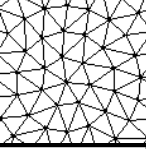




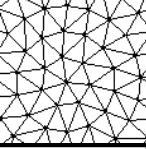
Finite Elements - A practical introduction



- Introduction
 - Why Finite Elements
 - Domains of Applications
 - Applications in Geophysics
 - Brief history
 - Examples
- Review of Matrix Algebra



Finite Elements - a definition

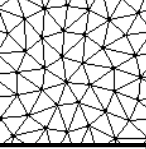


Finite elements ...

A general discretization
procedure of continuum
problems posed by
mathematically defined
statements



Finite Elements - the concept



Basic principle: building a complicated object with simple blocks (e.g. LEGO) or divide a complicated object into manageable small pieces.

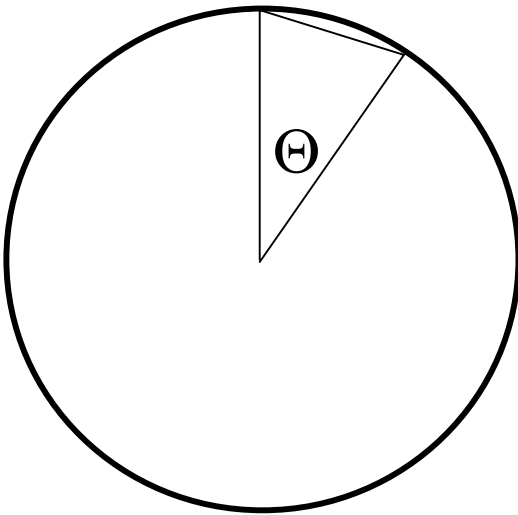
Example: **approximation of an area of a circle**

Area of one triangle: $S_i = \frac{1}{2} R^2 \sin \theta_i$

Area of the circle: $S_N = \sum_{i=1}^N S_i = \frac{1}{2} R^2 N \sin\left(\frac{2\pi}{N}\right)$

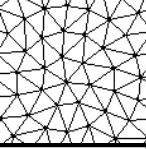
$$\rightarrow \pi R^2 \quad \text{as } N \rightarrow \infty$$

N total number of triangles





Finite Elements - the concept

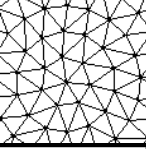


How to proceed in FEM analysis:

- Divide structure into pieces
- Describe behaviour of the physical quantities in each element
- Connect (assemble) the elements at the nodes to form an approximate system of equations for the whole structure
- Solve the system of equations involving unknown quantities at the nodes (e.g. displacements)
- Calculate desired quantities (e.g. strains and stresses) at selected elements



Finite Elements - Why?



FEM allows discretization of bodies with arbitrary shape. Originally designed for problems in static elasticity.

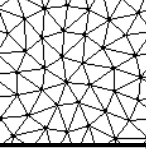
FEM is the most widely applied computer simulation method in engineering.

The required grid generation techniques are interfaced with graphical techniques (CAD).

Today numerous commercial FEM software is available (e.g. ANSYS, SMART)



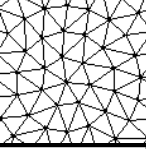
Finite Elements - Applications



- Mechanical, Aerospace, Civil, Automobile Engineering
- Structure analysis (static, dynamic, linear, nonlinear)
- Thermal and fluid flows
- Electromagnetics
- Geomechanics
- Biomechanics
- ...



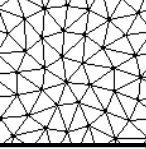
Finite Elements - Geophysics



- Crustal deformation
- Geophysical fluid mechanics
 - Geodynamics
 - Mantle Convection
- Electromagnetics
- Wave Propagation (FE and SE - spectral elements)
- Strong ground motion, earthquake engineering



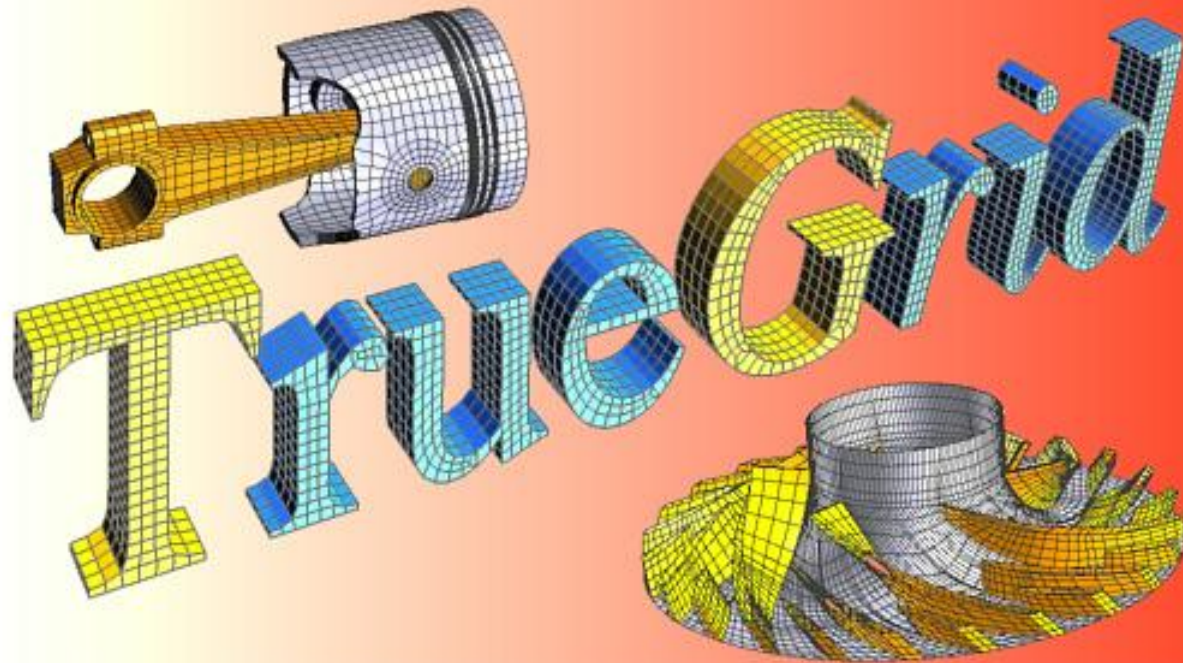
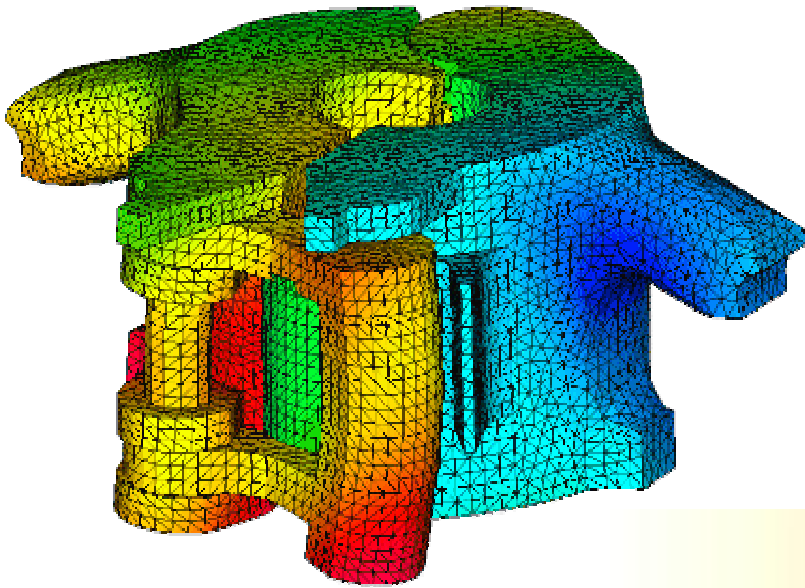
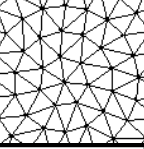
Finite Elements - History



- 1943 - Courant (Variational Methods)
- 1956 - Turner, Clough, Martin, Topp (Stiffness)
- 1960 - Clough ("Finite Elements", plane problems)
- 1970s - applications on mainframe computers
- 1980s - pre- and postprocessing on microcomputers
- 1990s - today applications to large structural systems

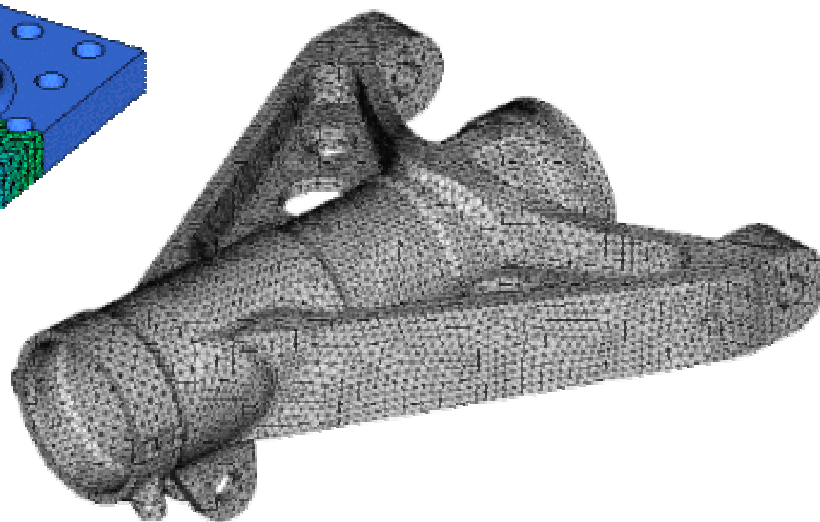
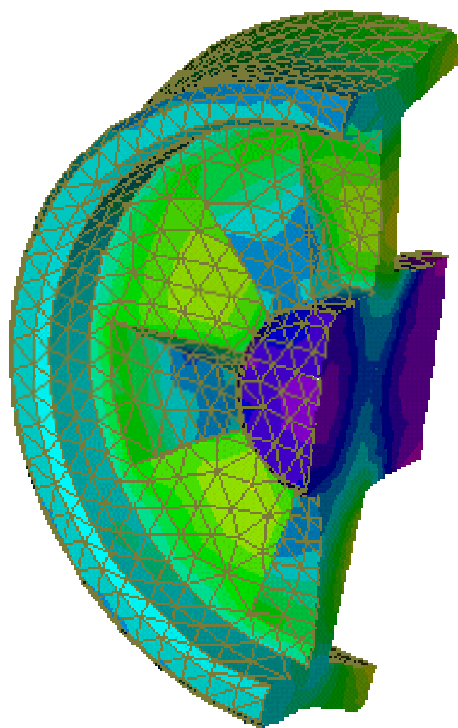
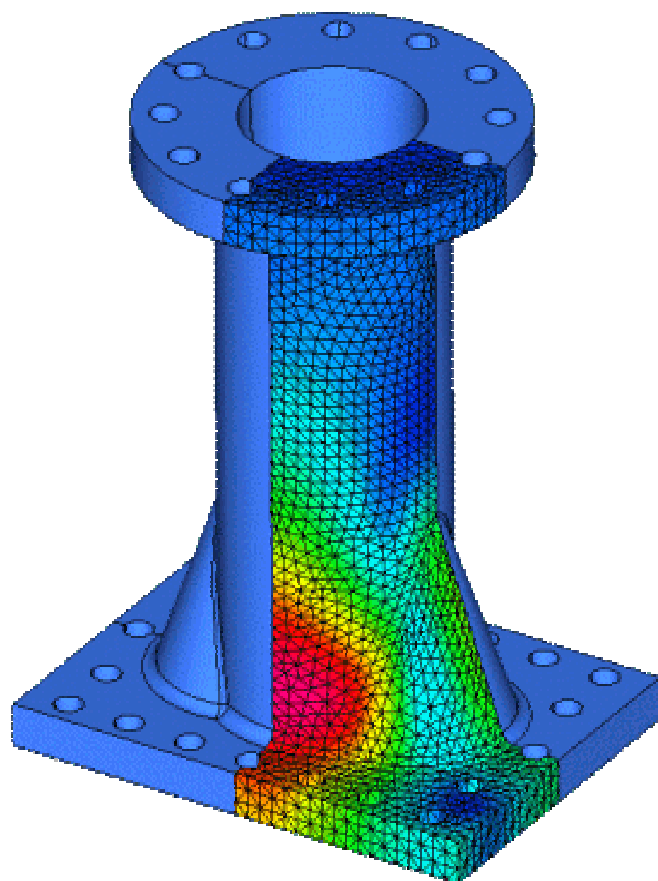
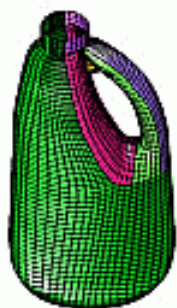
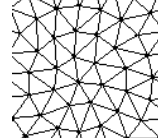


Finite Elements - Examples



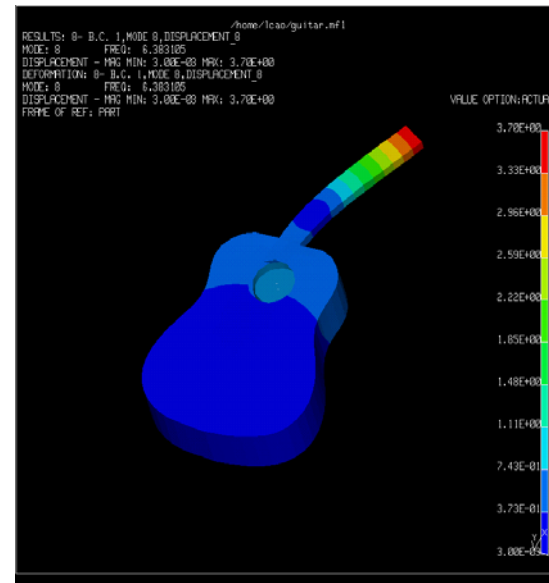
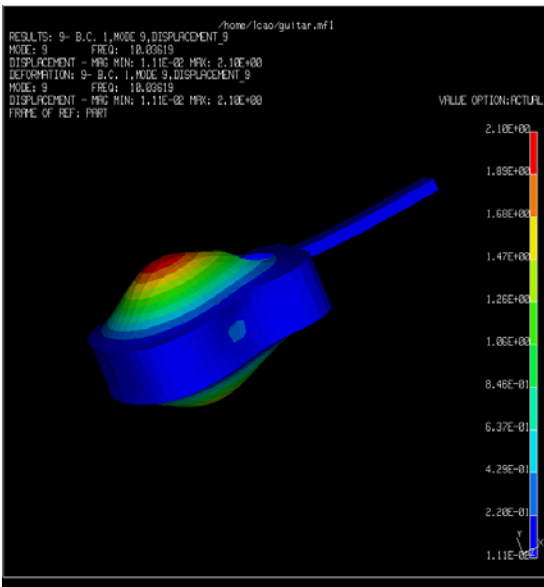
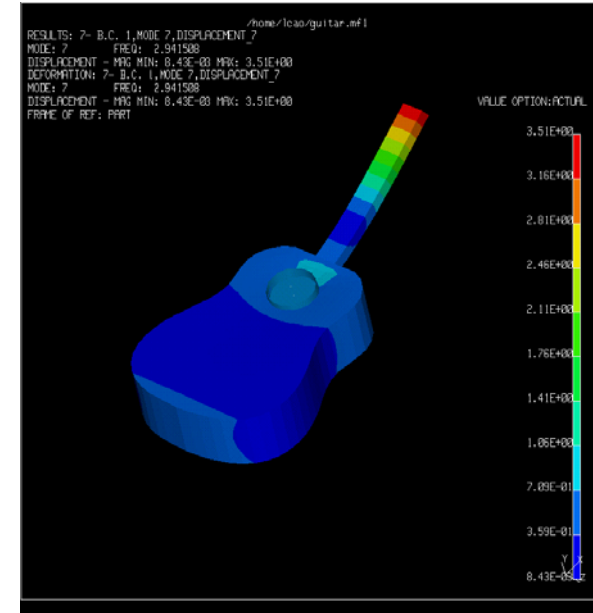
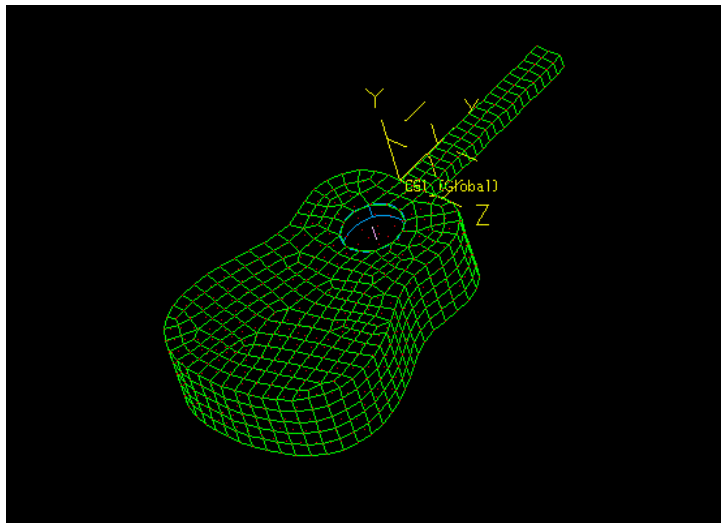


Finite Elements - Examples



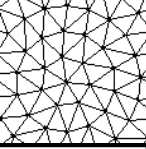


Finite Elements - Examples





Matrix Algebra - Linear Systems



Linear system of algebraic equations

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n} = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n} = b_2$$

.....

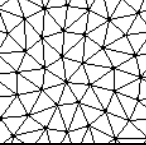
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn} = b_n$$

... where the $x_1, x_2, \dots, x_n$ are the unknowns ...
in matrix form

$$\mathbf{Ax} = \mathbf{b}$$



Matrix Algebra - Linear Systems



$$\mathbf{Ax} = \mathbf{b}$$

where

$$\mathbf{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{11} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

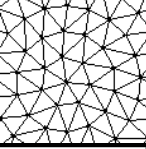
$$\mathbf{x} = \{x_i\} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$$

$$\mathbf{b} = \{b_i\} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix}$$

$\mathbf{A}$ is a $n \times n$ (square) matrix,
and $\mathbf{x}$ and $\mathbf{b}$ are column
vectors of dimension n



Matrix Algebra - Vectors



Row vectors

$$\mathbf{v} = [v_1 \quad v_2 \quad v_3]$$

Column vectors

$$\mathbf{w} = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}$$

Matrix addition and subtraction

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

with

$$c_{ij} = a_{ij} + b_{ij}$$

$$\mathbf{D} = \mathbf{A} - \mathbf{B}$$

with

$$d_{ij} = a_{ij} - b_{ij}$$

Matrix multiplication

$$\mathbf{C} = \mathbf{A}\mathbf{B}$$

with

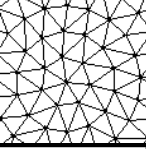
$$c_{ij} = \sum_{k=1}^m a_{ik} b_{kj}$$

where $\mathbf{A}$ (size $l \times m$) and $\mathbf{B}$ (size $m \times n$) and $i=1,2,\dots,l$ and $j=1,2,\dots,n$.

Note that in general $\mathbf{AB} \neq \mathbf{BA}$ but $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$



Matrix Algebra - Special



Transpose of a matrix

$$\mathbf{A} = [a_{ij}] \quad \mathbf{A}^T = [a_{ji}]$$

$$(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$$

Symmetric matrix

$$\mathbf{A} = \mathbf{A}^T$$

$$a_{ij} = a_{ji}$$

Identity matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$$\text{with } \mathbf{AI} = \mathbf{A}, \mathbf{Ix} = \mathbf{x}$$



Matrix Algebra - Determinants



The determinant of a square matrix A is a scalar number denoted $\det A$ or $|A|$, for example

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

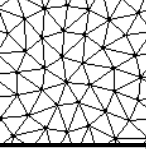
or

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$



Matrix Algebra - Inversion



A square matrix is singular if $\det \mathbf{A} = 0$. This usually indicates problems with the system (non-uniqueness, linear dependence, degeneracy ..)

Matrix Inversion

For a square and non-singular matrix $\mathbf{A}$ its inverse is defined such as

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

The **cofactor matrix** $\mathbf{C}$ of matrix $\mathbf{A}$ is given by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

where M_{ij} is the determinant of the matrix obtained by eliminating the i -th row and the j -th column of $\mathbf{A}$.

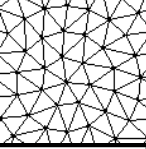
The inverse of $\mathbf{A}$ is then given by

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \mathbf{C}^T$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$



Matrix Algebra - Solution techniques



... the solution to a linear system of equations is given by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

The main task in solving a linear system of equations is finding the inverse of the coefficient matrix $\mathbf{A}$.

Solution techniques are e.g.

Gauss elimination methods
Iterative methods

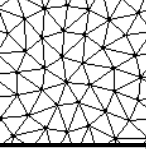
A square matrix is said to be **positive definite** if for any non-zero vector $\mathbf{x}$

$$\mathbf{x}^T \mathbf{A} \mathbf{x} > 0$$

... positive definite matrices are non-singular



Matrices - Differentiation and Integration



Let

$$\mathbf{A}(t) = [a_{ij}(t)]$$

Then the differentiation of this matrix w.r.t. time is

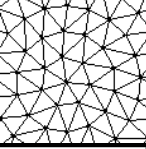
$$\frac{d}{dt} \mathbf{A}(t) = \left[\frac{da_{ij}(t)}{dt} \right]$$

Likewise integration is defined by

$$\int \mathbf{A}(t) dt = \left[\int a_{ij}(t) dt \right]$$



Finite elements - elastostatics



The **finite element method** was originally derived for **static problems in elasticity**. It is informative to follow this historic route to introduce the basic concepts of **elements**, **stiffness matrix**, etc.

To introduce this concept we only need:

Hooke's law: $F = Ds$

The principle of **force balance** $F_{\text{tot}} = F_1 + F_2 + F_3 + F_4 + \dots$

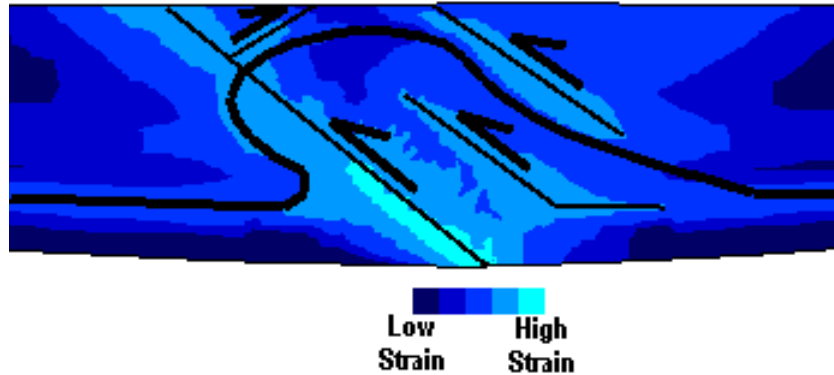
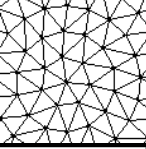
The concept of **work** $W = \int F ds$

and **strain energy** $W = \frac{1}{2} \int \epsilon_{ij} c_{ijkl} \epsilon_{kl}$

... and a blackboard



Crustal Deformation



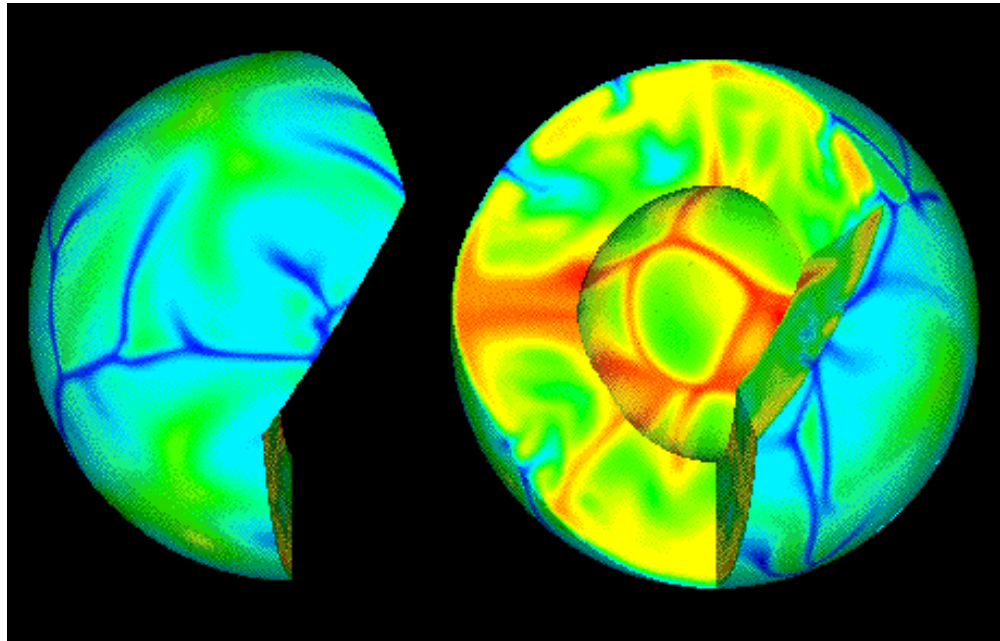
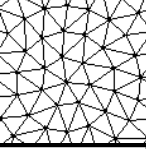
A piece of continental crust is forced to shorten as a velocity discontinuity is imposed along its base simulating subduction of the underlying mantle; in this example, the continental crust is assumed to be strongly rheologically layered leading to the formation of sub-horizontal decollements at mid-crustal levels and above the Moho. ...

From: J. Braun, Canberra

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Mantle Convection

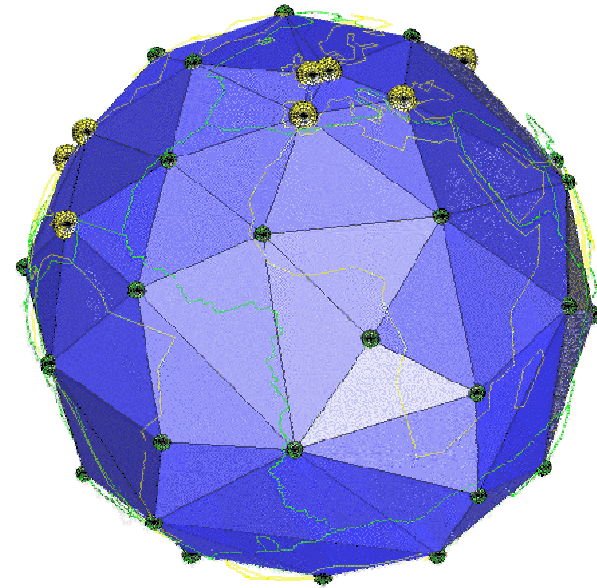
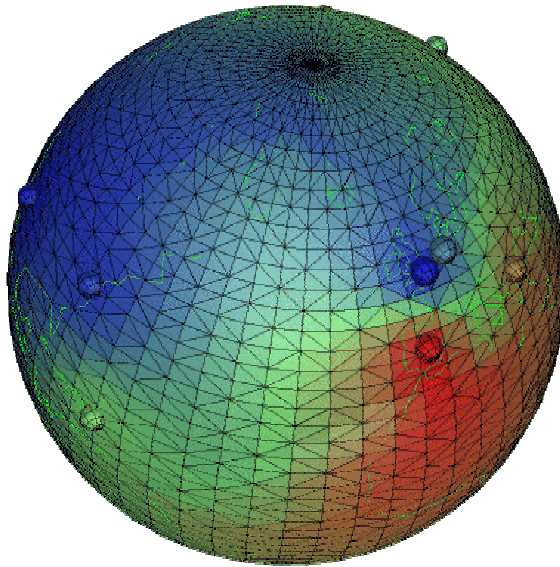
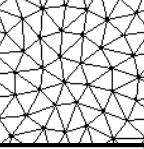


Thermal convection is modelled in 3-D a finite-element technique. The mesh has about 10 million grid points.
From Peter Bunge, Princeton.

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Electromagnetics



Gridding of a spherical surface used in FE modelling of the Earth's magnetic field. (From A. Schultz, Cambridge)

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