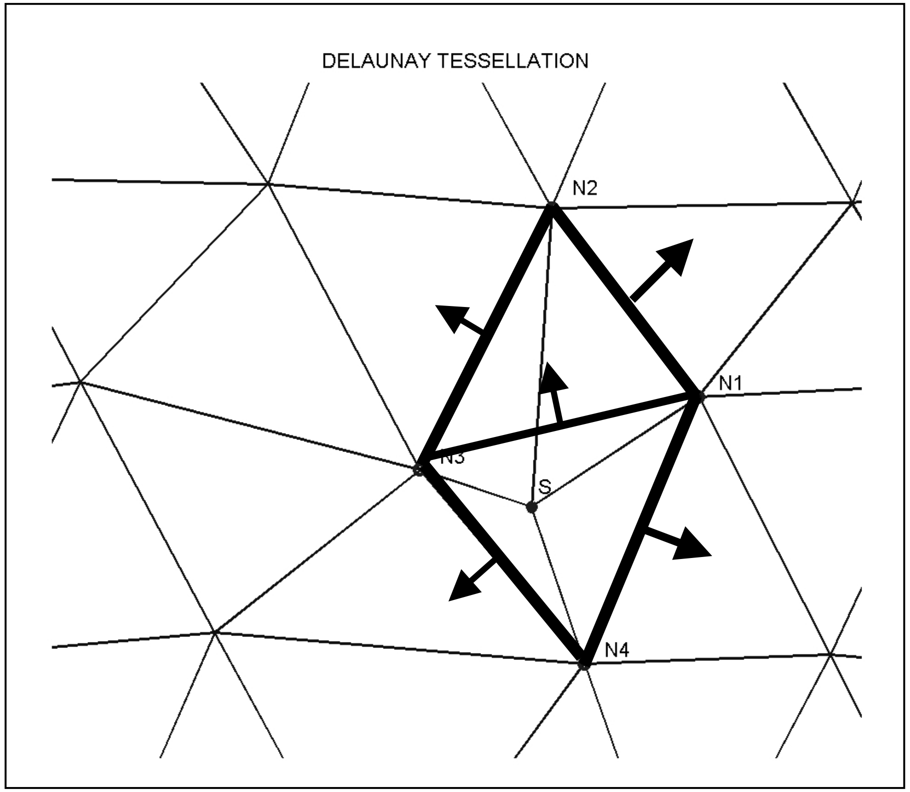
Triangulation



Voronoi Cells

Interpolation (and differential weights for natural neighbours are calculated using overlapping Voronoi cells).

Method 2: The Finite Volume Method



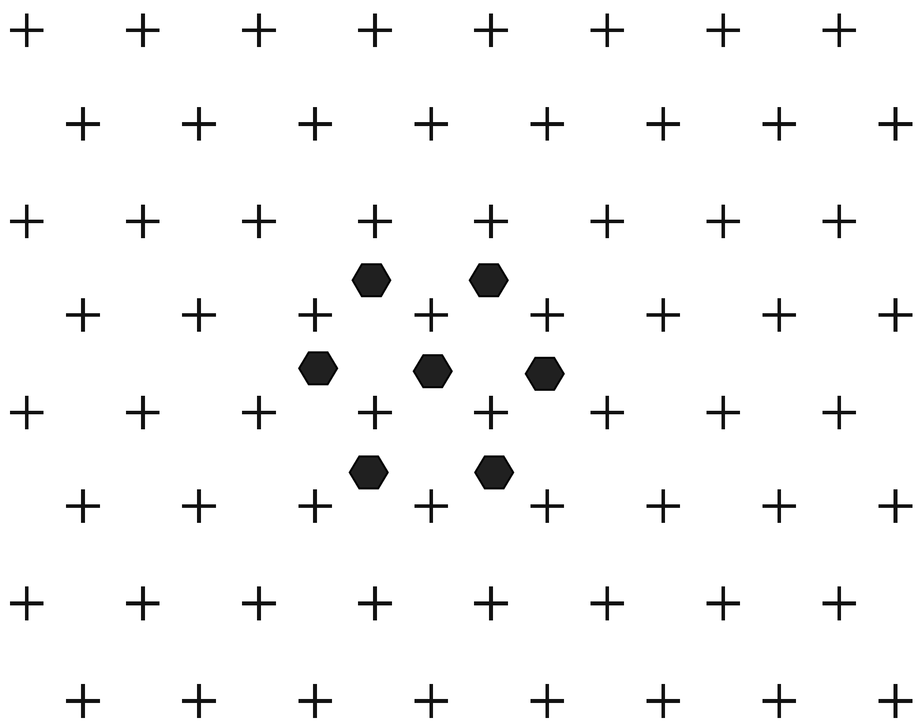
The Finite Volume method is based on a discretization of Gauss' Law

$$\partial_i f = \frac{1}{\Delta S} \sum_{j=1}^{NN} \Delta L_{ij} n_{ij} f_j$$

Note that the position of point S is irrelevant!

Surprising result! Using only three points is more accurate than using all natural neighbours!

Method 3: Hexagonal grids (reference)



Differentiation weights for exact hexagonal grids were obtained by Magnier et al. (1994). This is called a *minimal* grid.

Test Function

Test function f_p on primary grid points x_i:

$$f_p(x_i) = \sin(\underline{k}\,\underline{x}_i - wt)$$

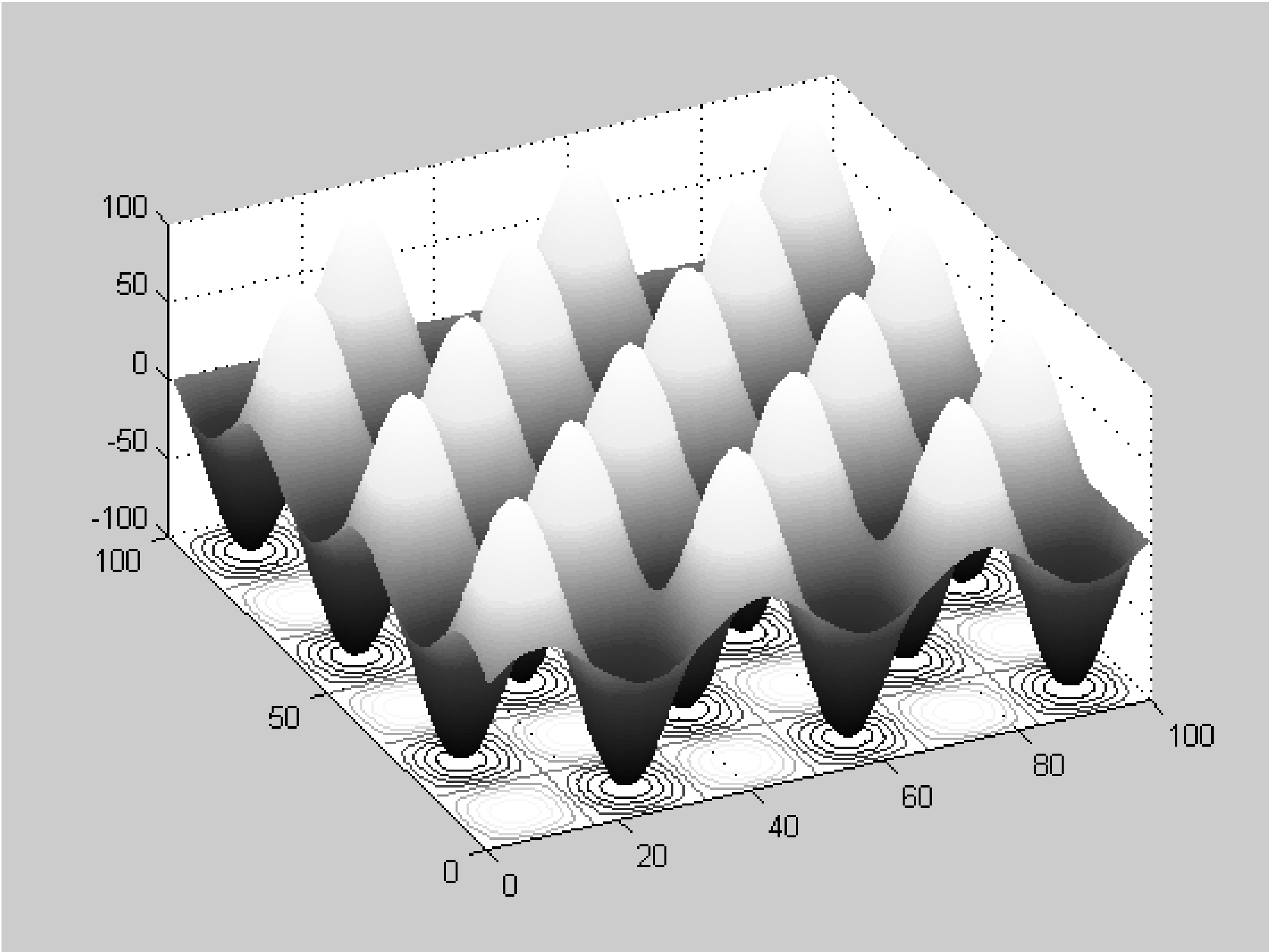
Analytical deriv ative $f^{(j)}$ on secondary grid points x_k:

$$f_s^{(j)}(x_k) = k_j \cos(\underline{k}\,\underline{x}_k - wt)$$

Error of numerical derivative on sec. grid

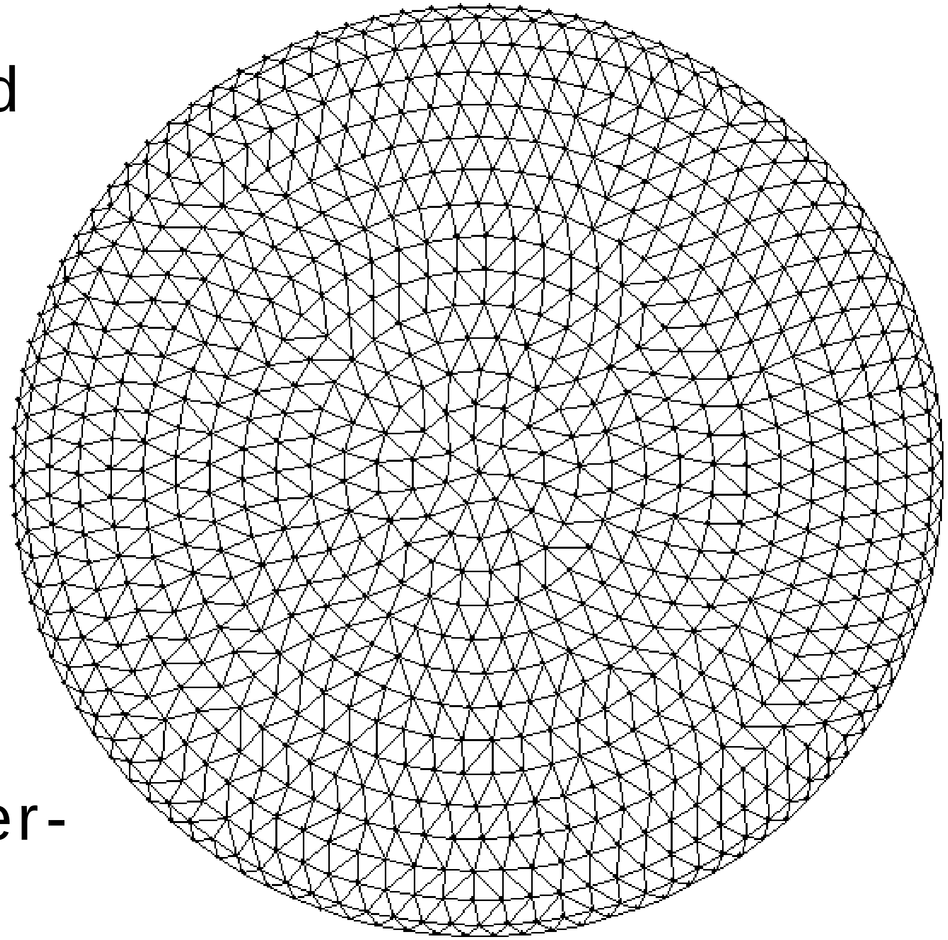
$$\varepsilon(\underline{k}, q_{mean}) = \frac{\sum_k (\tilde{f}^{(j)}(x_k) - f(x_k))^2}{\sum_k f^2(x_k)}$$

Test Function



Cylindrical Grid - Triangulation

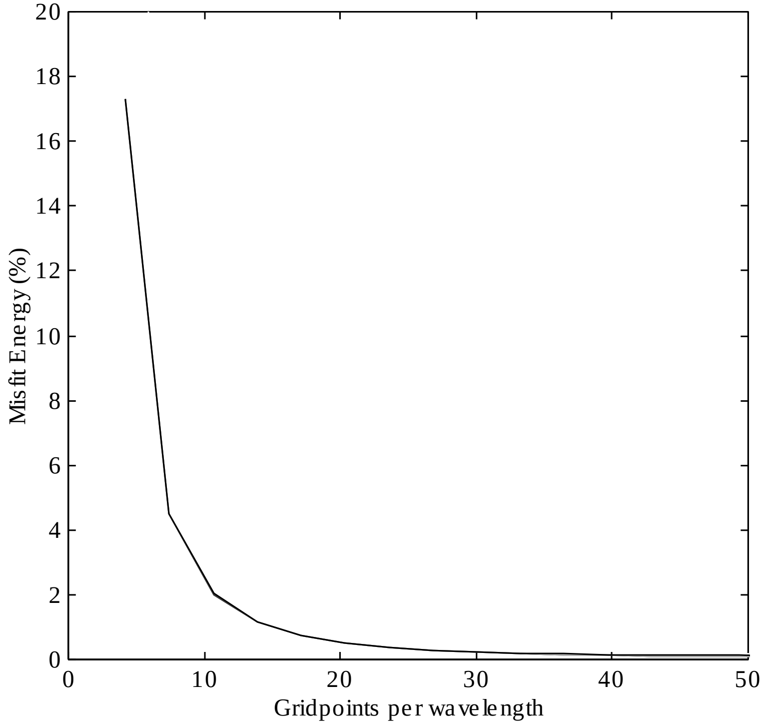
Equations are solved in Cartesian form.



Grid density can be tied to the velocity model to avoid over-/under-sampling.

Error of space derivative

Hexagonal grid - perfect equilateral triangles

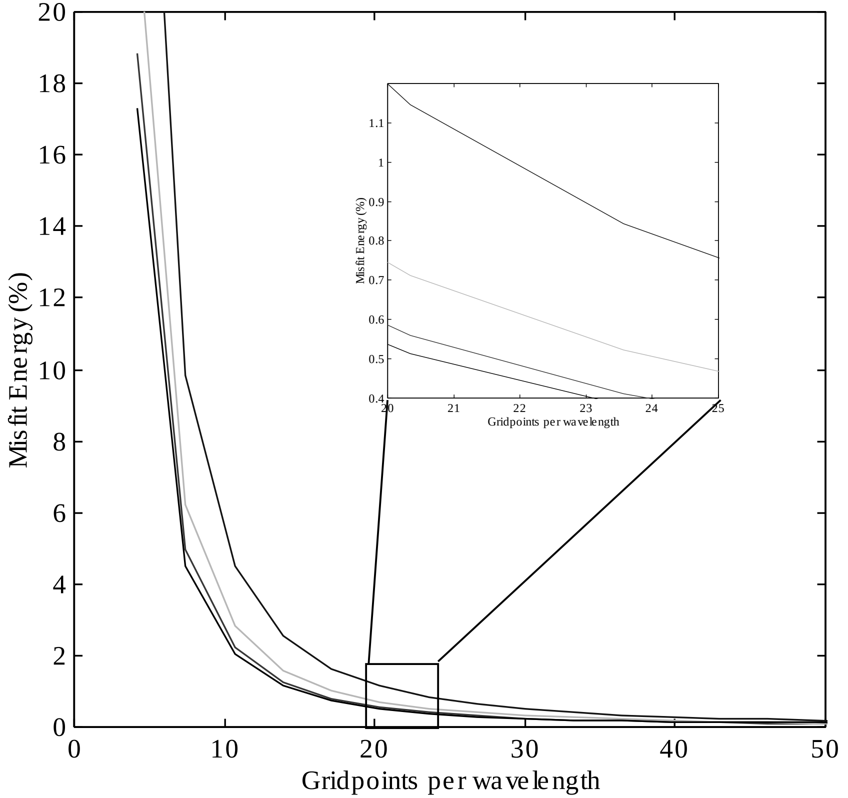


black - Magnier
green - NN
blue - FV(NN)
red - FV (3 points)

all methods are identical !

Error of space derivative

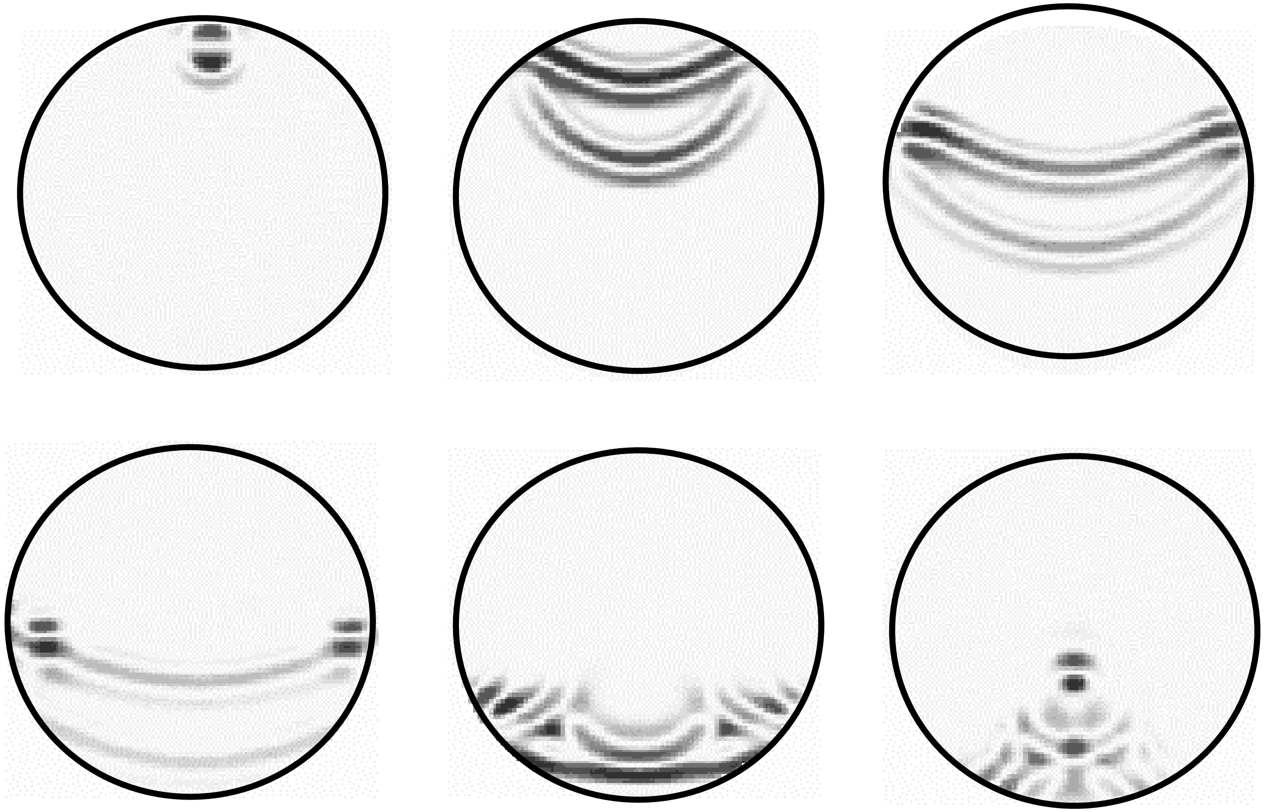
Irregular Grid - $q^{mean} = 0.8$



black - Magnier
green - NN
blue - FV(NN)
red - FV (3 points)

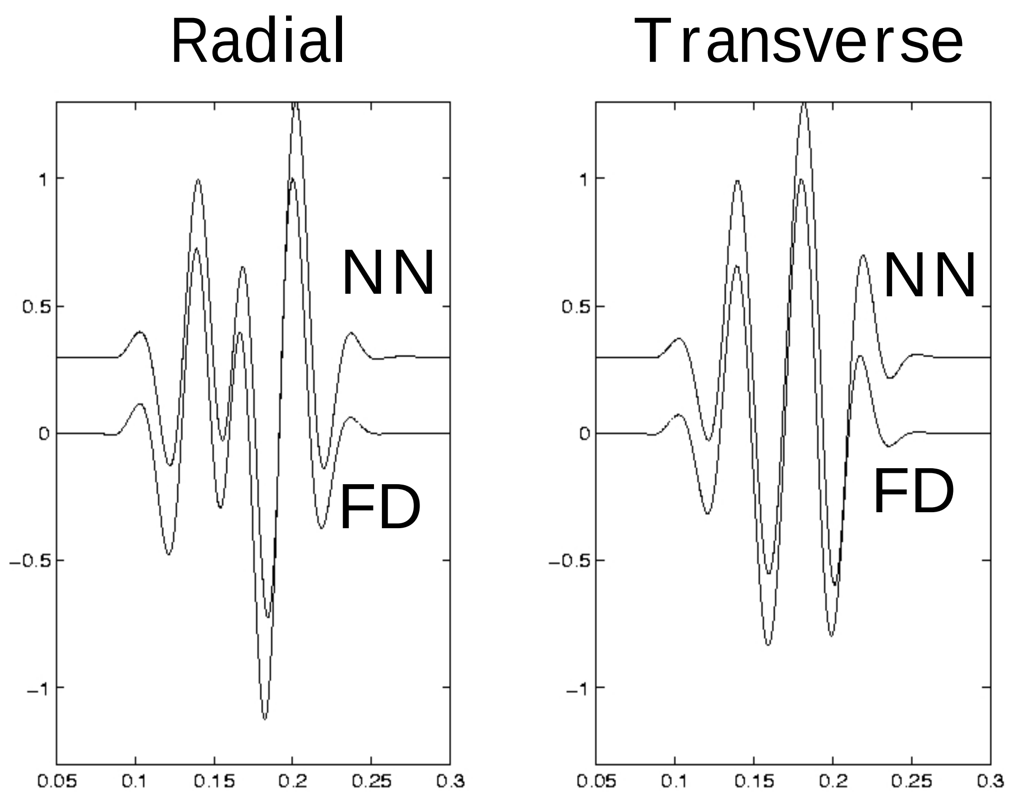
FV(3) is the winner !

Cylindrical Grid - Snaps

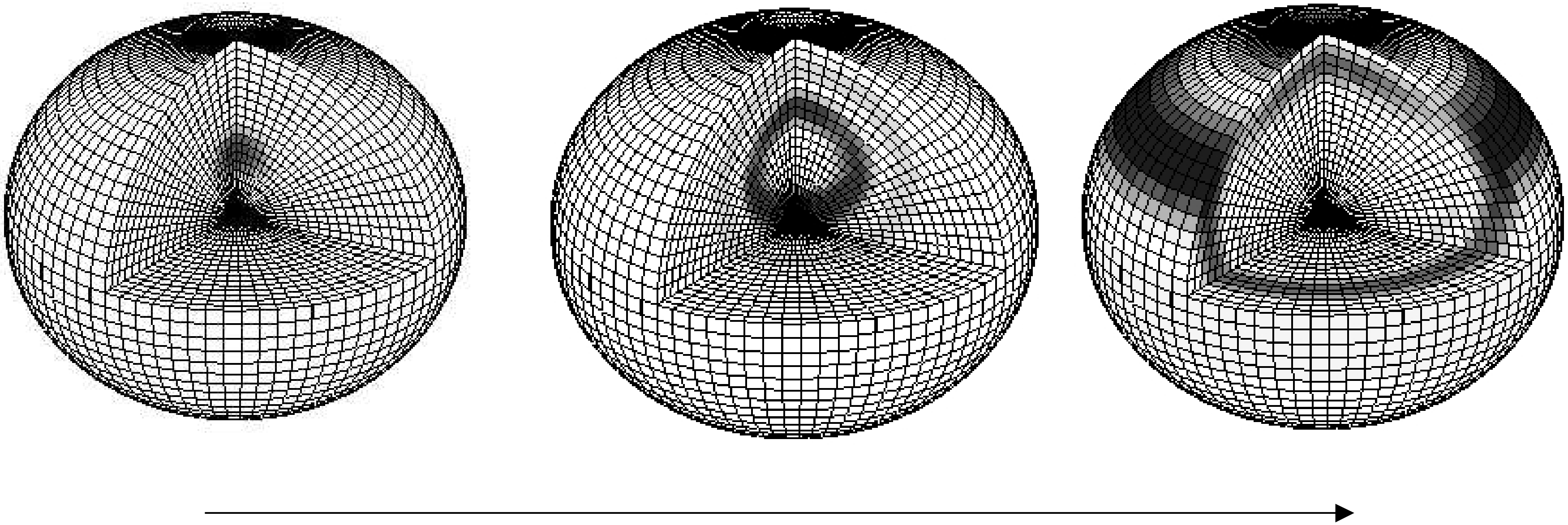


Acoustic wave propagation in a cylinder

Comparison with FD



Waves in a Sphere



Time

Pressure Waves - Homogeneous Sphere