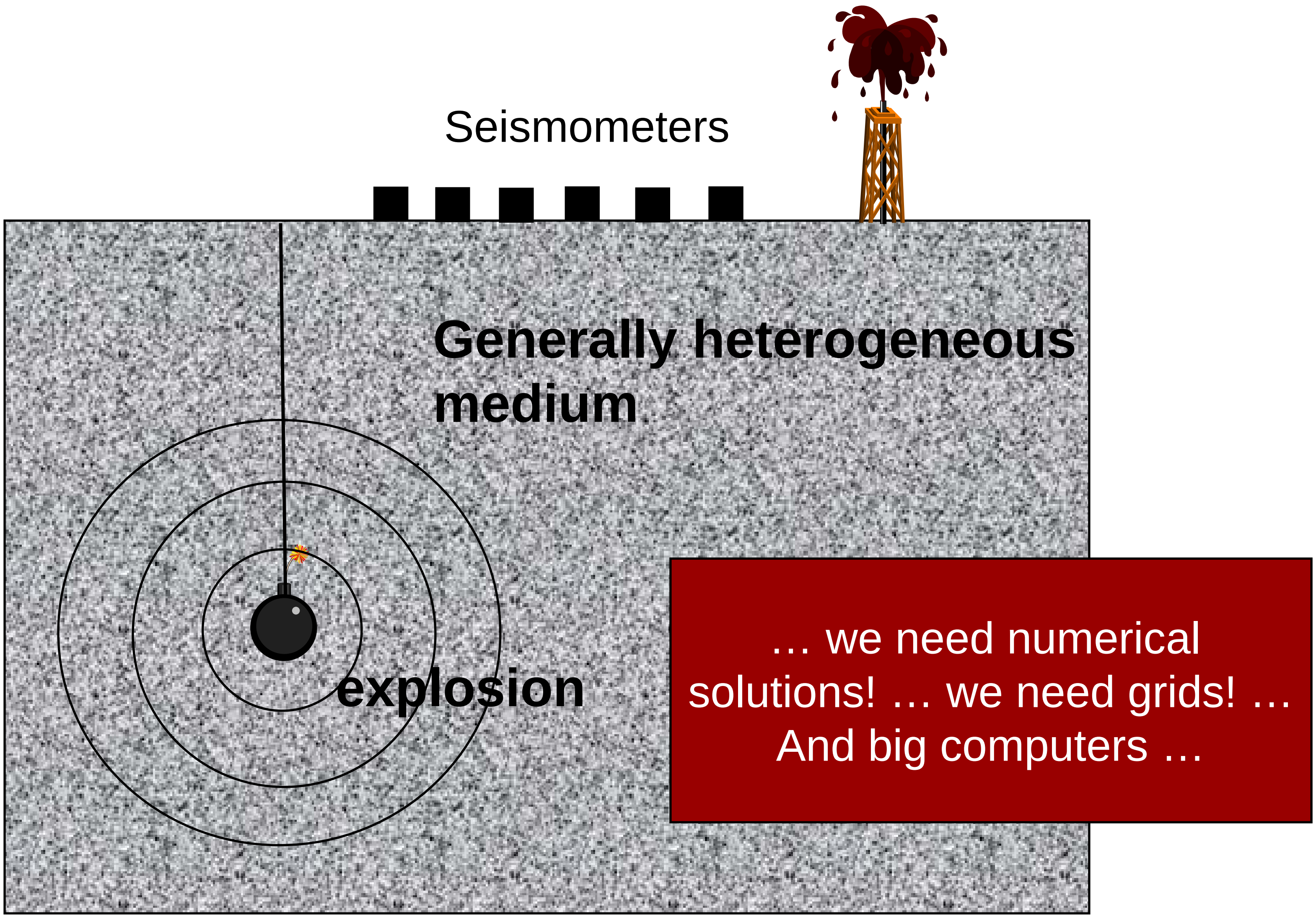
Generating synthetic seismograms

- Short introduction
- Finite-differences
- Solutions of the acoustic wave equation
  - difference equations
- Stability and numerical dispersion
- The Fourier method

Scope: Understand how to calculate synthetics using finite differences, we will later analyse the data (time series) in the spectral domain

Why numerical methods

Example: seismic wave propagation



Partial Differential Equations in Geophysics

$$\partial_t^2 p = c^2 \Delta p + s$$

$$\Delta = (\partial_x^2 + \partial_y^2 + \partial_z^2)$$

P	pressure
c	acoustic wave speed
s	sources

The acoustic wave equation

- seismology
- acoustics
- oceanography
- meteorology

$$\partial_t C = k\Delta C - \mathbf{v} \bullet \nabla C - RC + p$$

C	tracer concentration
k	diffusivity
v	flow velocity
R	reactivity
p	sources

Diffusion, advection, Reaction

- geodynamics
- oceanography
- meteorology
- geochemistry
- sedimentology
- geophysical fluid dynamics

Numerical methods: properties

Finite differences →

- time-dependent PDEs
- seismic wave propagation
- geophysical fluid dynamics
- Maxwell's equations
- Ground penetrating radar

-> robust, simple concept, easy to parallelize, regular grids, explicit method

Finite elements →

- static and time-dependent PDEs
- seismic wave propagation
- geophysical fluid dynamics
- all problems

-> implicit approach, matrix inversion, well founded, irregular grids, more complex algorithms, engineering problems

Finite volumes →

- time-dependent PDEs
- seismic wave propagation
- mainly fluid dynamics

-> robust, simple concept, irregular grids, explicit method

Other Numerical methods:

Particle-based methods → - lattice gas methods
- molecular dynamics
- granular problems
- fluid flow
- earthquake simulations

-> very heterogeneous problems, nonlinear problems

Boundary element methods → - problems with boundaries (rupture)
- based on analytical solutions
- only discretization of planes

--> good for problems with special boundary conditions (rupture, cracks, etc)

*Pseudo*spectral methods → - orthogonal basis functions, special case of FD
- spectral accuracy of space derivatives
- wave propagation, GPR

-> regular grids, explicit method, problems with strongly heterogeneous media

What is a finite difference?

Common definitions of the derivative of f(x):

$$\partial_x f = \lim_{dx \to 0} \frac{f(x+dx) - f(x)}{dx}$$

$$\partial_x f = \lim_{dx \to 0} \frac{f(x) - f(x-dx)}{dx}$$

$$\partial_x f = \lim_{dx \to 0} \frac{f(x+dx) - f(x-dx)}{2dx}$$

These are all correct definitions in the limit dx->0.

But we want dx to remain **FINITE**

What is a finite difference?

The equivalent ***approximations*** of the derivatives are:

$$\partial_x f \approx \frac{f(x+dx)-f(x)}{dx}$$ forward difference

$$\partial_x f \approx \frac{f(x)-f(x-dx)}{dx}$$ backward difference

$$\partial_x f \approx \frac{f(x+dx)-f(x-dx)}{2dx}$$ centered difference

What about the second or higher derivatives?

Higher Derivatives with FD

$$\partial_x f^+ \approx \frac{f(x+dx) - f(x)}{dx}$$

$$\partial_x f^- \approx \frac{f(x) - f(x-dx)}{dx}$$

$$\partial_x^2 f \approx \frac{\partial_x f^+ - \partial_x f^-}{dx}$$

$$\boxed{\partial_x^2 f \approx \frac{f(x+dx) - 2f(x) + f(x-dx)}{dx^2}}$$

Second Derivative

Other derivation via Taylor Series (Exercise).

How good are the FD approximations?

$$\approx \neq =$$

This leads us to Taylor series....

Taylor Series

Taylor series are expansions of a function f(x) for some finite distance dx to f(x+dx)

$$f(x \pm dx) = f(x) \pm \mathrm{dx} f^{'}(x) + \frac{dx^2}{2!} f^{''}(x) \pm \frac{dx^3}{3!} f^{'''}(x) + \frac{dx^4}{4!} f^{''''}(x) \pm \ldots$$

What happens, if we use this expression for

$$\partial_x f^{+} \approx \frac{f(x+dx) - f(x)}{dx}$$

?

Taylor Series

... that leads to :

$$\frac{f(x+dx)-f(x)}{dx}=\frac{1}{dx}\left[\mathrm{dx}f^{'}(x)+\frac{dx^2}{2!}f^{''}(x)+\frac{dx^3}{3!}f^{'''}(x)+\ldots\right]$$

$$=f^{'}(x)+O(dx)$$

The error of the first derivative using the *forward* formulation is *of order dx*.

Is this the case for other formulations of the derivative? Let’s check!

Taylor Series

... with the *centered* formulation we get:

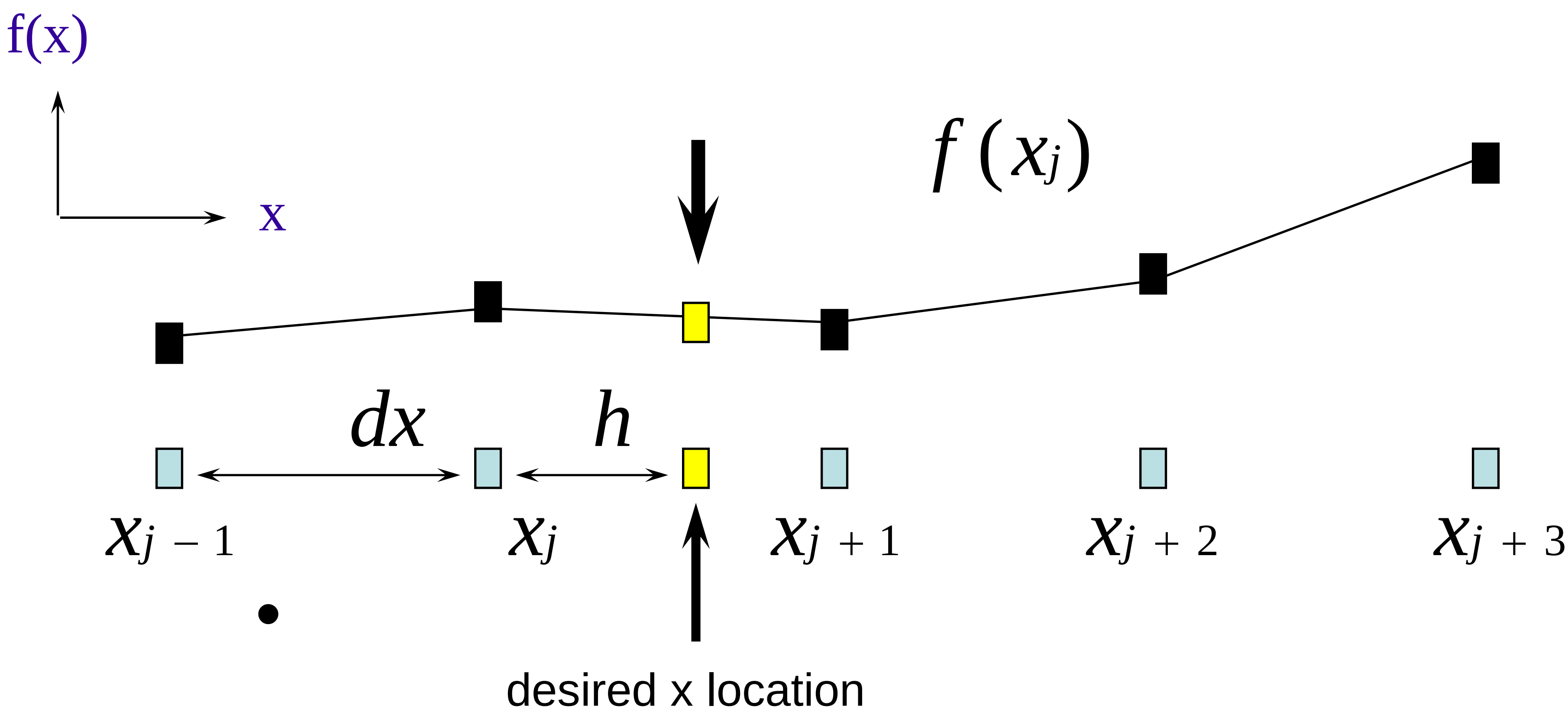
$$\frac{f(x+dx/2)-f(x-dx/2)}{dx}=\frac{1}{dx}\left[\mathrm{dx}f^{'}(x)+\frac{dx^{3}}{3!}f^{'''}(x)+\ldots\right]$$

$$=f^{'}(x)+O(dx^{2})$$

The error of the first derivative using the centered approximation is *of order* dx^2.

This is an **important** results: it DOES matter which formulation we use. The centered scheme is more accurate!

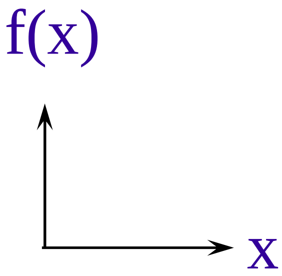
Alternative Derivation of FD



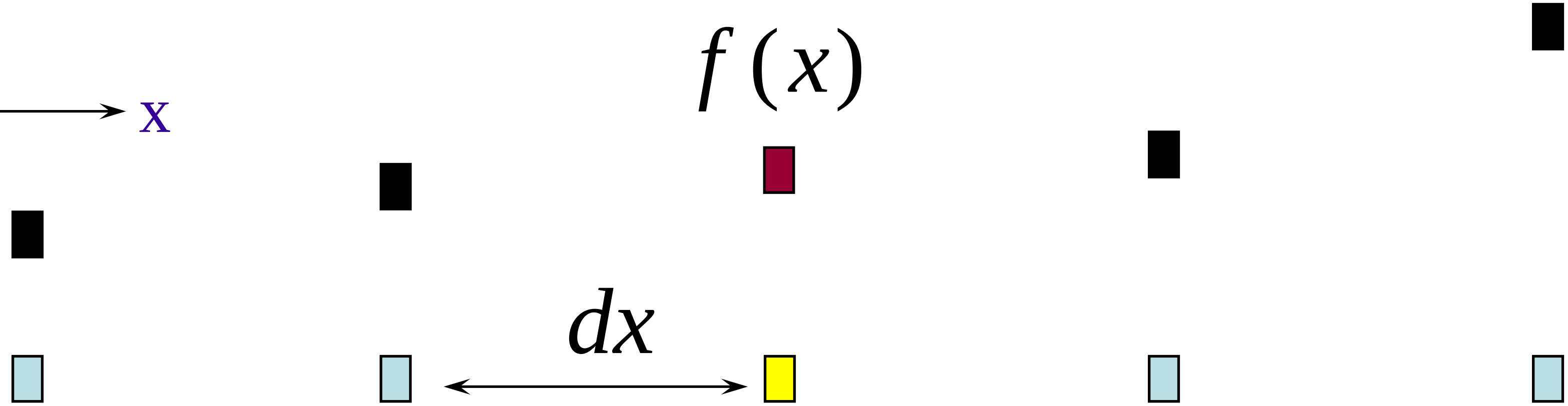
What is the (approximate) value of the function or its (first, second ..) derivative at the desired location ?

How can we calculate the weights for the neighboring points?

Alternative Derivation of FD



Lets' try Taylor's Expansion



$$f(x+dx) = f(x) + f'(x)dx \qquad (1)$$

$$f(x-dx) = f(x) - f'(x)dx \qquad (2)$$

we are looking for something like

$$f^{(i)}(x) \approx \sum_{j=1,L} w_j^{(i)} f(x_{index(j)})$$

Alternative Derivation of FD

$$af^{+} \approx af + af'dx \quad + \quad bf^{-} \approx bf - bf'dx$$

$$\Rightarrow af^{+} + bf^{-} \approx (a+b)f + (a-b)f'dx$$

Interpolation	Derivative
$\boxed{a - b = 0}$	$\boxed{a + b = 0}$
$\Downarrow$	$\Downarrow$
$f \approx \frac{1}{2} f^{-} + \frac{1}{2} f^{+}$	$f' \approx \frac{f^{+} - f^{-}}{2dx}$
$w_1 = 0.5, w_2 = 0.5$	$w_1 = -\frac{1}{2dx}, w_2 = \frac{1}{2dx}$
Interpolation weights	Derivative weights

Our first FD algorithm (ac1d.m) !

$$\partial_t^2 p = c^2 \Delta p + s$$

$$\Delta = (\partial_x^2 + \partial_y^2 + \partial_z^2)$$

P	pressure
c	acoustic wave speed
s	sources

Problem: Solve the 1D acoustic wave equation using the finite Difference method.

Solution:

$$p(t+dt) = \frac{c^2 dt^2}{dx^2}\left[p(x+dx) - 2p(x) + p(x-dx)\right] + 2p(t) - p(t-dt) + sdt^2$$

Problems: Stability

$$p(t+dt) = \frac{c^2 dt^2}{dx^2}\left[p(x+dx) - 2p(x) + p(x-dx)\right] + 2p(t) - p(t-dt) + sdt^2$$

Stability: Careful analysis using harmonic functions shows that a stable numerical calculation is subject to special conditions (conditional stability). This holds for many numerical problems.

$$c\frac{dt}{dx} \le \varepsilon \approx 1$$

The 1-D wave equation

$$\rho(x)\partial_t^2 u(x,t) = \partial_x\left[E(x)\partial_x u(x,t)\right]$$

Elastic parameters E(x) vary only in one direction.

$$E(x) = \mu(x)$$ shear waves

$$E(x) = \lambda(x) + 2\mu(x)$$ P waves

with the corresponding velocities

$$v_S = \sqrt{\frac{\mu}{\rho}}$$ shear waves

$$v_P = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$ P waves

The 1-D wave equation

We want to avoid having to take derivatives of the material parameters (why?). This can be achieved by using a *velocity-stress* formulation, which leads to the following simultaneous equations:

$$\boxed{\begin{aligned} \partial_t \dot{u} &= \frac{1}{\rho(x)} \partial_x \tau \\ \partial_t \tau &= E(x) \partial_x \dot{u} \end{aligned}}$$

where

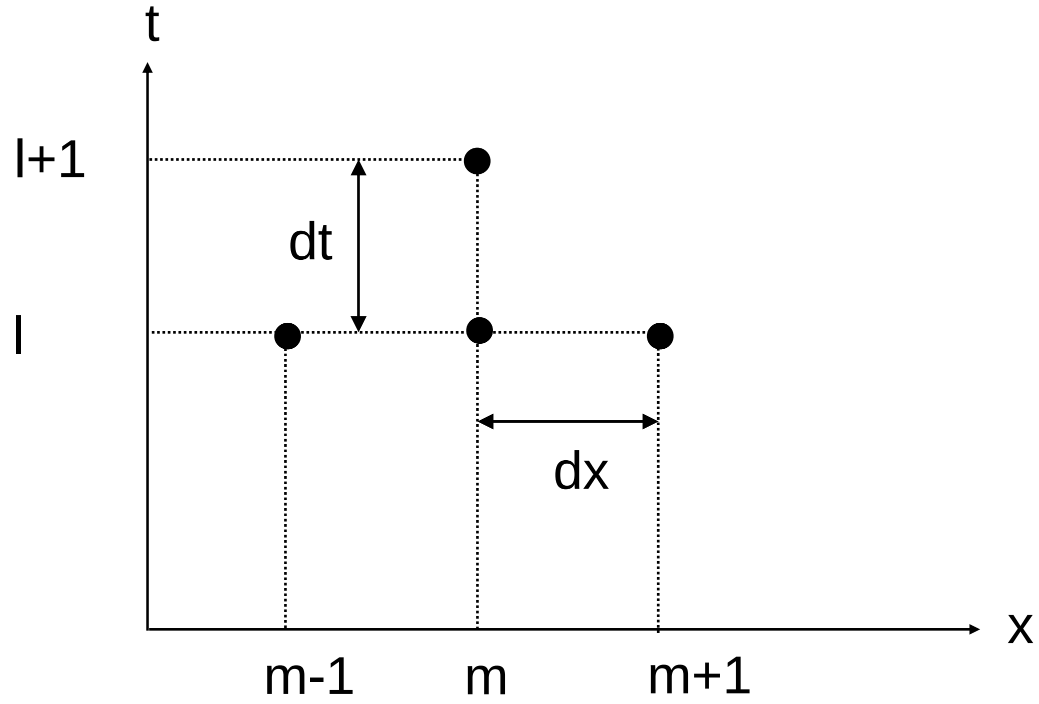
$$\tau = E(x) \partial_x u \qquad \text{stress}$$

The 1-D wave equation - FD scheme

Let us try to use one of the previously introduced FD schemes: central difference for space and forward difference for time

Discretization: $(l\mathrm{dt}, m\mathrm{dx})$

dx space increment, dt time increment



The 1-D wave equation - FD scheme

... leading to the following scheme:

forward

$$\frac{\dot{u}_m^{l+1} - \dot{u}_m^{l}}{dt} = \frac{1}{\rho_m} \frac{\tau_{m+1}^{l} - \tau_{m-1}^{l}}{2dx}$$

centered

forward

$$\frac{\tau_m^{l+1} - \tau_m^{l}}{dt} = E_m \frac{\dot{u}_{m+1}^{l} - \dot{u}_{m-1}^{l}}{2dx}$$

centered

like in the continuous case, we can make the following *Ansatz:*

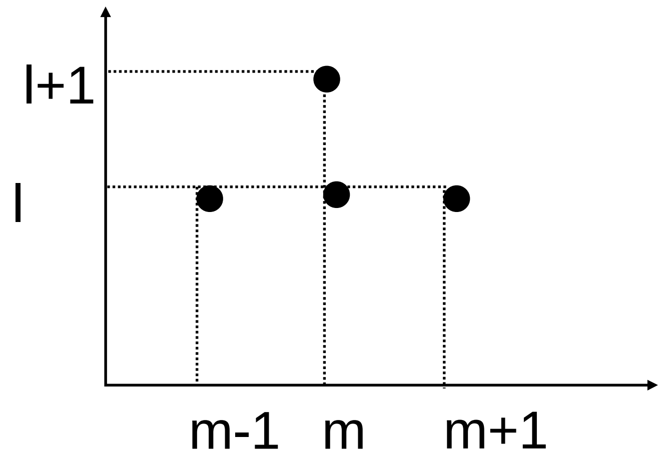
$$f(x,t) = A\exp(ikx - iwt)$$

which in the discrete world is :

$$f_{lm} = A\exp(ikm\mathrm{dx} - iwl\mathrm{dt})$$

The 1-D wave equation - FD scheme

... in practical terms: first solve



$$\dot{u}_m^{l+1} = dt\left[\frac{1}{\rho_m}\frac{\tau_{m+1}^l - \tau_{m-1}^l}{2dx}\right] + \dot{u}_m^l$$

then solve

$$\tau_m^{l+1} = dt\left[E_m\frac{\dot{u}_{m+1}^l - \dot{u}_{m-1}^l}{2dx}\right] + \tau_m^l$$

The 1-D wave equation - FD scheme

... let us assume a signal is propagating:

$$f(\tau_m^l) = A\exp(ikm\mathrm{dx} - iwl\mathrm{dt})$$

$$f(\dot{u}_m^l) = B\exp(ikm\mathrm{dx} - iwl\mathrm{dt})$$

we now put this *Ansatz* into the following equations ...

$$\frac{\dot{u}_m^{l+1} - \dot{u}_m^l}{dt} = \frac{1}{\rho_m}\frac{\tau_{m+1}^l - \tau_{m-1}^l}{2dx}$$

$$\frac{\tau_m^{l+1} - \tau_m^l}{dt} = E_m\frac{\dot{u}_{m+1}^l - \dot{u}_{m-1}^l}{2dx}$$

The 1-D wave equation - FD scheme

...after some algebra (hours later) ...

$$\boxed{\exp(-iwdt) = 1 \pm i\sqrt{\frac{E_m}{\rho_m}}\left(\frac{dt}{dx}\right)\sin kdx}$$

What does this result tell us about the numerical solution?

$$\left|\exp(-iwdt)\right| > 1$$

for any choice of dt and dx! So ω must be complex.
But then for example:

$$f(\tau_m^l) = A\exp(ikm\mathrm{dx} - iwl\mathrm{dt}) = A\exp(ikm)\exp(-w^* ldt)$$

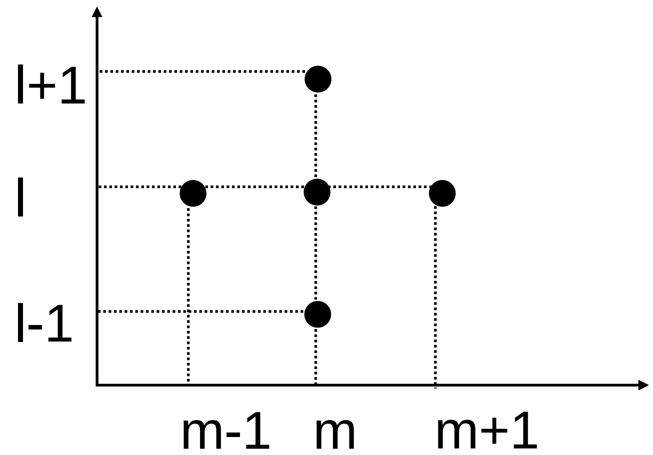
will grow exponentially as, ω* is real.

The 1-D wave equation - FD scheme

Can we find a scheme that works?
Let us use a centered scheme in time:

$$\frac{\dot{u}_m^{l+1} - \dot{u}_m^{l-1}}{2dt} = \frac{1}{\rho_m} \frac{\tau_{m+1}^{l} - \tau_{m-1}^{l}}{2dx}$$

$$\frac{\tau_m^{l+1} - \tau_m^{l-1}}{2dt} = E_m \frac{\dot{u}_{m+1}^{l} - \dot{u}_{m-1}^{l}}{2dx}$$



And again we use the following Ansatz to investigate the behavior of the numerical solution:

$$f(\tau_m^l) = A\exp(ikm\mathrm{dx} - iwl\mathrm{dt})$$

$$f(\dot{u}_m^l) = B\exp(ikm\mathrm{dx} - iwl\mathrm{dt})$$

The 1-D wave equation - FD scheme

...again after some algebra (minutes later) ...

$$\boxed{\sin wdt = \pm\sqrt{\frac{E_m}{\rho_m}}\left(\frac{dt}{dx}\right)\sin kdx}$$

... has real solutions as long as

$$\sqrt{\frac{E_m}{\rho_m}}\left(\frac{dt}{dx}\right)\leq 1$$

... knowing that for example ...

$$\sqrt{\frac{E_m}{\rho_m}} = v_p$$ P-wave velocity

The 1-D wave equation - FD scheme

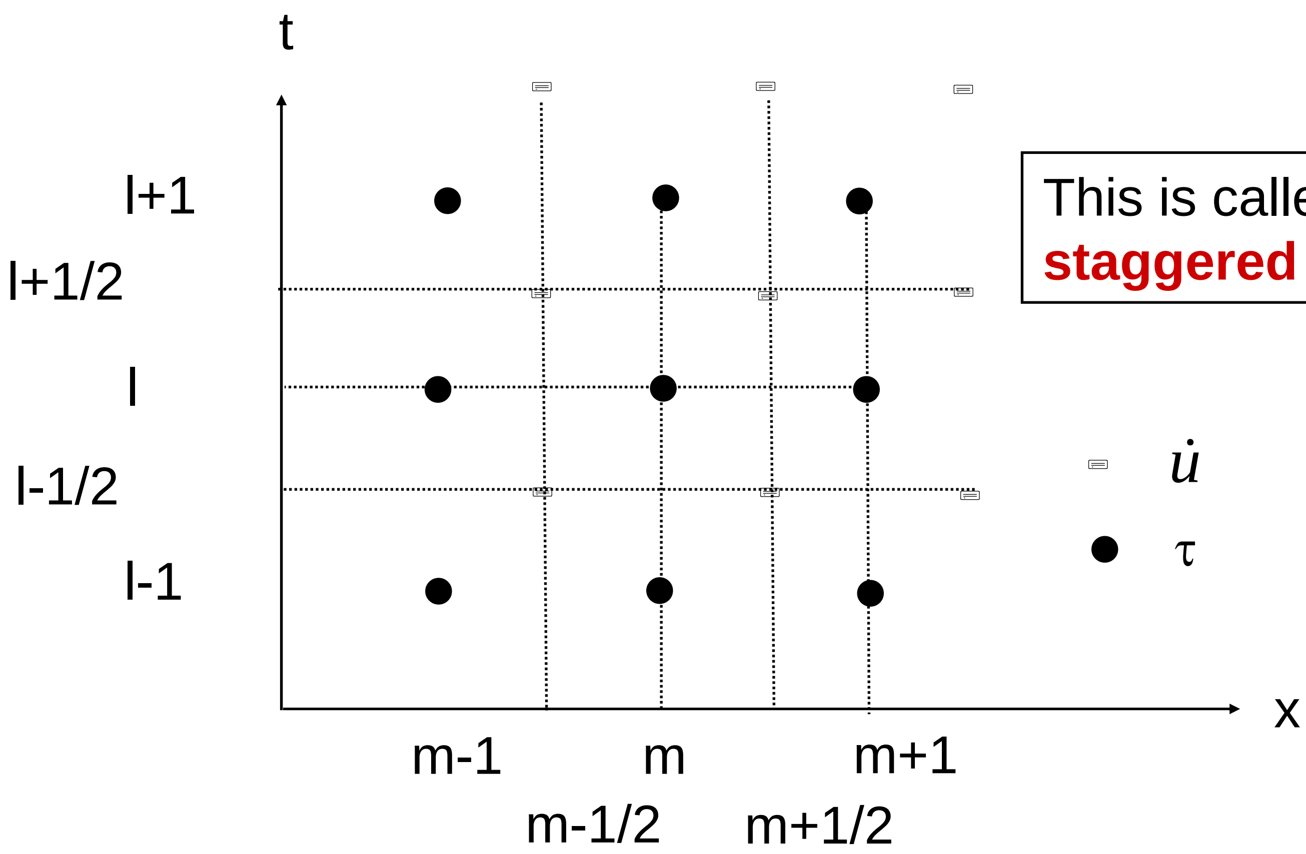
... we arrive at maybe the most important result for FD schemes applied to the wave equation:

$$\mathbf{v_{P,S}\left(\frac{dt}{dx}\right) \leq 1}$$

$v_{P,S}$ is the locally homogeneous velocity. This is called a *conditionally stable* finite-difference scheme. Finding the right combination of dt and dx for a practical application, where the velocities vary in the medium is one of the most important tasks.

The 1-D wave equation - FD scheme

There is an even better scheme!



This is called a **staggered** scheme

... leading to the FD scheme:

$$\frac{\dot{u}_m^{l+1/2} - \dot{u}_m^{l-1/2}}{dt} = \frac{1}{\rho_m} \frac{\tau_{m+1/2}^{l} - \tau_{m-1/2}^{l}}{dx}$$

$$\frac{\tau_{m+1/2}^{l+1} - \tau_{m+1/2}^{l}}{dt} = E_{m+1/2} \frac{\dot{u}_{m+1}^{l+1/2} - \dot{u}_m^{l+1/2}}{dx}$$

And again we use the following *Ansatz* to investigate the behaviour of the numerical solution:

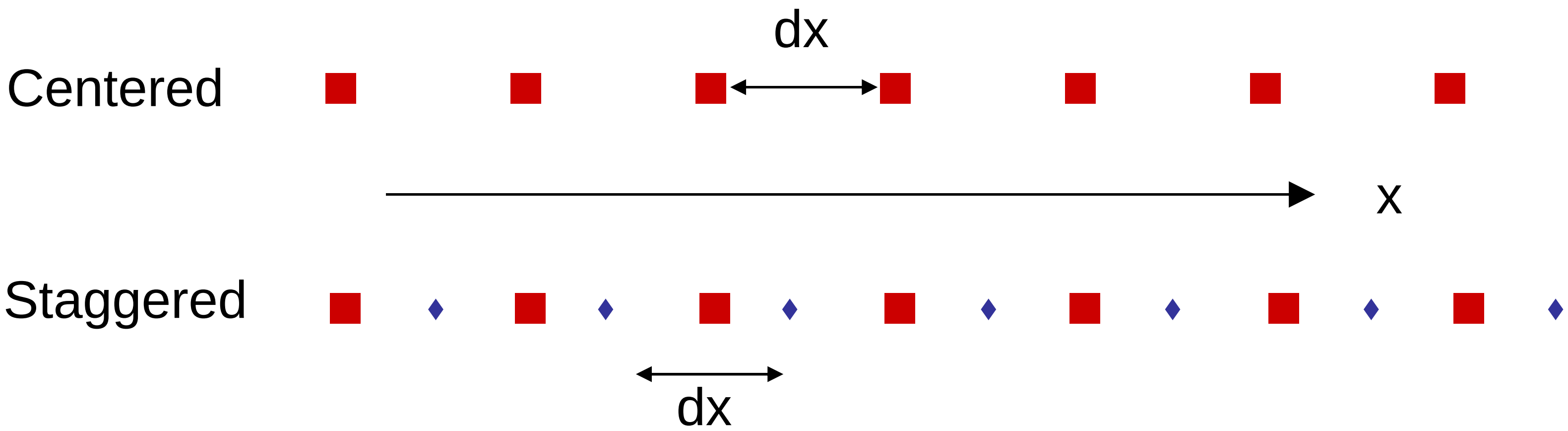
$$f(\tau_m^l) = A\exp(ikm\mathrm{dx} - iwl\mathrm{dt})$$

$$f(\dot{u}_m^l) = B\exp(ikm\mathrm{dx} - iwl\mathrm{dt})$$

Find the corresponding stability condition (Exercise)!

Staggered Grids

Which scheme is more accurate?



centered: $$\partial_x f \approx \frac{f(x+dx) - f(x-dx)}{2dx}$$

staggered: $$\partial_x f \approx \frac{f(x+dx/2) - f(x-dx/2)}{dx}$$

Because the error is O(h^2), the error of the centered scheme is 4 times larger.

Numerical Dispersion

What does the stability criterion tell us about the quality of the numerical solution?

$$\sin\frac{\omega dt}{2} = \pm\sqrt{\frac{E_{m+1/2}}{\rho_m}}\left(\frac{dt}{dx}\right)\sin\frac{kdx}{2}$$

To answer this we need the concept of ***phase velocity*.**

Remember we assumed a harmonic oscillation with frequency ω and wavenumber k, for example

$$y(x,t) = \sin(kx - \omega t) = \sin(k(x - \frac{\omega}{k}t)) = \sin(\omega(\frac{k}{\omega}x - t))$$

where the phase velocity is

$$c_{phase} = \frac{\omega}{k}$$

Numerical Dispersion

$$\sin\frac{\omega dt}{2} = \pm\sqrt{\frac{E_{m+1/2}}{\rho_m}}\left(\frac{dt}{dx}\right)\sin\frac{kdx}{2}$$

we can first assume that dt and dx are very small, in this case :

$$\sin(x) \approx x$$ for small x

then

$$\frac{\omega}{k} = \sqrt{\frac{E_{m+1/2}}{\rho_m}} = c$$ wave speed

for small dt and dx we simulate the correct velocity:
The scheme is **convergent**.

Numerical Dispersion

How about the general case?

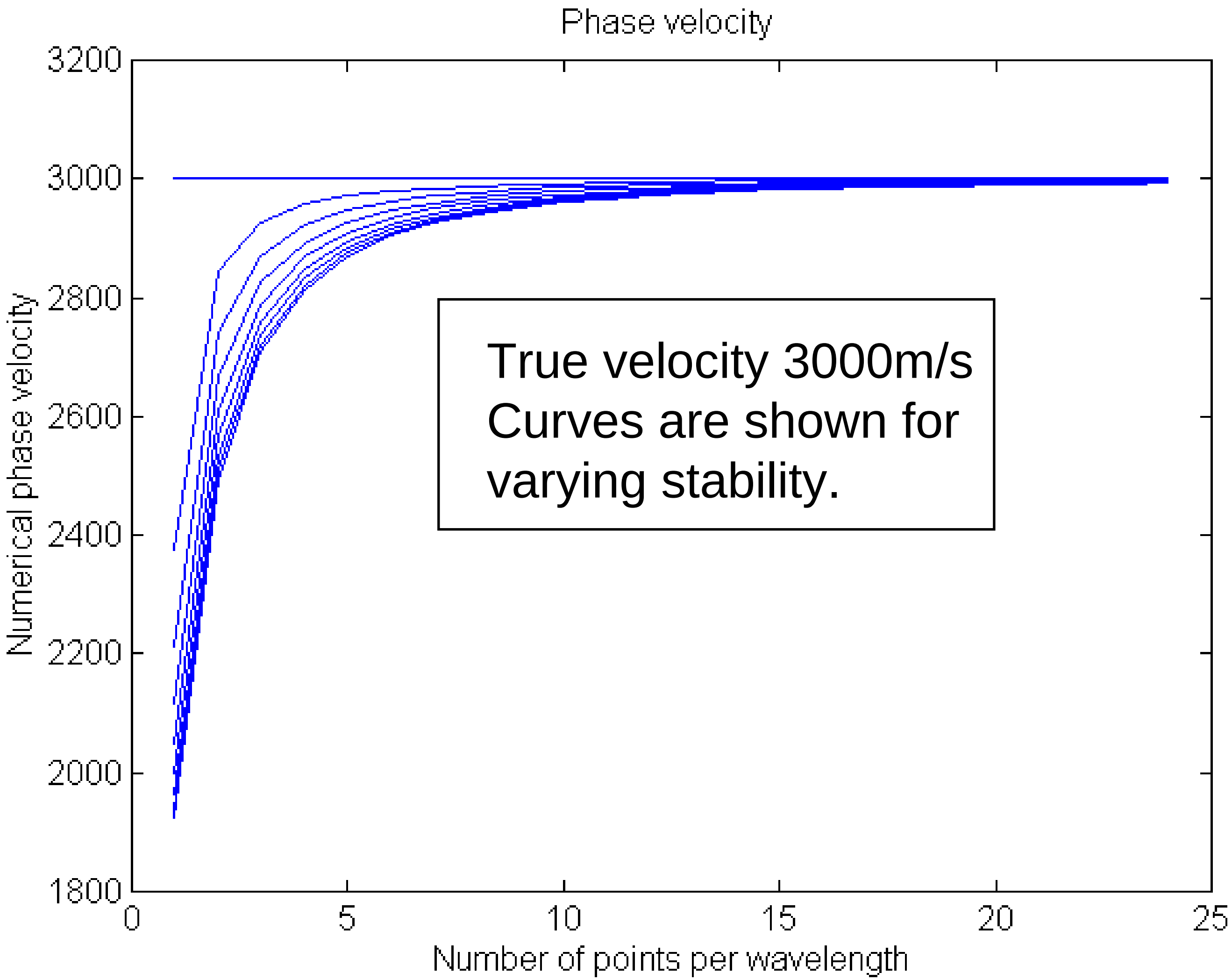
$$\sin\frac{\omega dt}{2} = \pm\sqrt{\frac{E_{m+1/2}}{\rho_m}}\left(\frac{dt}{dx}\right)\sin\frac{kdx}{2}$$

using $k = \dfrac{2\pi}{\lambda}$ we obtain

$$\boxed{c(\lambda) = \frac{\omega}{k} = \frac{\lambda}{\pi dt}\sin^{-1}\left(c_0\frac{dt}{dx}\sin\frac{\pi dx}{\lambda}\right)}$$

This formula expresses our *numerical* phase velocity as a function of the wave speed and the propagating wavelength.

Numerical Phase Velocity



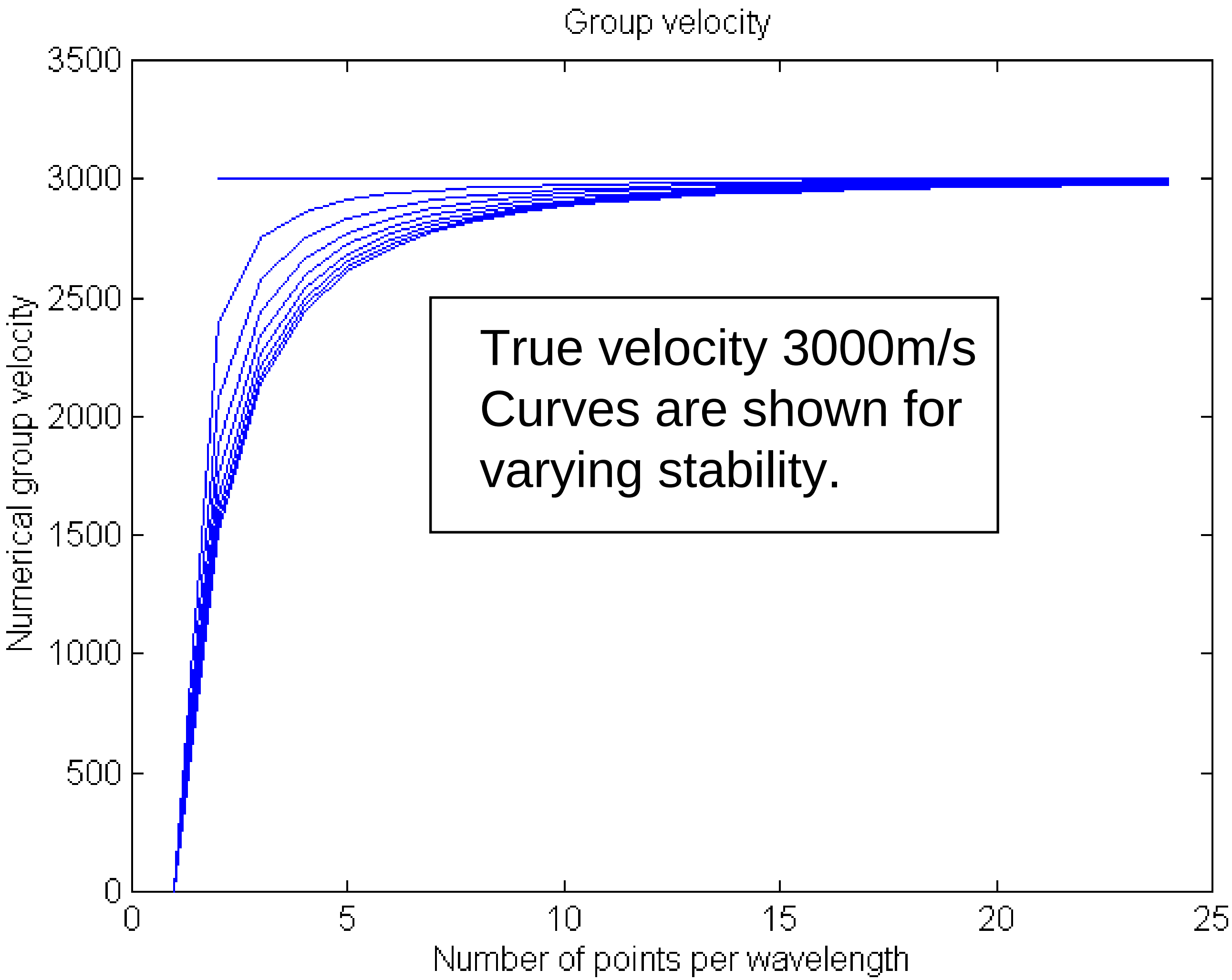
Numerical Dispersion

What we really measure in a seismogram is the group velocity:

$$\frac{\partial \omega}{\partial k} = \frac{c \cos \dfrac{\pi dx}{\lambda}}{\left[1 - \left(c \dfrac{dt}{dx} \sin \dfrac{\pi dx}{\lambda}\right)^2\right]^{1/2}}$$

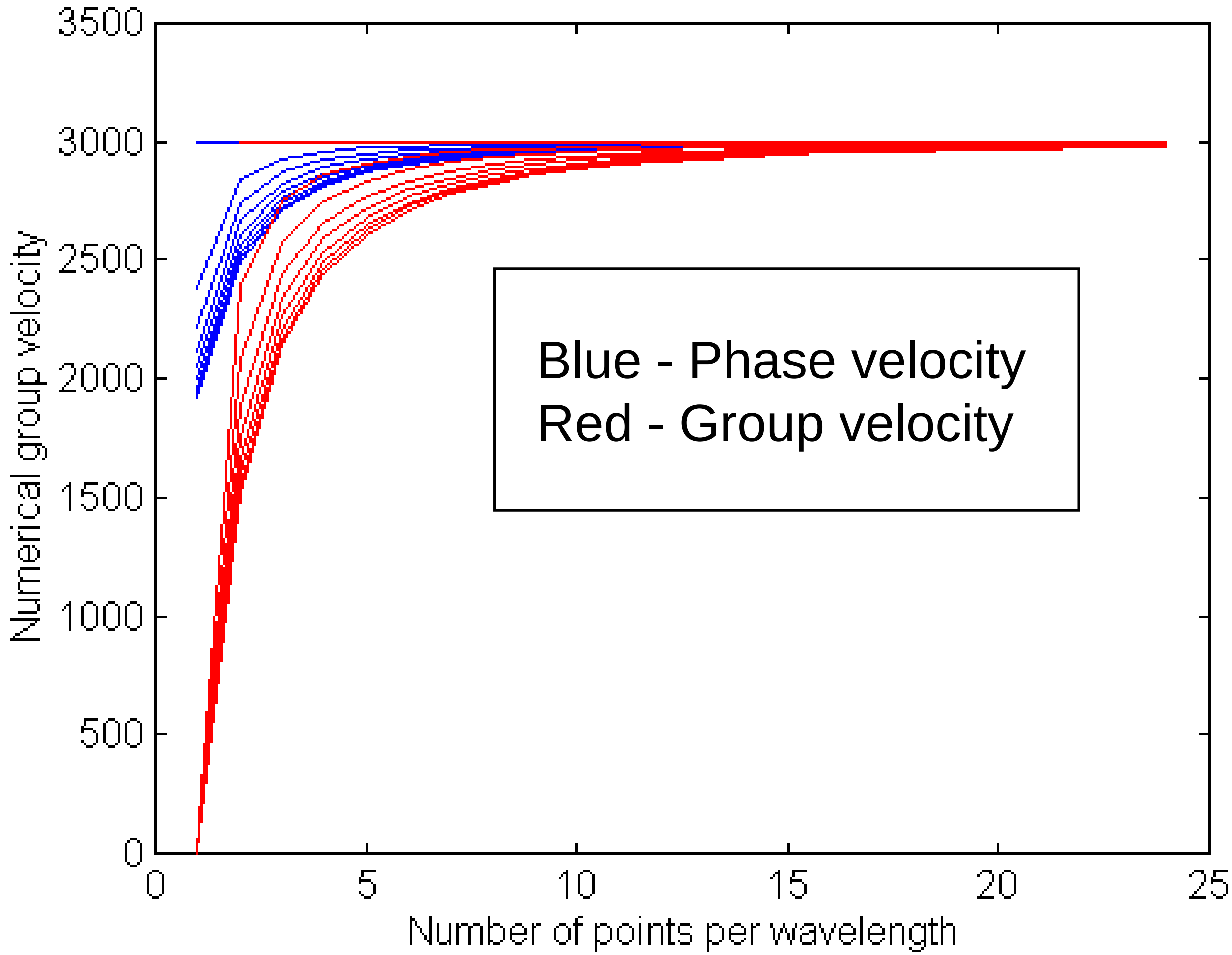
This formula expresses our *numerical* group velocity as a function of the wave speed and the propagating wavelength.

Numerical Group Velocity

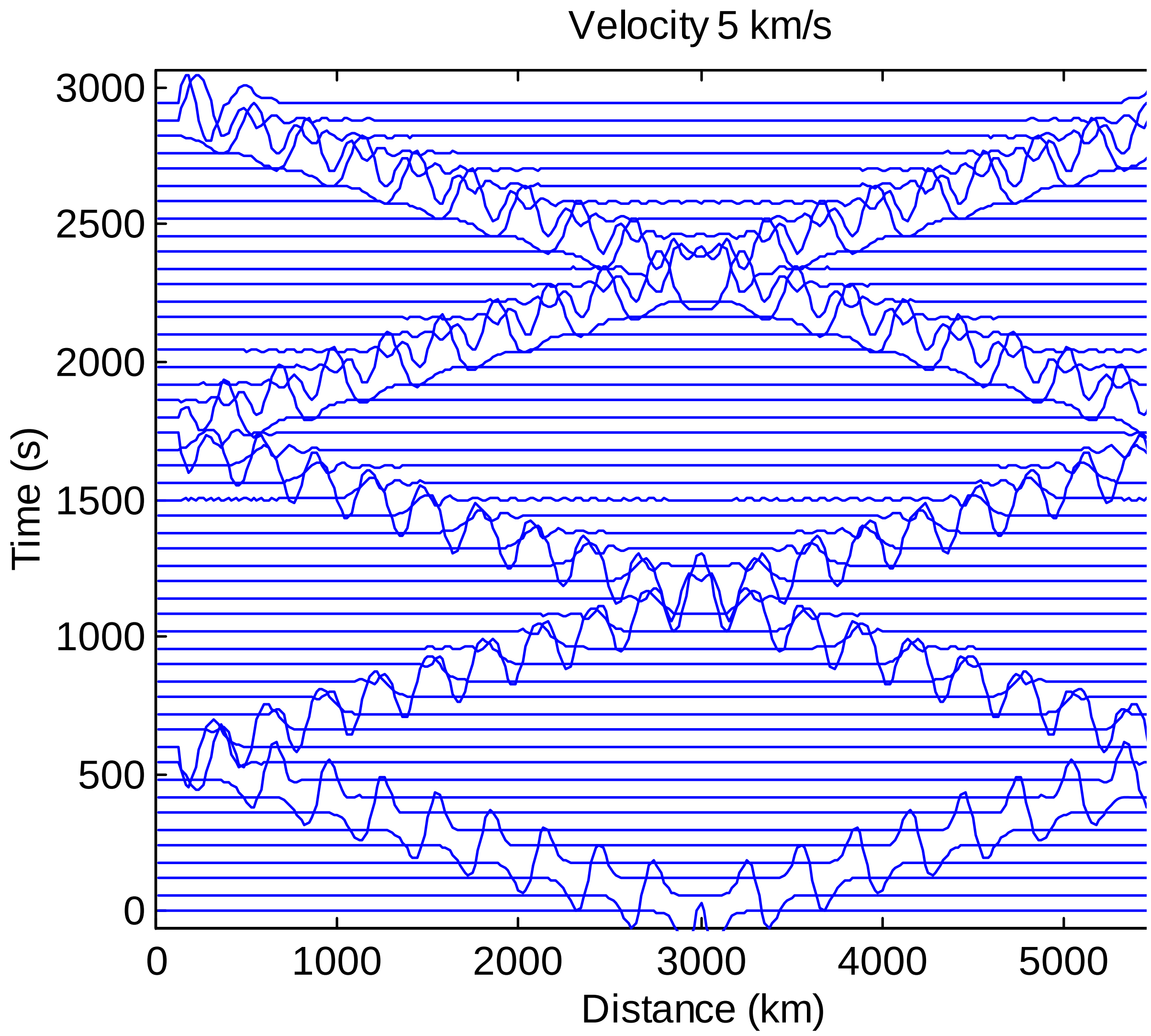


True velocity 3000m/s
Curves are shown for varying stability.

Numerical Dispersion

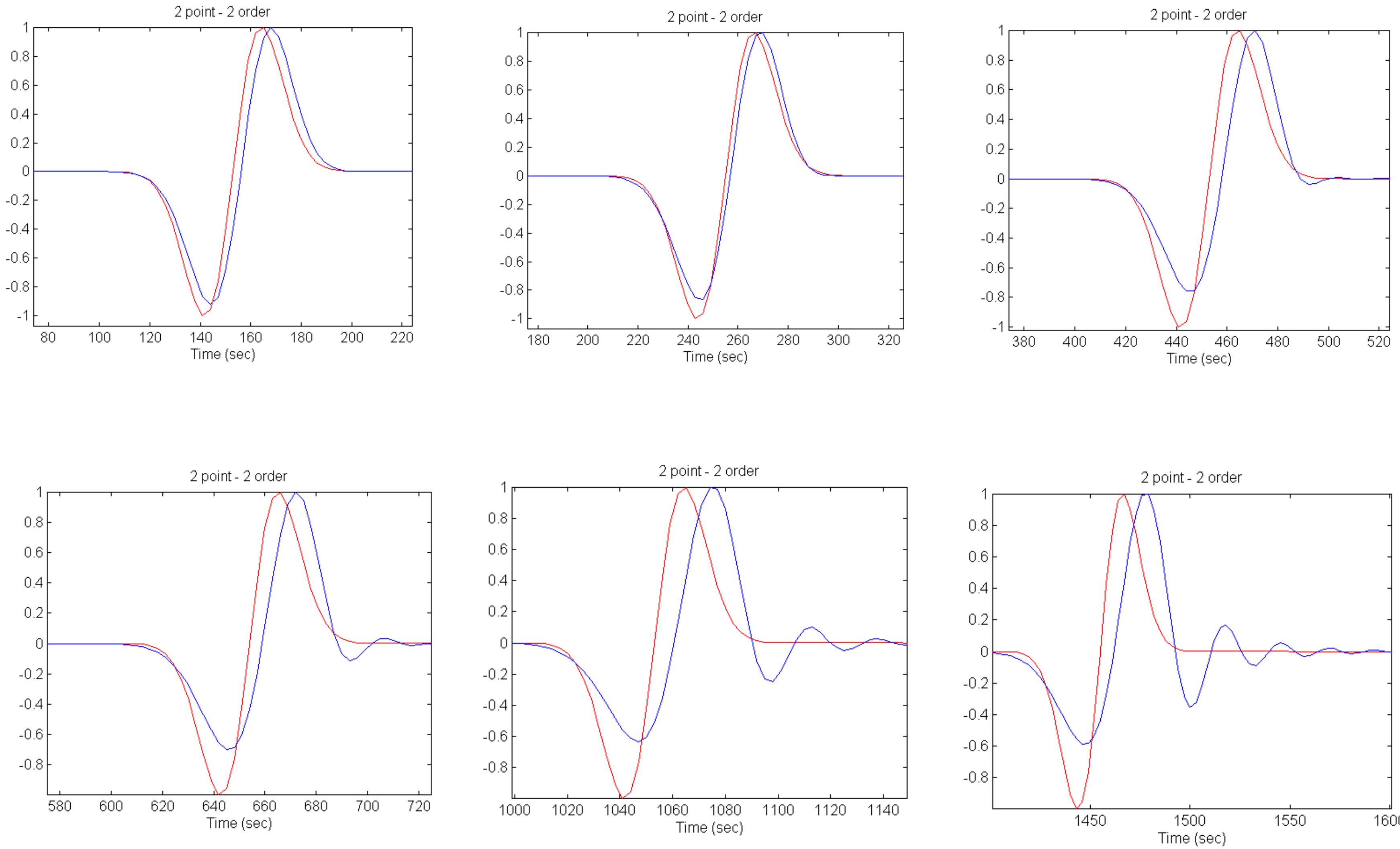


Example



Velocity 5 km/s
3000
2500
2000
1500
1000
500
0
Time (s)
0
1000
2000
3000
4000
5000
Distance (km)

Numerical Dispersion



FD Summary

Depending on the choice of the FD scheme (e.g. forward, backward, centered) a numerical solution may be more or less accurate.

Explicit finite difference solutions to differential equations are often *conditionally stable*. The correct choice of the space or time increment is crucial to enable accurate solutions.

Sometimes it is useful to employ so-called *staggered grids* where the fields are defined on seperate grids which may improve the overall accuracy of the scheme.

The Fourier Method

- What is a *pseudo*-spectral Method?

- Fourier Derivatives

- The Fast Fourier Transform (FFT)

- The Acoustic Wave Equation with the Fourier Method

- Comparison with the Finite-Difference Method

- Dispersion and Stability of Fourier Solutions

Pseudospectral methods

Spectral solutions to time-dependent PDEs are formulated in the frequency-wavenumber domain and solutions are obtained in terms of spectra (e.g. seismograms). This technique is particularly interesting for geometries where partial solutions in the ω-k domain can be obtained analytically (e.g. for layered models).

In the pseudo-spectral approach - in a finite-difference like manner - the PDEs are solved pointwise in physical space (x-t). However, the space derivatives are calculated using orthogonal functions (e.g. Fourier Integrals, Chebyshev polynomials). They are either evaluated using matrix-matrix multiplications or the fast Fourier transform (FFT).

Spectral derivative

.. let us recall the definition of the derivative using Fourier integrals ...

$$\partial_x f(x) = \partial_x \left(\int_{-\infty}^{\infty} F(k) e^{-ikx} dk \right)$$

$$= -\int_{-\infty}^{\infty} ikF(k) e^{-ikx} dk$$

... we could either ...

1) perform this calculation in the space domain by convolution

2) actually transform the function f(x) in the k-domain and back

Acoustic wave equation

let us take the acoustic wave equation with variable density

$$\frac{1}{\rho c^2}\partial_t^2 p = \partial_x\left(\frac{1}{\rho}\partial_x p\right)$$

the left hand side will be expressed with our standard centered finite-difference approach

$$\frac{1}{\rho c^2 dt^2}\left[p(t+dt) - 2p(t) + p(t-dt)\right] = \partial_x\left(\frac{1}{\rho}\partial_x p\right)$$

... leading to the extrapolation scheme ...

… pseudospectral approximation …

$$p(t+dt) = \rho c^2 dt^2 \boxed{\partial_x \left(\frac{1}{\rho} \partial_x p \right)} + 2p(t) - p(t-dt)$$

where the space derivatives will be calculated using the Fourier Method.
The highlighted term will be calculated as follows:

$$P_j^n \rightarrow \text{FFT} \rightarrow \hat{P}_\upsilon^n \rightarrow ik_\upsilon \hat{P}_\upsilon^n \rightarrow \text{FFT}^{-1} \rightarrow \partial_x P_j^n$$

multiply by 1/ρ

$$\frac{1}{\rho}\partial_x P_j^n \rightarrow \text{FFT} \rightarrow \left(\frac{1}{\rho}\partial_x \hat{P} \right)_\upsilon^n \rightarrow ik_\upsilon \left(\frac{1}{\rho}\partial_x \hat{P} \right)_\upsilon^n \rightarrow \text{FFT}^{-1} \rightarrow \partial_x \left(\frac{1}{\rho}\partial_x P_j^n \right)$$

... then extrapolate ...

… pseudospectral approximation …

$$p(t+dt) = \rho c^2 dt^2 \boxed{\left(\partial_x \left(\frac{1}{\rho} \partial_x p \right) + \partial_y \left(\frac{1}{\rho} \partial_y p \right) + \partial_z \left(\frac{1}{\rho} \partial_z p \right) \right)} + 2p(t) - p(t-dt)$$

.. where the following algorithm applies to each space dimension ...

$$P_j^n \rightarrow \mathrm{FFT} \rightarrow \hat{P}_\upsilon^n \rightarrow ik_\upsilon \hat{P}_\upsilon^n \rightarrow \mathrm{FFT}^{-1} \rightarrow \partial_x P_j^n$$

$$\frac{1}{\rho}\partial_x P_j^n \rightarrow \mathrm{FFT} \rightarrow \left(\frac{1}{\rho}\partial_x \hat{P} \right)_\upsilon^n \rightarrow ik_\upsilon \left(\frac{1}{\rho}\partial_x \hat{P} \right)_\upsilon^n \rightarrow \mathrm{FFT}^{-1} \rightarrow \partial_x \left(\frac{1}{\rho}\partial_x P_j^n \right)$$

Matlab code

let us compare the core of the algorithm - the calculation of the derivative (Matlab code)

```
function df=fder1d(f,dx,nop)
% fDER1D(f,dx,nop) finite difference
% second derivative

nx=max(size(f));

n2=(nop-1)/2;

if nop==3; d=[1 -2 1]/dx^2; end
if nop==5; d=[-1/12 4/3 -5/2 4/3 -1/12]/dx^2; end

df=[1:nx]*0;

for i=1:nop;
df=df+d(i).*cshift1d(f,-n2+(i-1));
end
```

Matlab code

... and the first derivative using FFTs ...

```
function df=sder1d(f,dx)
% SDER1D(f,dx) spectral derivative of vector
nx=max(size(f));


% initialize k
kmax=pi/dx;
dk=kmax/(nx/2);
for i=1:nx/2, k(i)=(i)*dk; k(nx/2+i)=-kmax+(i)*dk; end
k=sqrt(-1)*k;

% FFT and IFFT
ff=fft(f); ff=k.*ff; df=real(ifft(ff));
```

.. simple and elegant ...

Dispersion

... with the usual *Ansatz*

$$p_j^n = e^{i(kjdx - n\omega dt)}$$

we obtain

$$\partial_x^2 p_j^n = -k^2 e^{i(kjdx - \omega ndt)}$$

$$\partial_t^2 p_j^n = -\frac{4}{dt^2}\sin^2\frac{\omega dt}{2} e^{i(kjdx - \omega ndt)}$$

... leading to

$$k = \frac{2}{cdt}\sin\frac{\omega dt}{2}$$

Dispersion

$$k = \frac{2}{cdt} \sin \frac{\omega dt}{2}$$

$$\omega = \frac{2}{dt} \sin^{-1}\left(\frac{kcdt}{2}\right)$$

What are the consequences?

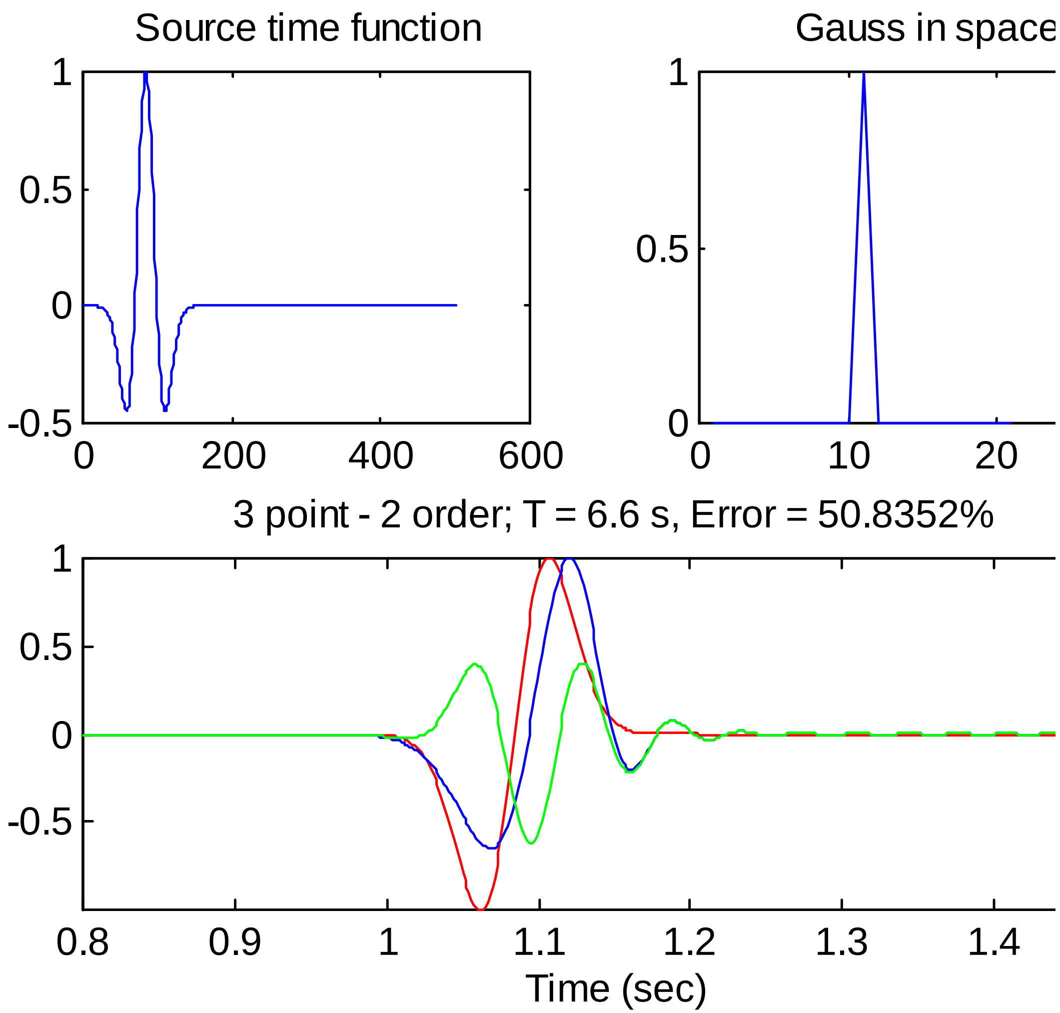
a) when dt << 1, $\sin^{-1}$ (kcdt/2) ≈kcdt/2 and w/k=c
   -> practically no dispersion

b) the argument of asin must be smaller than one.

$$\frac{k_{\max} cdt}{2} \leq 1$$

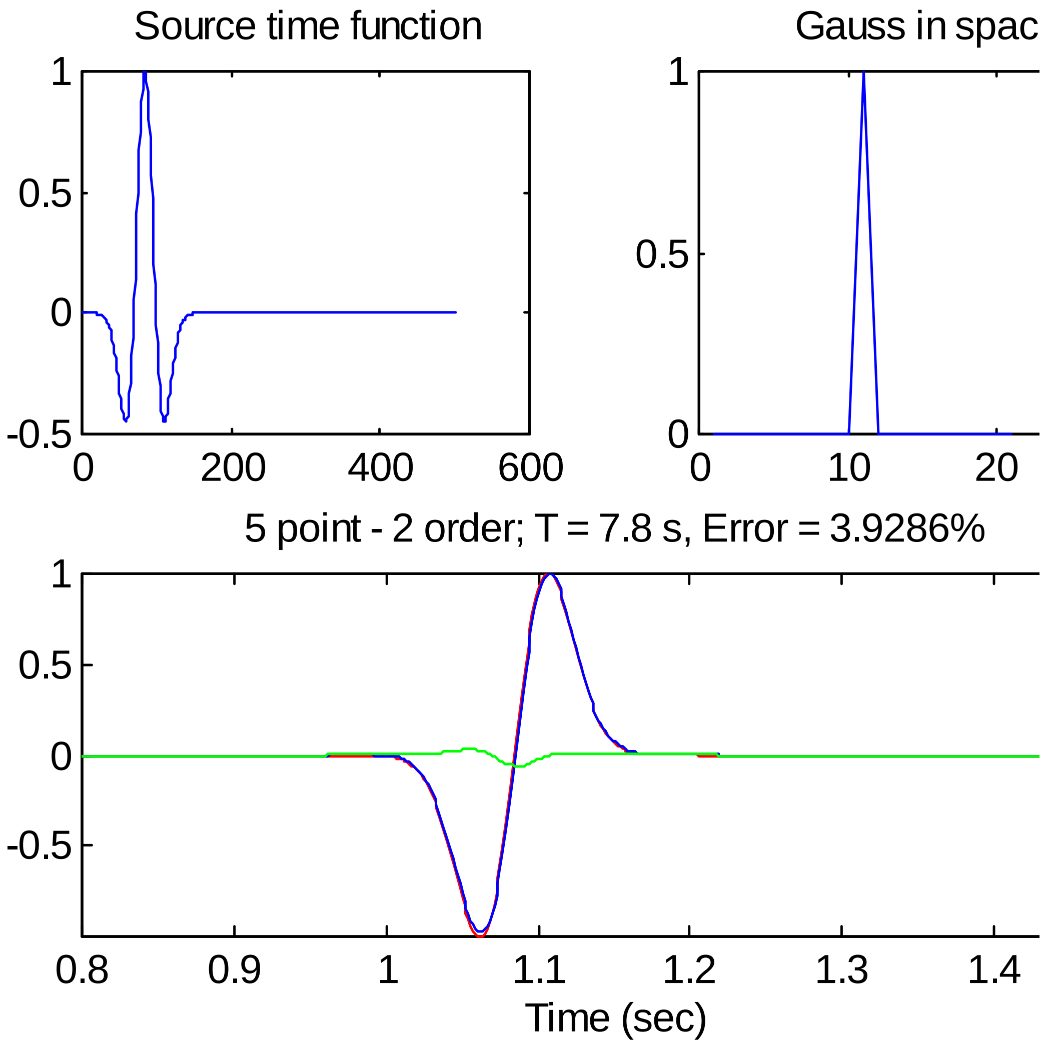
$$cdt / dx \leq 2 / \pi \approx 0.636$$

Matlab code: lectsac.m



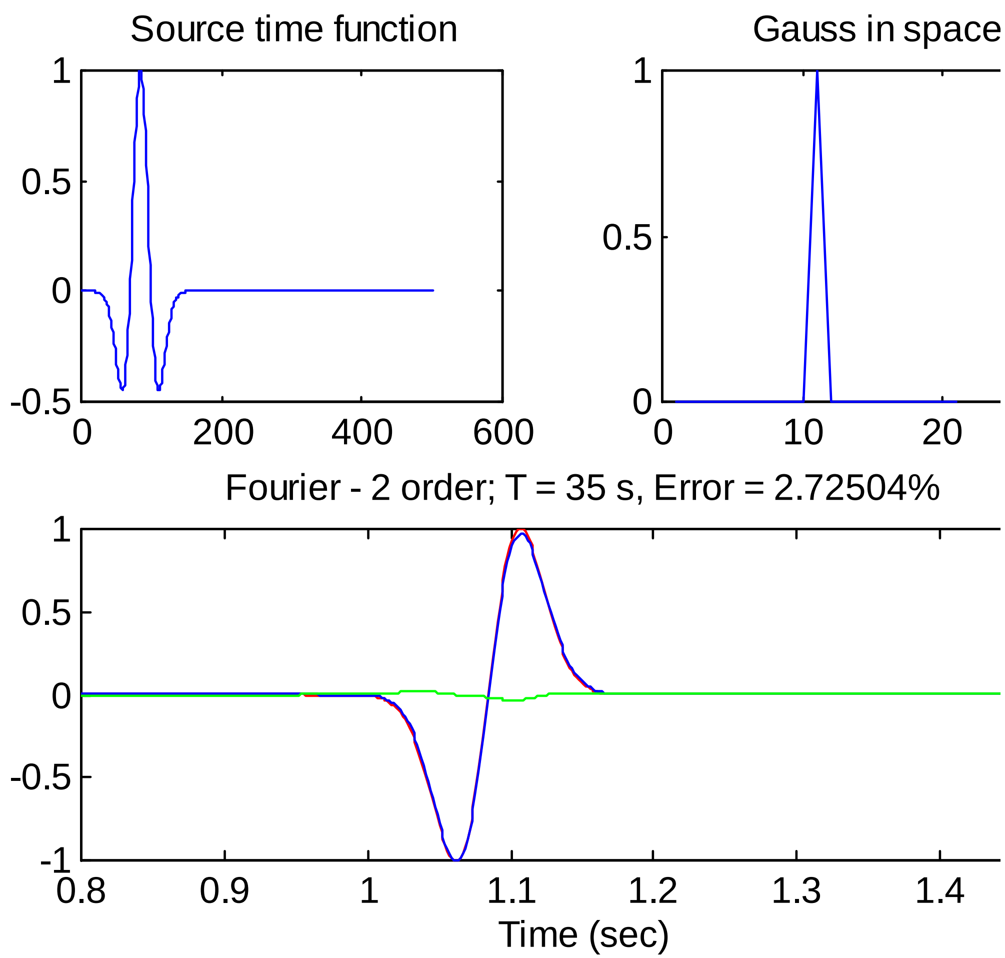
Example of acoustic 1D wave simulation.
FD 3 -point operator
red-analytic; blue-numerical; green-difference

Results



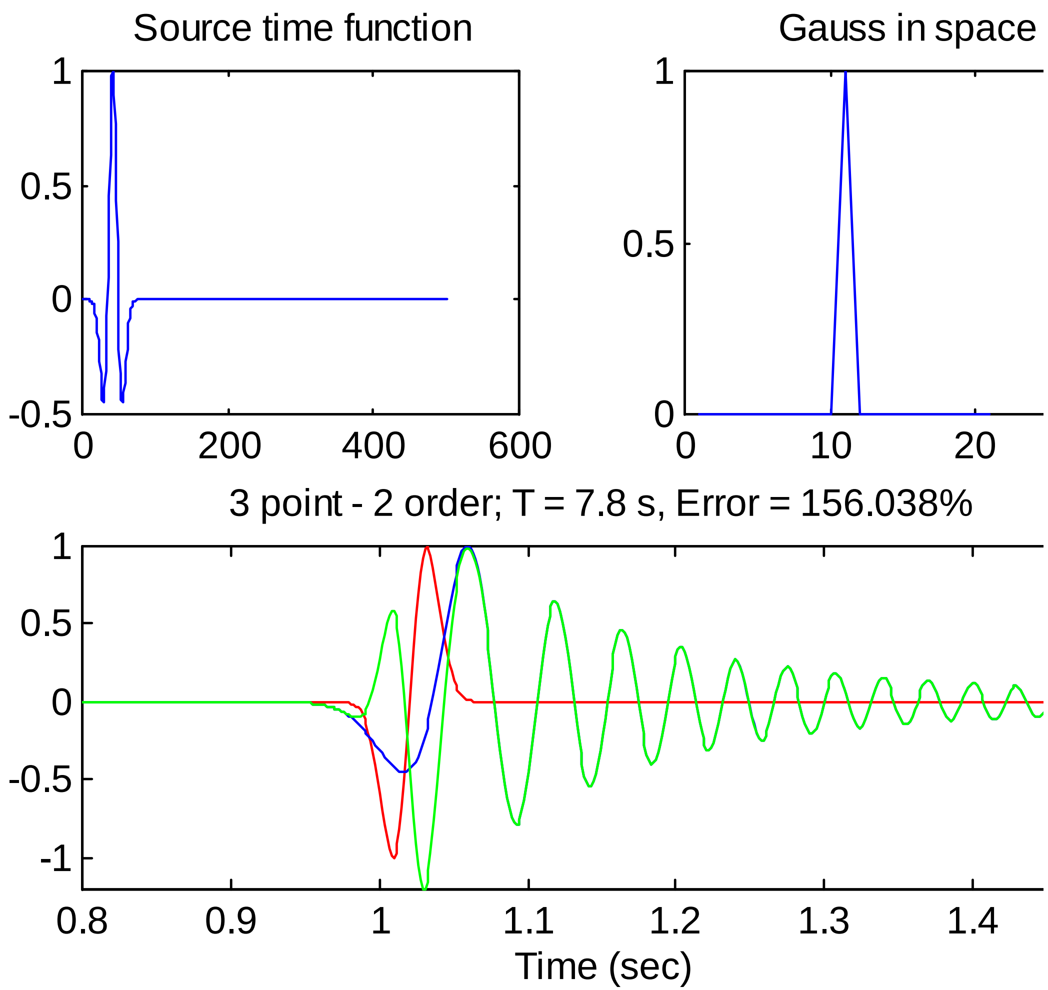
Example of acoustic 1D wave simulation.
FD 5 -point operator
red-analytic; blue-numerical; green-difference

Results



Example of acoustic 1D wave simulation.
Fourier operator
red-analytic; blue-numerical; green-difference

Results



Example of acoustic 1D wave simulation.
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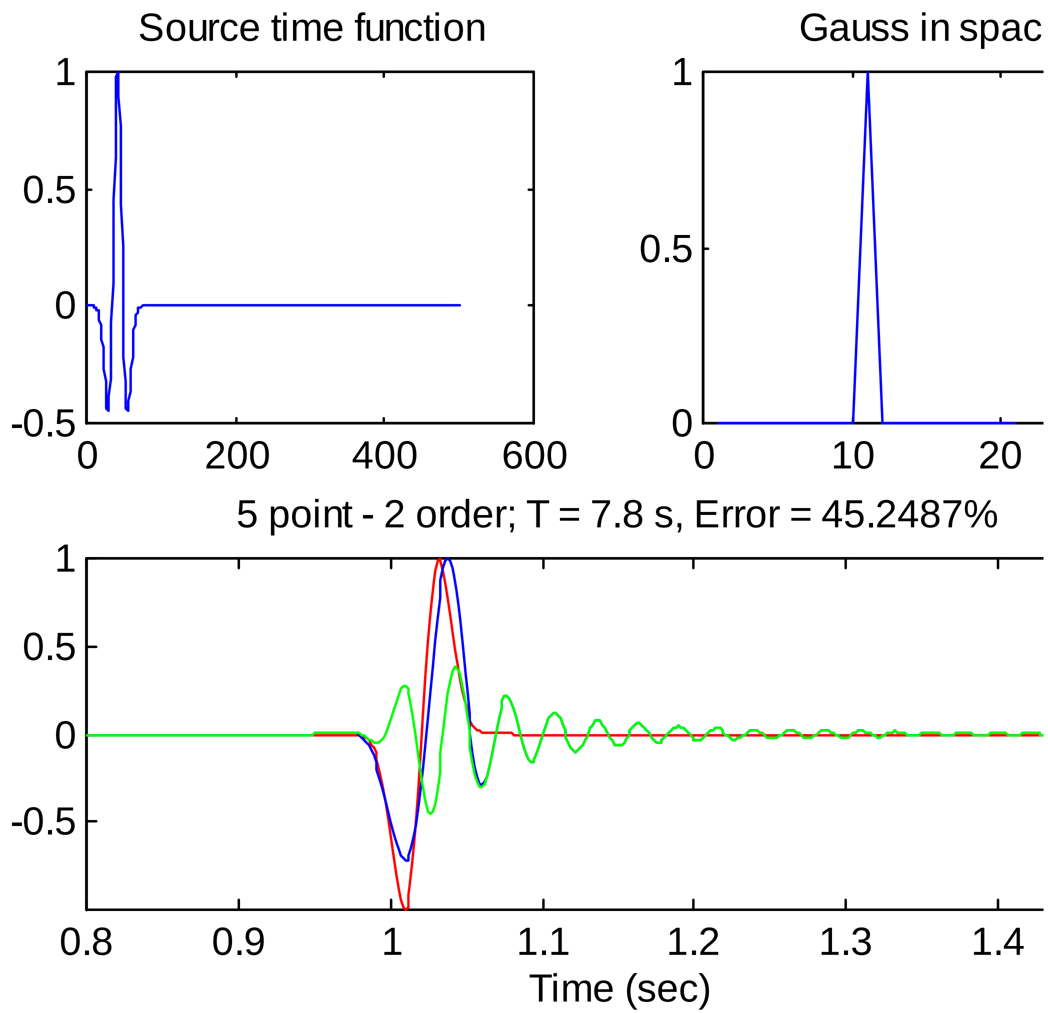

Example of acoustic 1D wave simulation.
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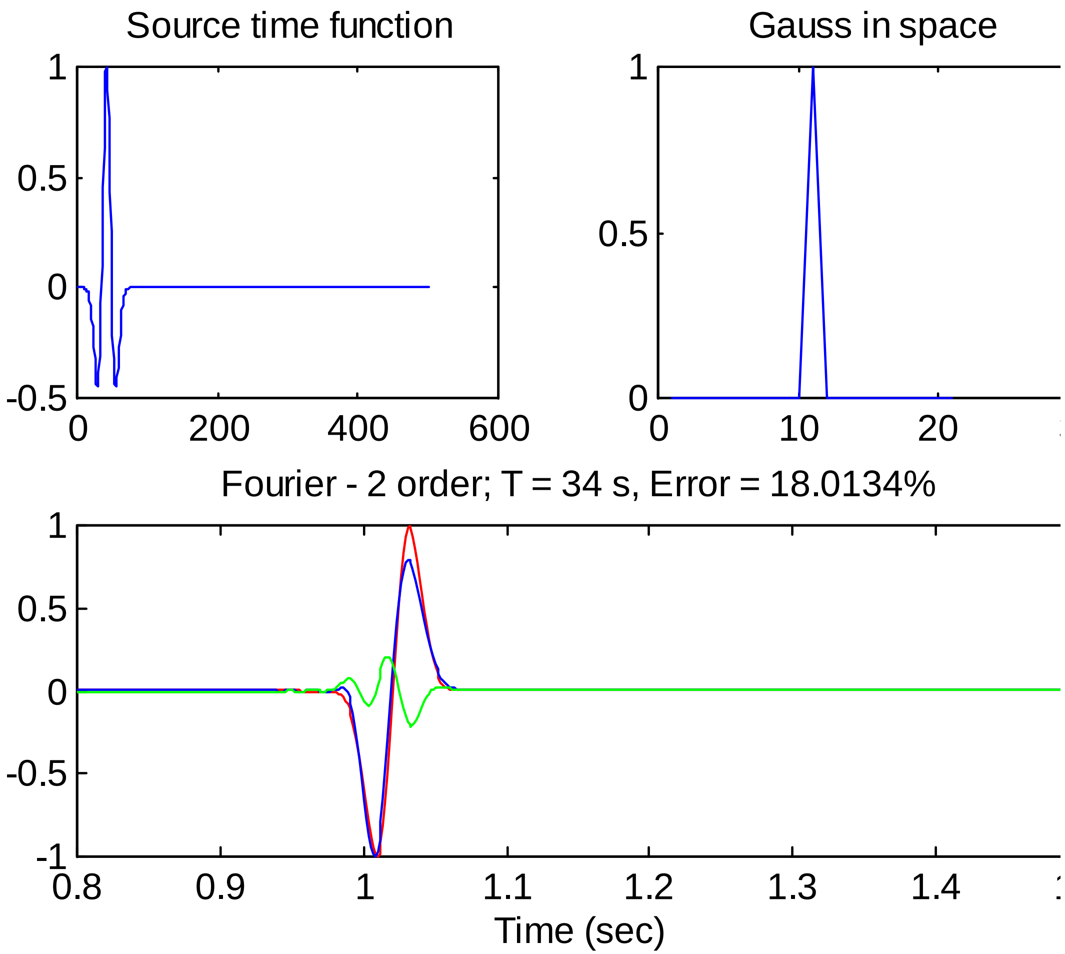
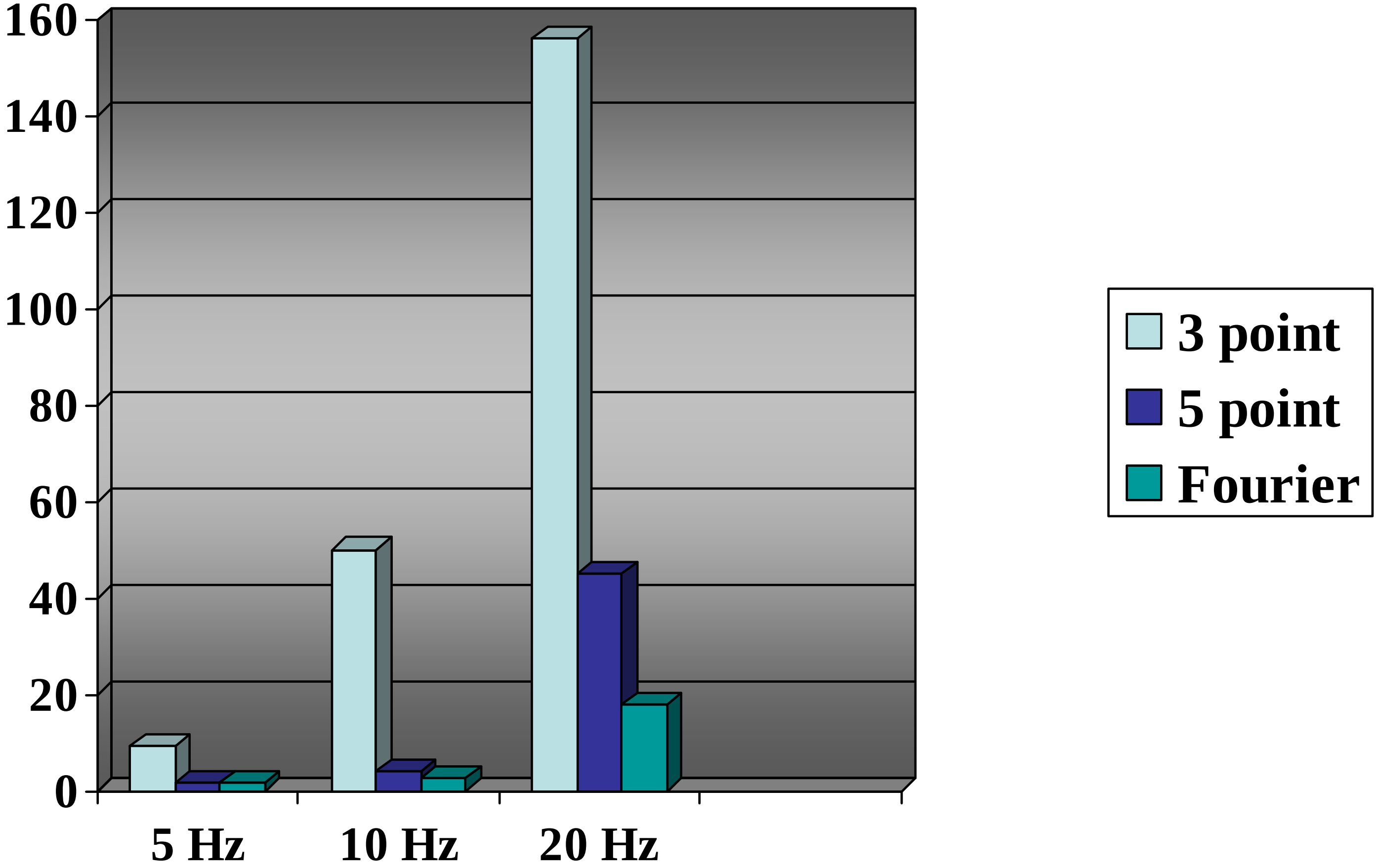


Example of acoustic 1D wave simulation.
Fourier operator
red-analytic; blue-numerical; green-difference

Computation speed

Difference (%) between numerical and analytical solution
as a function of propagating frequency



Simulation time
5.4s
7.8s
33.0s

Green's function

The concept of Green's Functions (impulse responses) plays an important role in the solution of partial differential equations. It is also useful for numerical solutions. Let us recall the acoustic wave equation

$$\partial_t^2 p = c^2 \Delta p$$

with Δ being the Laplace operator. We now introduce a delta source in space and time

$$\partial_t^2 p = \delta(\underline{x})\delta(t) + c^2 \Delta p$$

the formal solution to this equation is

$$p(\underline{x}, t) = \frac{1}{4\pi c^2} \frac{\delta(t - |\underline{x}| / c)}{|\underline{x}|}$$

(Full proof given in Aki and Richards, Quantitative Seismology, Freeman+Co, 1981, p. 65)

Green's function?

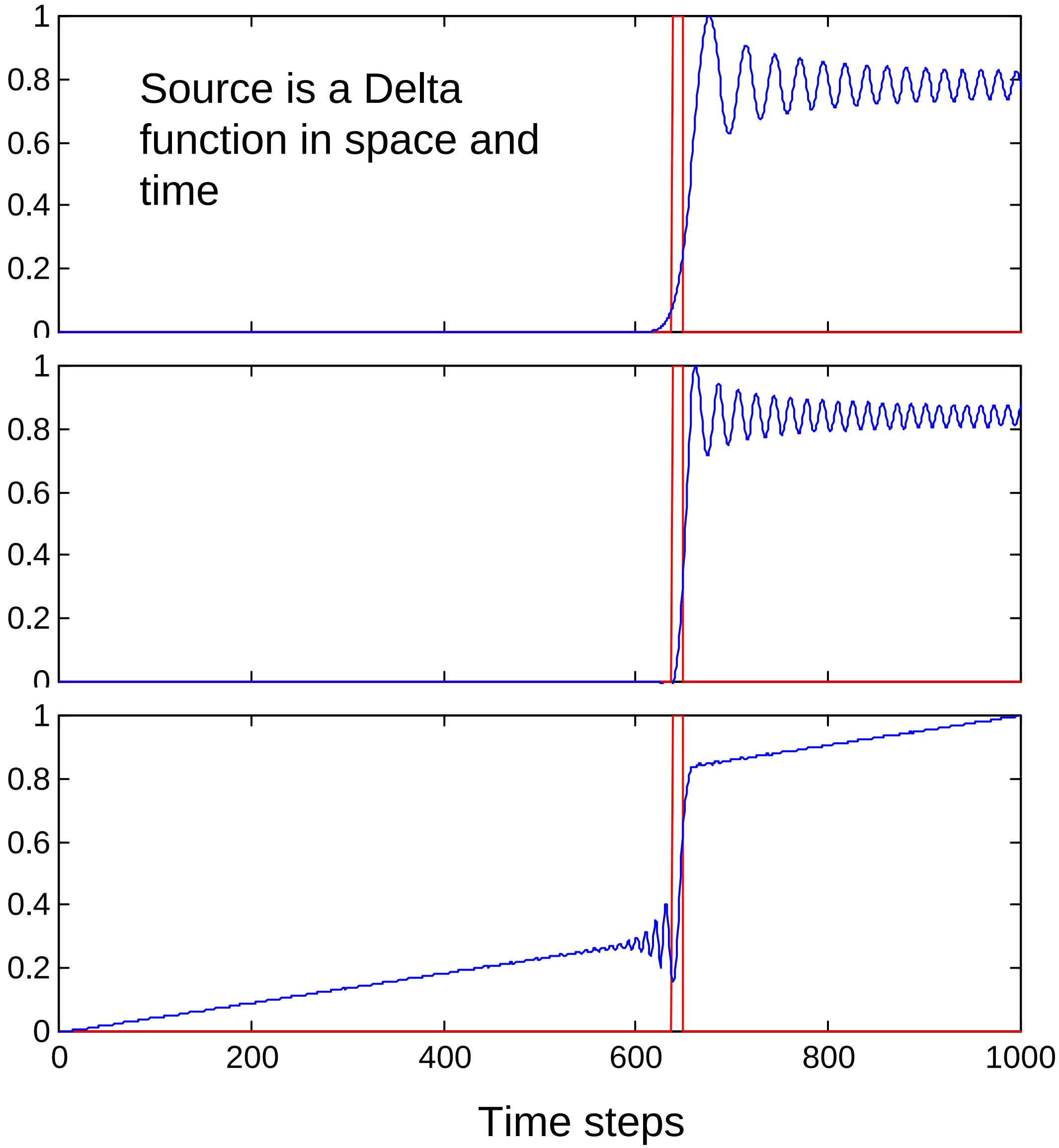
$$p(\underline{x}, t) = \frac{1}{4\pi c^2} \frac{\delta(t - |\underline{x}| / c)}{|\underline{x}|}$$

In words this means (in 1D and 3D but not in 2D, why?) , that in homogeneous media the same source time function which is input at the source location will be recorded at a distance r, but with amplitude proportional to 1/r.

An arbitrary source can evidently be constructed by summing up different delta - solutions. Can we use this property in our numerical simulations?

What happens if we solve our numerical system with delta functions as sources?

Heaviside



3 point operator

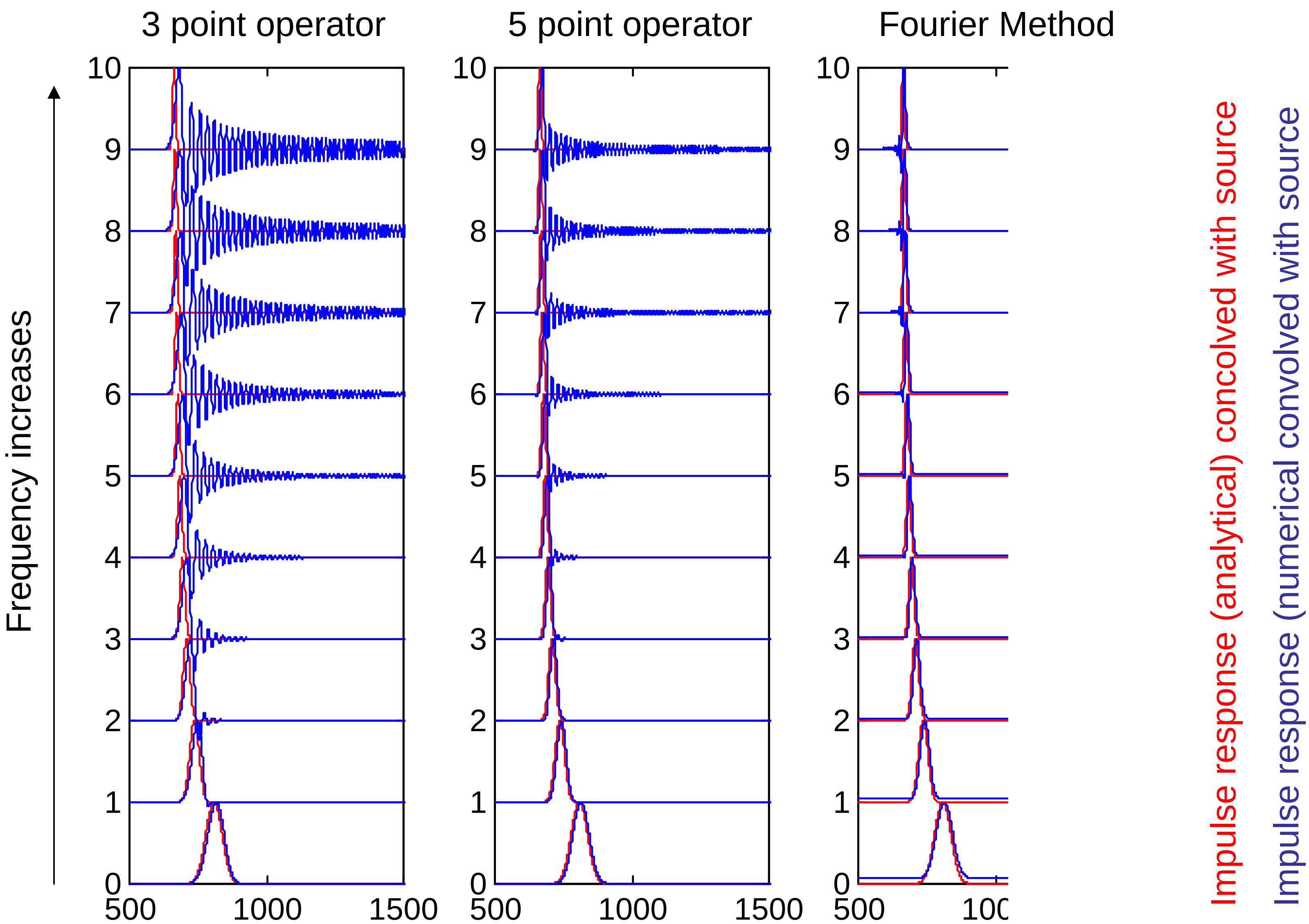
5 point operator

Fourier Method

Impulse response (analytical)

Impulse response (numerical

FD vs. Fourier



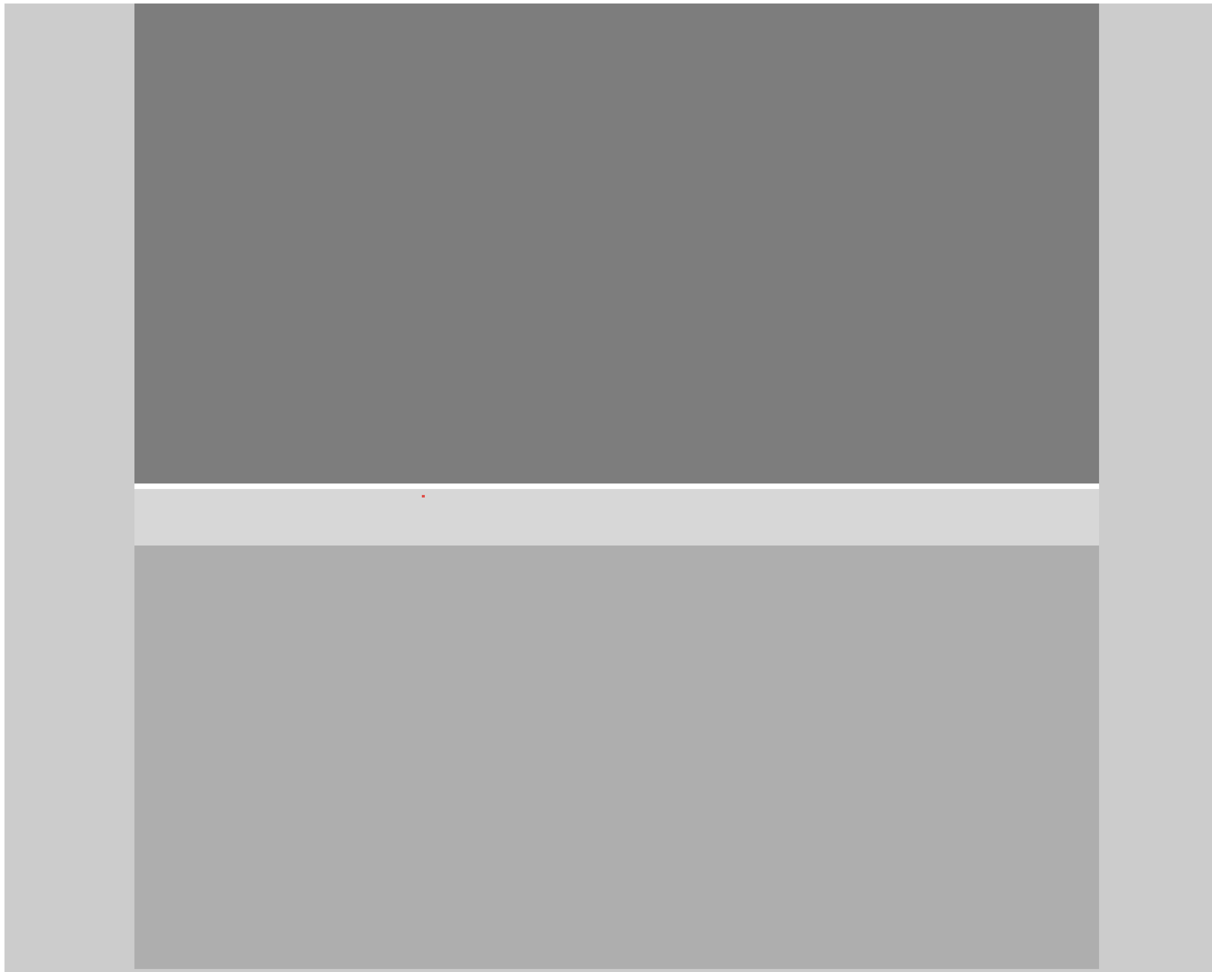
2-D acoustic wave propagation ac2d.m

Editor - F:\Heiner\Matlab\Ac2d\ac2d.m

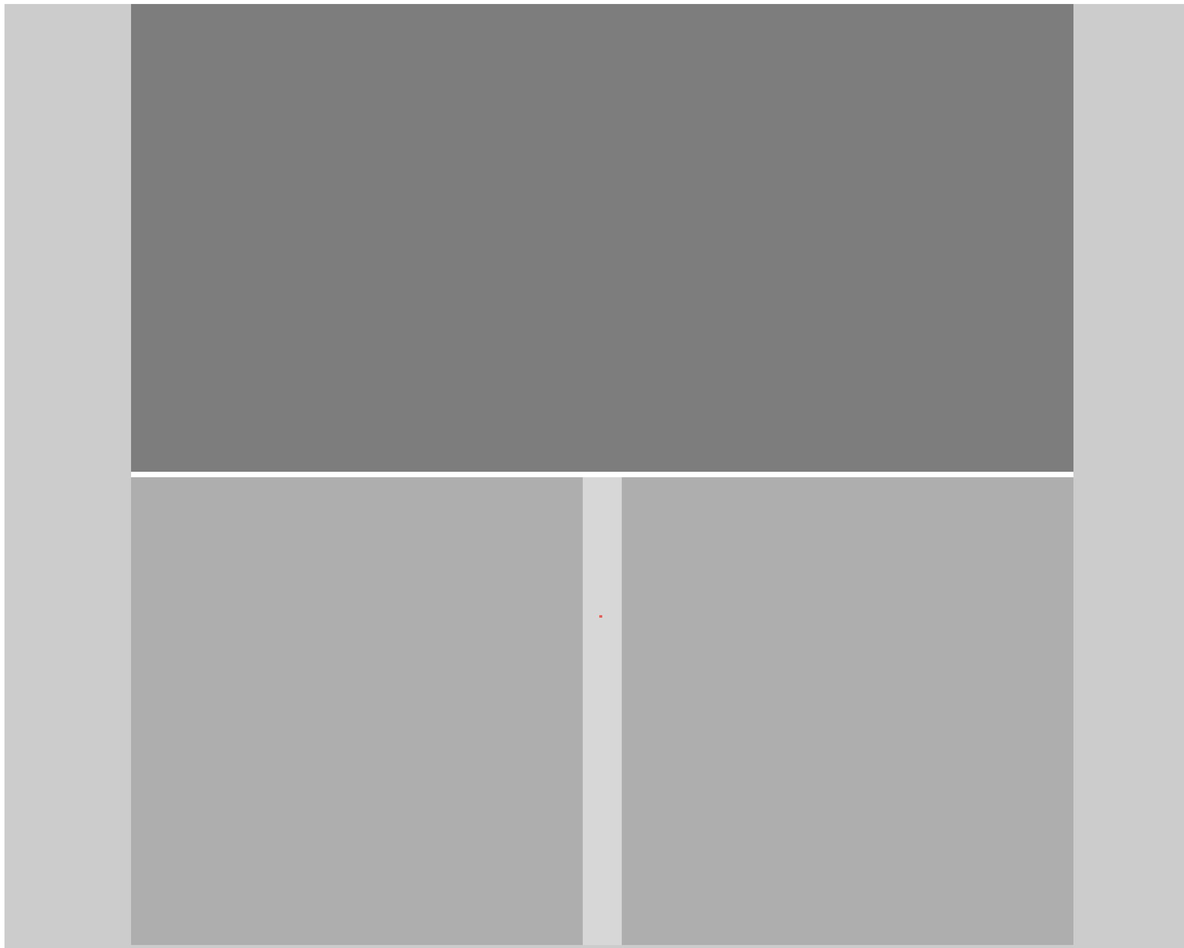
```
% Time stepping

for it=1:nt,

   % FD

   disp(sprintf(' Time step : %i',it));

   for j=3:nx-2,
      for k=3:nz-2,
         d2px(j,k)=(-1/12*p(j+2,k)+4/3*p(j+1,k)-5/2*p(j,k)+4/3*p(j-1,k)-1/12*p(j-2,k))/dx^2; % space derivative
         d2pz(j,k)=(-1/12*p(j,k+2)+4/3*p(j,k+1)-5/2*p(j,k)+4/3*p(j,k-1)-1/12*p(j,k-2))/dx^2; % space derivative
      end
   end
   pnew=2*p-pold+c.*c.*(d2px+d2pz)*dt^2;                 % time extrapolation
   pnew(nx/4,nz/4)=pnew(nx/4,nz/4)+src(it)*dt^2;      % add source term
   pold=p;                                           % time levels
   p=pnew;

   if rem(it,isnap)== 0,
   % Display
   imagesc(x,z,p), axis equal
   title(' FD ')
   drawnow
   end
end
```

Snapshots und Seismogramme:
homogenes Medium

Snapshots und Seismogramme: Niedriggeschwindigkeitsschicht



Snapshots und Seismogramme:
Störungszone (Verwerfung)



Snapshots und Seismogramme: Punktstreuung

Summary: Fourier Method

The Fourier Method can be considered as the limit of the finite-difference method as the length of the operator tends to the number of points along a particular dimension.

The space derivatives are calculated in the wavenumber domain by multiplication of the spectrum with *ik.* The inverse Fourier transform results in an exact space derivative up to the Nyquist frequency.

The use of Fourier transform imposes some constraints on the smoothness of the functions to be differentiated. Discontinuities lead to Gibb's phenomenon.

As the Fourier transform requires periodicity this technique is particular useful where the physical problems are periodical (e.g. angular derivatives in cylindrical problems).