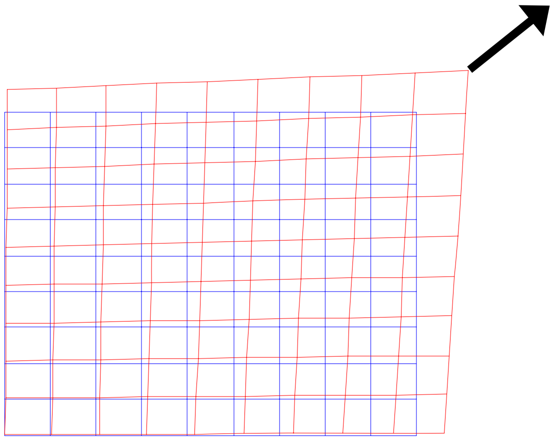
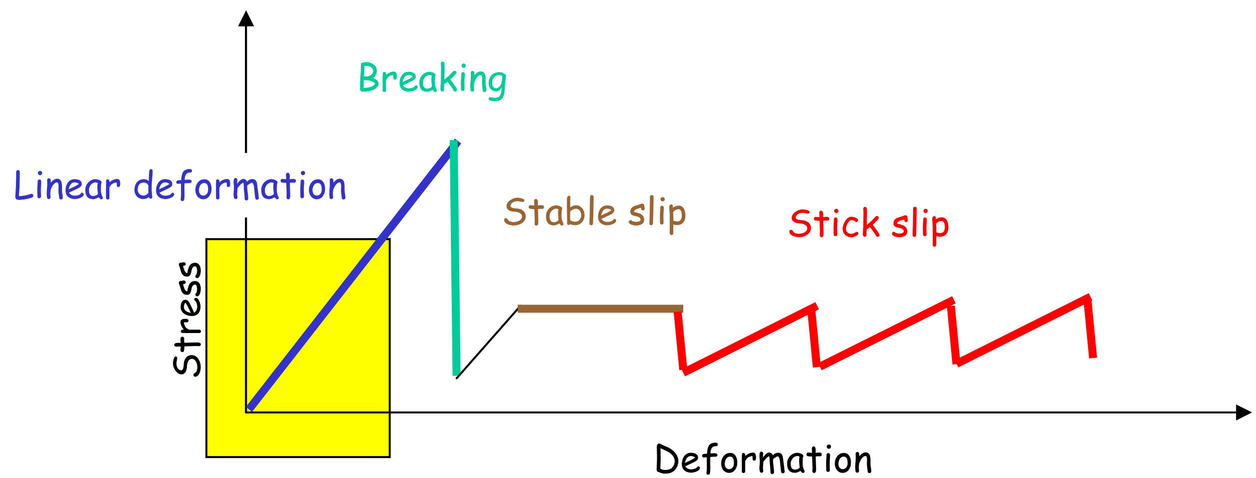
Elasticity and Seismic Waves

- Some mathematical basics
- Strain-displacement relation
  - Linear elasticity
  - Strain tensor – meaning of its elements
- Stress-strain relation (Hooke's Law)
  - Stress tensor
  - Symmetry
  - Elasticity tensor
  - Lame's parameters
- Equation of Motion
  - P and S waves
  - Plane wave solutions

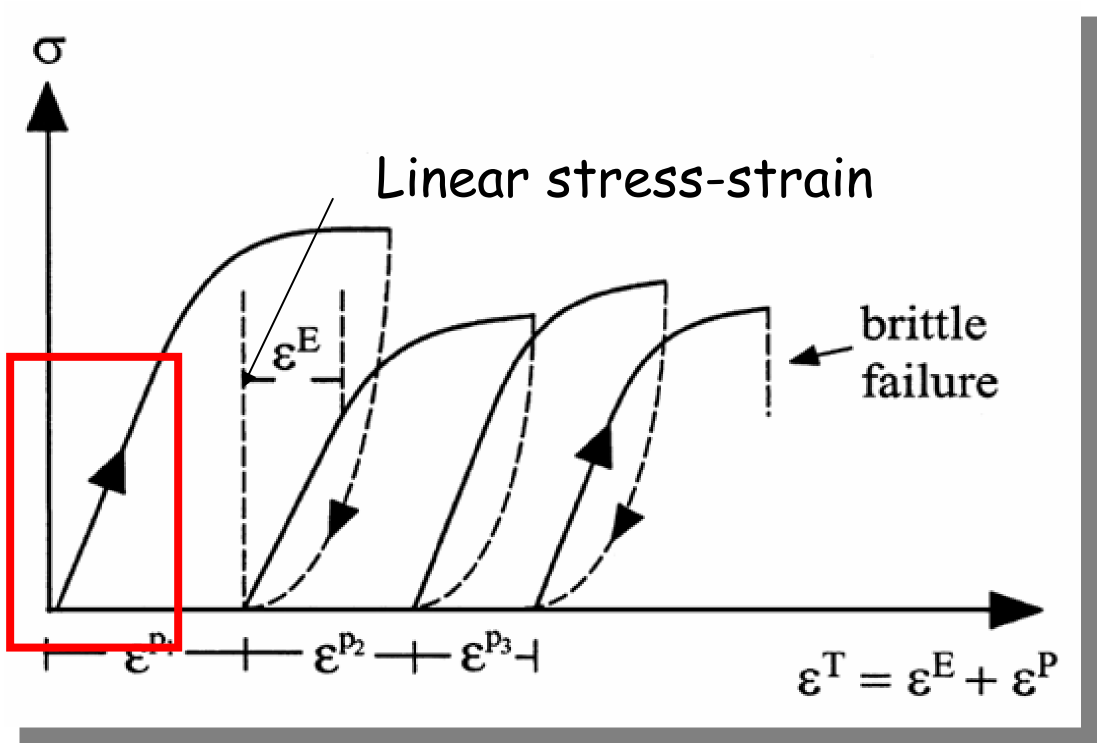


Stress-strain regimes

- Linear elasticity (teleseismic waves)
- rupture, breaking
- stable slip (aseismic)
- stick-slip (with sudden ruptures)



Linear and non-linear stress and strain

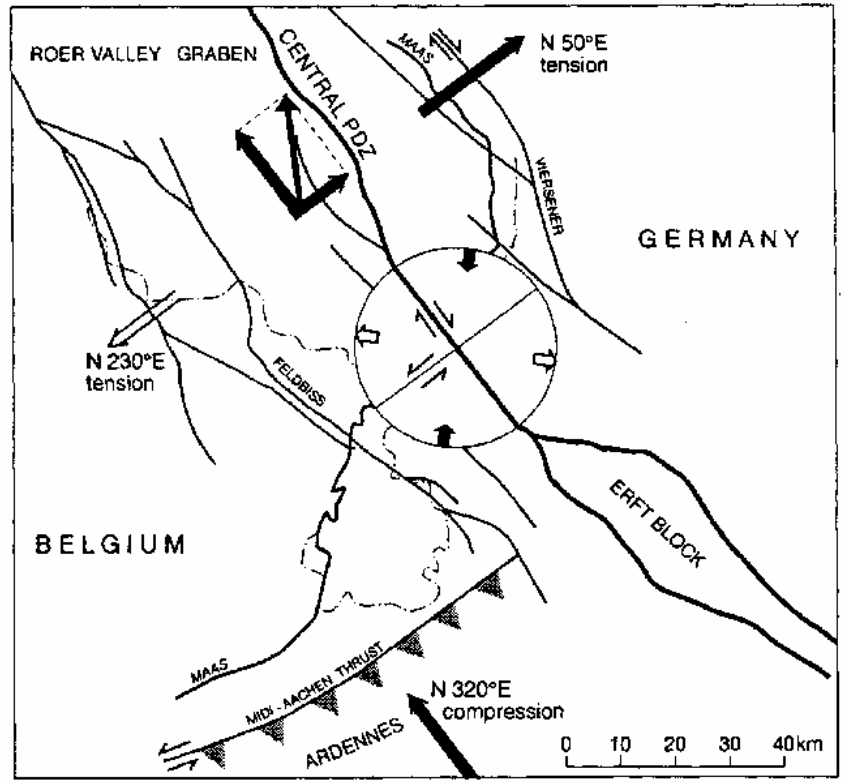


Stress vs. strain for a loading cycle with rock that breaks. For wave propagation problems assuming linear elasticity is usually sufficient.

Principal stress, hydrostatic stress

Horizontal stresses are influenced by tectonic forces (regional and local). This implies that usually there are two uneven horizontal principal stress directions.

Example: Cologne Basin



When all three orthogonal principal stresses are equal we speak of hydrostatic stress.

Elasticity Theory

A time-dependent perturbation of an elastic medium (e.g. a rupture, an earthquake, a meteorite impact, a nuclear explosion etc.) generates elastic waves emanating from the source region. These disturbances produce local changes in stress and strain.

To understand the propagation of elastic waves we need to describe kinematically the deformation of our medium and the resulting forces (stress). The relation between deformation and stress is governed by elastic constants.

The time-dependence of these disturbances will lead us to the elastic wave equation as a consequence of conservation of energy and momentum.

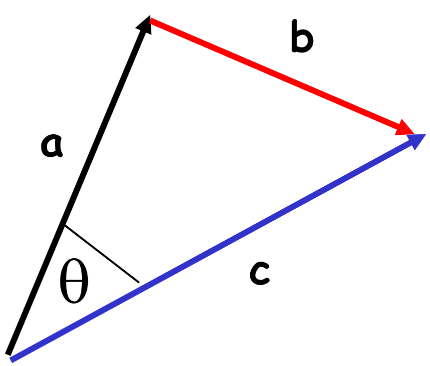
Some mathematical basics - Vectors

The mathematical description of deformation processes heavily relies on vector analysis. We therefore review the fundamental concepts of vectors and tensors.

Usually vectors are written in boldface type, x is a scalar but **y** is a vector, y_i are the scalar components of a vector

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \qquad a\mathbf{y} = \begin{pmatrix} ay_1 \\ ay_2 \\ ay_3 \end{pmatrix} \qquad a\mathbf{y} + b\mathbf{x} = \begin{pmatrix} ay_1 + bx_1 \\ ay_2 + bx_2 \\ ay_3 + bx_3 \end{pmatrix}$$

Scalar or Dot Product



c=a+b
b=c-a
a=c-b

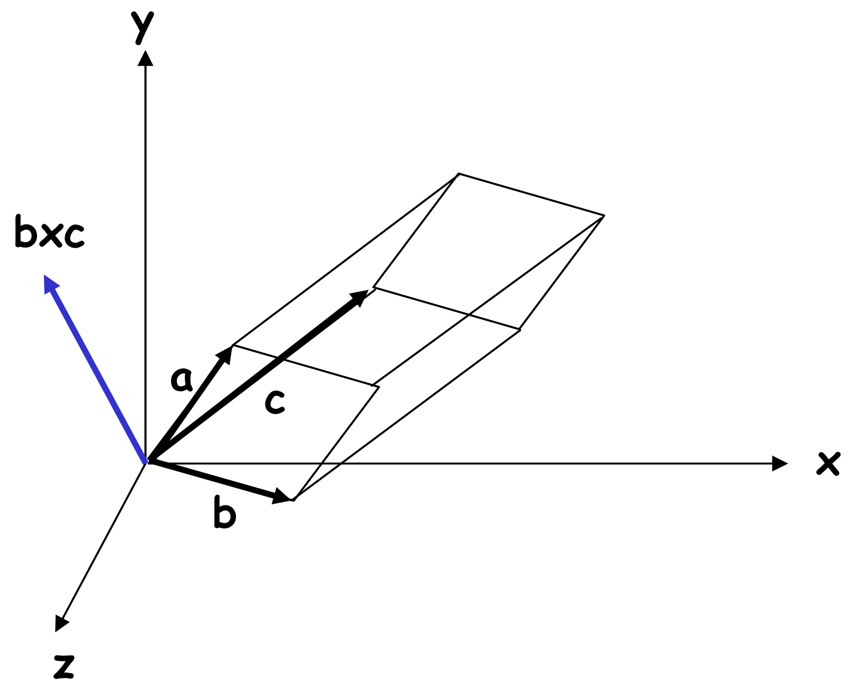
$$\mathbf{a} \bullet \mathbf{b} = (a_1 b_1 + a_2 b_2 + a_3 b_3) = |a||b|\cos\theta$$

$$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Vectors – Triple Product

The vector cross product is defined as:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$



The triple scalar product is defined as

$$\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c})$$

which is a scalar and represents the volume of the parallelepiped defined by **a**,**b**, and **c**.
It is also calculated like a determinant:

$$\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vectors – Gradient

Assume that we have a scalar field $\Phi(x)$, we want to know how it changes with respect to the coordinate axes, this leads to a vector called the **gradient of** Φ

$$\nabla\Phi = \begin{pmatrix} \partial_x \Phi \\ \partial_y \Phi \\ \partial_z \Phi \end{pmatrix}$$

With the **nabla operator** $\nabla = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$ and $\partial_x = \frac{\partial}{\partial x}$

The gradient is a vector that points in the direction of maximum rate of change of the scalar function $\Phi(x)$.

What happens if we have a vector field?

Vectors – Divergence + Curl

The **divergence** is the scalar product of the nabla operator with a vector field **V(x)**. The divergence of a vector field is a scalar!

$$\nabla \bullet \mathbf{V} = \partial_x V_x + \partial_y V_y + \partial_z V_z$$

Physically the divergence can be interpreted as the net flow out of a volume (or change in volume). E.g. the divergence of the seismic wavefield corresponds to compressional waves.

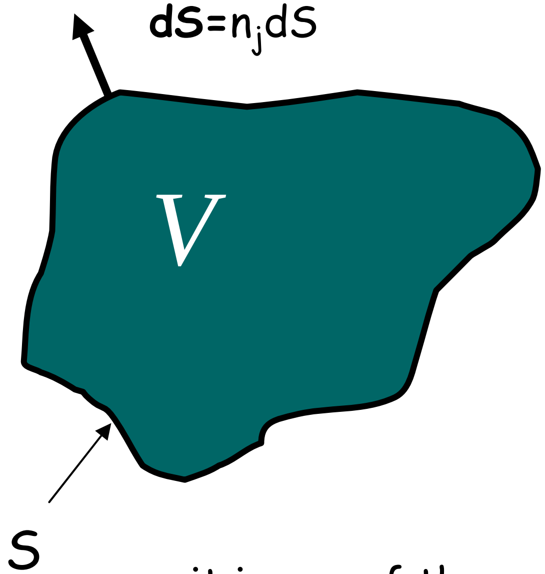
The **curl** is the vector product of the nabla operator with a vector field **V(x)**. The curl of a vector field is a vector!

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_{\mathbf{x}} & \partial_{\mathbf{y}} & \partial_{\mathbf{z}} \\ V_{\mathbf{x}} & V_{\mathbf{y}} & V_{\mathbf{z}} \end{vmatrix} = \begin{pmatrix} \partial_y V_z - \partial_z V_y \\ \partial_z V_x - \partial_x V_z \\ \partial_x V_y - \partial_y V_x \end{pmatrix}$$

The curl of a vector field represents the rotational part of that field (e.g. shear waves in a seismic wavefield)

Vectors – Gauss' Theorem

Gauss' theorem is a relation between a volume integral over the divergence of a vector field **F** and a surface integral over the values of the field at its surface S:

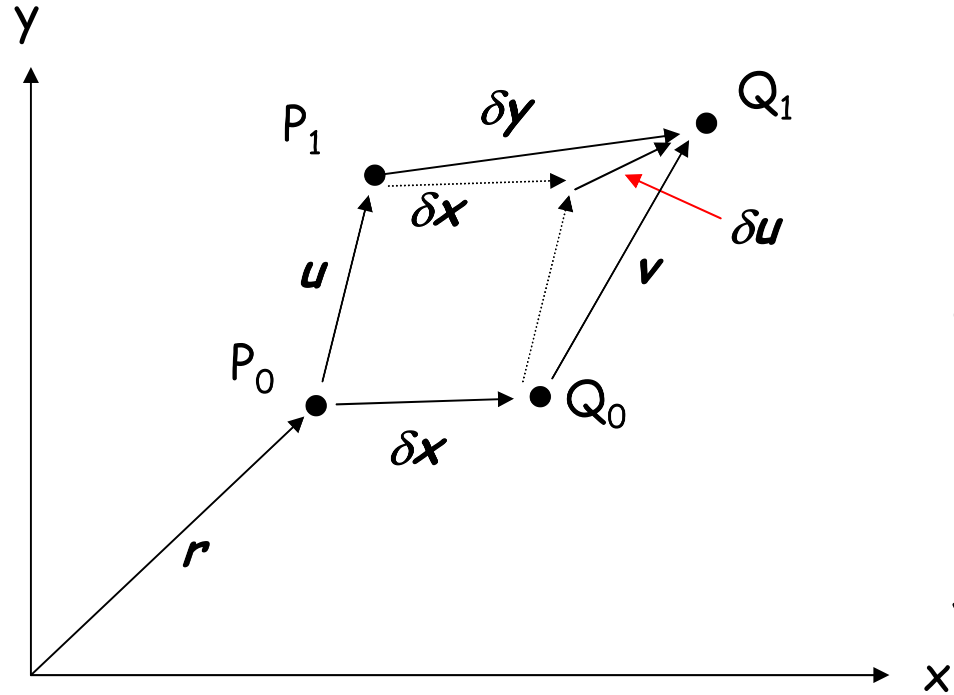


$$\int_S \mathbf{F} \bullet \mathbf{dS} = \int_V \nabla \bullet \mathbf{F} \, dV$$

… it is one of the most widely used relations in mathematical physics. The physical interpretation is again that the value of this integral can be considered the net flow out of volume V.

Deformation

Let us consider a point P_0 at position r relative to some fixed origin and a second point Q_0 displaced from P_0 by dx

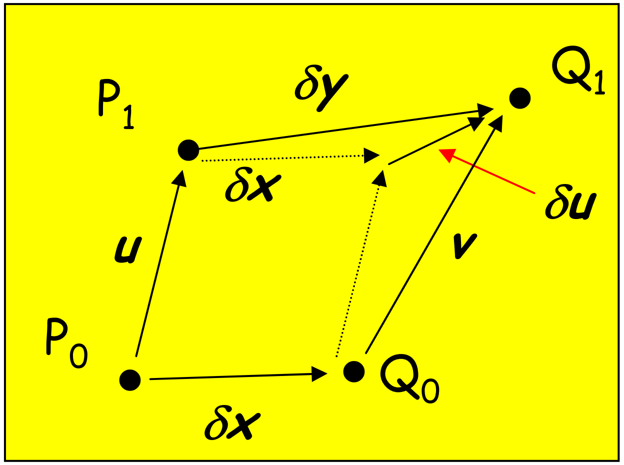


Unstrained state:
Relative position of point P_0 w.r.t. Q_0 is $\delta \boldsymbol{x}$.

Strained state:
Relative position of point P_0 has been displaced a distance $\boldsymbol{u}$ to P_1 and point Q_0 a distance $\boldsymbol{v}$ to Q_1.

Relative positive of point P_1 w.r.t. Q_1 is $\delta \boldsymbol{y} = \delta \boldsymbol{x} + \delta \boldsymbol{u}$. The change in relative position between Q and P is just $\delta \boldsymbol{u}$.

Linear Elasticity



The relative displacement in the unstrained state is u(r). The relative displacement in the strained state is v=u(r+ δx).

So finally we arrive at expressing the relative displacement due to strain:

$$\delta \mathbf{u} = \mathbf{u}(\mathbf{r} + \delta \mathbf{x}) - \mathbf{u}(\mathbf{r})$$

We now apply Taylor's theorem in 3-D to arrive at:

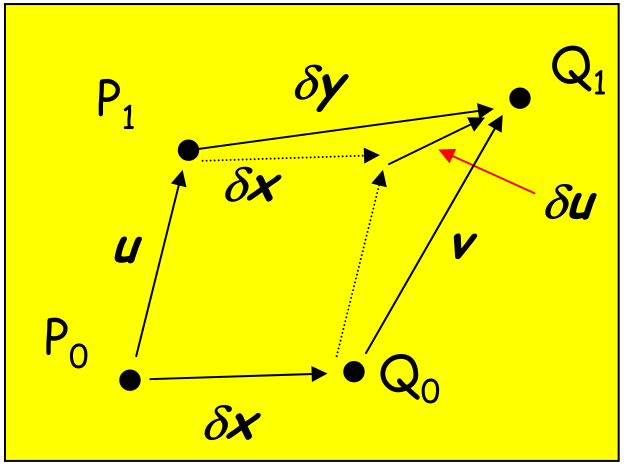
$$\delta u_i = \frac{\partial u_i}{\partial x_k} \delta x_k$$

What does this equation mean?

Linear Elasticity – symmetric part

The partial derivatives of the vector components

$$\frac{\partial u_i}{\partial x_k}$$



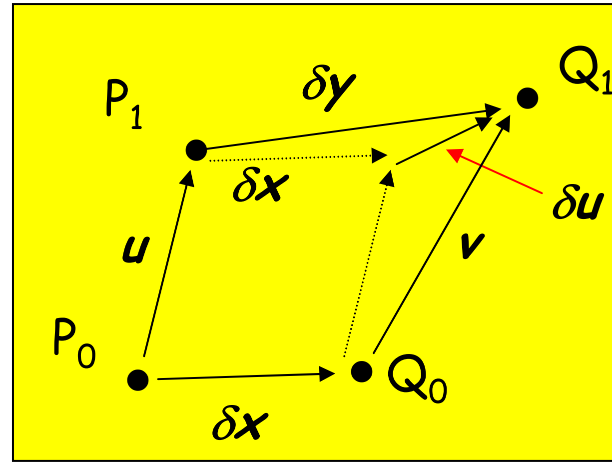
represent a second-rank tensor which can be resolved into a symmetric and anti-symmetric part:

$$\delta u_i = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)\delta x_k - \frac{1}{2}\left(\frac{\partial u_k}{\partial x_i} - \frac{\partial u_i}{\partial x_k}\right)\delta x_k$$

- symmetric
- deformation

- antisymmetric
- pure rotation

Linear Elasticity - deformation tensor



The symmetric part is called the
deformation tensor

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

and describes the relation between deformation and displacement in linear elasticity. In 2-D this tensor looks like

$$\varepsilon_{ij} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{\partial u_y}{\partial y} \end{bmatrix}$$

Deformation tensor – its elements

Through eigenvector analysis the meaning of the elements of the deformation tensor can be clarified:

The deformation tensor assigns each point – represented by position vector **y** a new position with vector **u** (summation over repeated indices applies):

$$u_i = \varepsilon_{ij} y_j$$

The eigenvectors of the deformation tensor are those y's for which the tensor is a scalar, the eigenvalues λ:

$$u_i = \lambda y_i$$

The eigenvalues λ can be obtained solving the system:

$$\left| \varepsilon_{ij} - \lambda \delta_{ij} \right| = 0$$

Deformation tensor – its elements

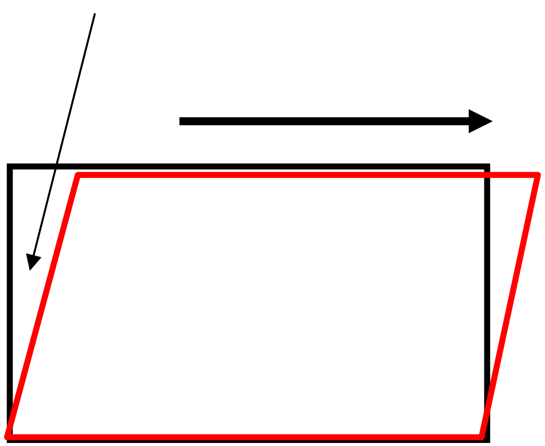
Thus

$$u_1 = \lambda_1 y_1 \qquad u_2 = \lambda_2 y_2 \qquad u_3 = \lambda_3 y_3$$

... in other words ...
the eigenvalues are the relative change of length along the three coordinate axes

$$\lambda_1 = \frac{u_1}{y_1}$$

shear angle



In arbitrary coordinates the diagonal elements are the relative change of length along the coordinate axes and the off-diagonal elements are the infinitesimal shear angles.

Deformation tensor – trace

The trace of a tensor is defined as the sum over the diagonal elements. Thus:

$$\varepsilon_{ii} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

This trace is linked to the volumetric change after deformation. Before deformation the volume was V_{0}. Because the diagonal elements are the relative change of lengths along each direction, the new volume after deformation is

$$V = (1 + \varepsilon_{xx})(1 + \varepsilon_{yy})(1 + \varepsilon_{zz})$$

... and neglecting higher-order terms ...

$$V = 1 + \varepsilon_{ii} = V_0 + \varepsilon_{ii}$$

$$\Theta = \frac{\Delta V}{V_0} = \varepsilon_{ii} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \mathrm{div} u = \nabla \bullet u$$

Deformation tensor - applications

The fact that we have linearised the strain-displacement relation is quite severe. It means that the elements of the strain tensor should be <<1. Is this the case in seismology?

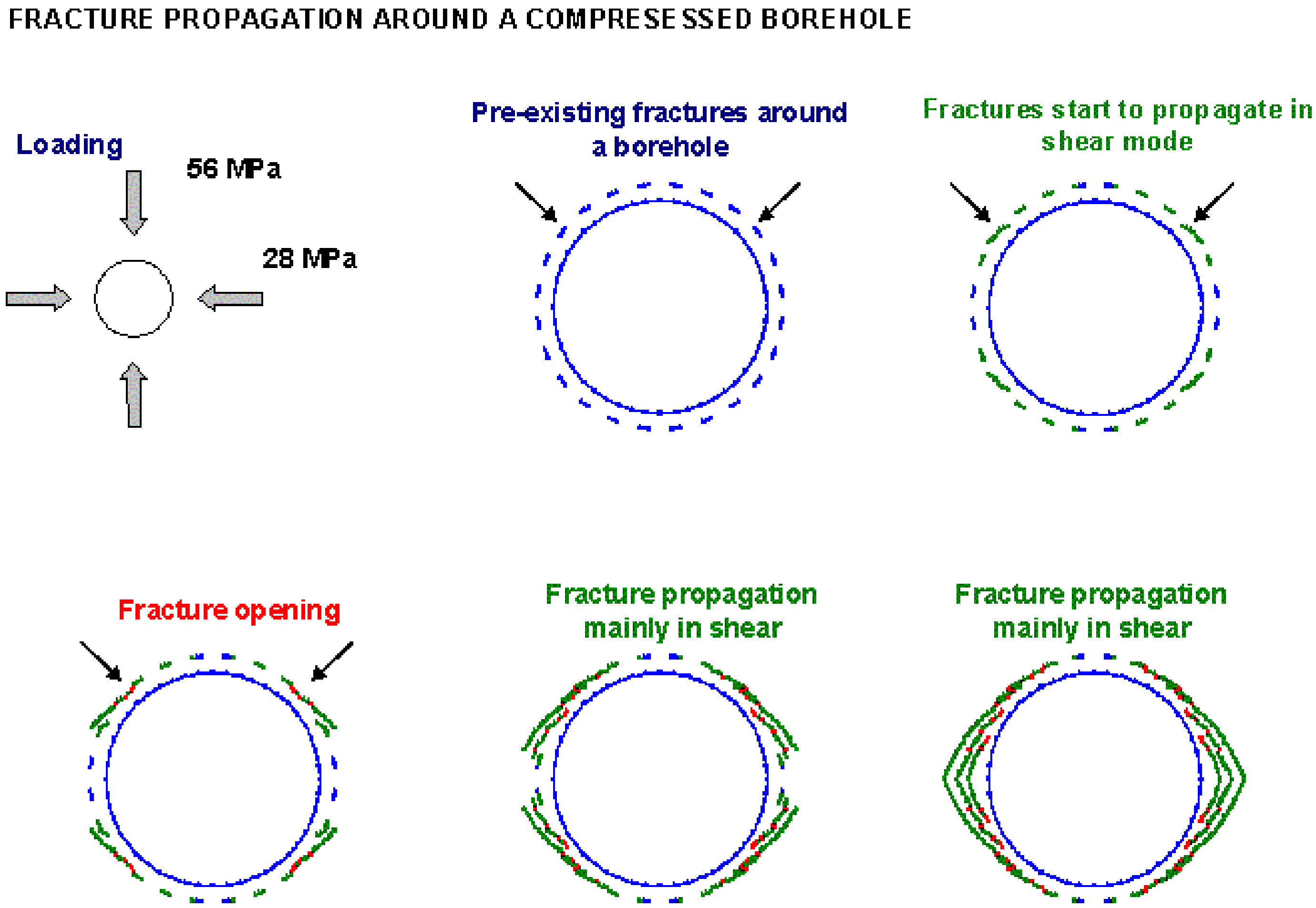
Let's consider an example. The 1999 Taiwan earthquake (M=7.6) was recorded in FFB. The maximum ground displacement was 1.5mm measured for surface waves of approx. 30s period. Let us assume a phase velocity of 5km/s. How big is the strain at the Earth's surface, give an estimate !

The answer is that ε would be on the order of 10^{-7} <<1. This is typical for global seismology if we are far away from the source, so that the assumption of infinitesimal displacements is acceptable.

For displacements closer to the source this assumption is not valid. There we need a finite strain theory. Strong motion seismology is an own field in seismology concentrating on effects close to the seismic source.

Strainmeter

Borehole breakout



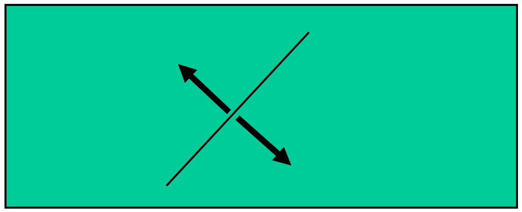
Source: www.fracom.fi

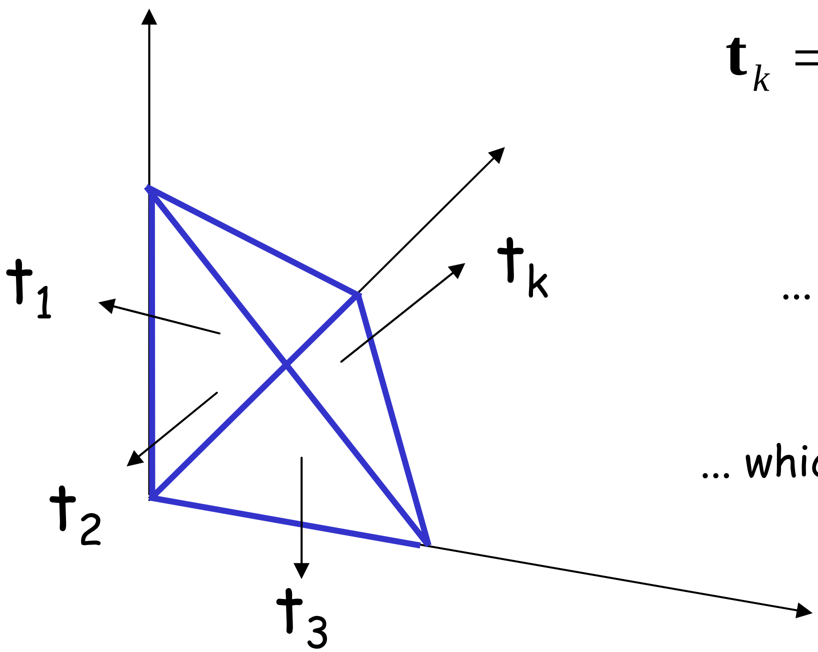
Stress - traction

In an elastic body there are restoring forces if deformation takes place. These forces can be seen as acting on planes inside the body. **Forces divided by an areas are called stresses.**
In order for the deformed body to remain deformed these forces have to compensate each other. We will see that the relationship between the stress and the deformation (strain) is linear and can be described by tensors.

The tractions t_k along axis k are

$$\mathbf{t}_k = \begin{pmatrix} t_{k1} \\ t_{k2} \\ t_{k3} \end{pmatrix}$$





... and along an arbitrary direction

$$\mathbf{t} = \mathbf{t}_i n_i$$

... which – using the summation convention yields ..

$$\mathbf{t} = \mathbf{t}_1 n_1 + \mathbf{t}_2 n_2 + \mathbf{t}_3 n_3$$

Stress tensor

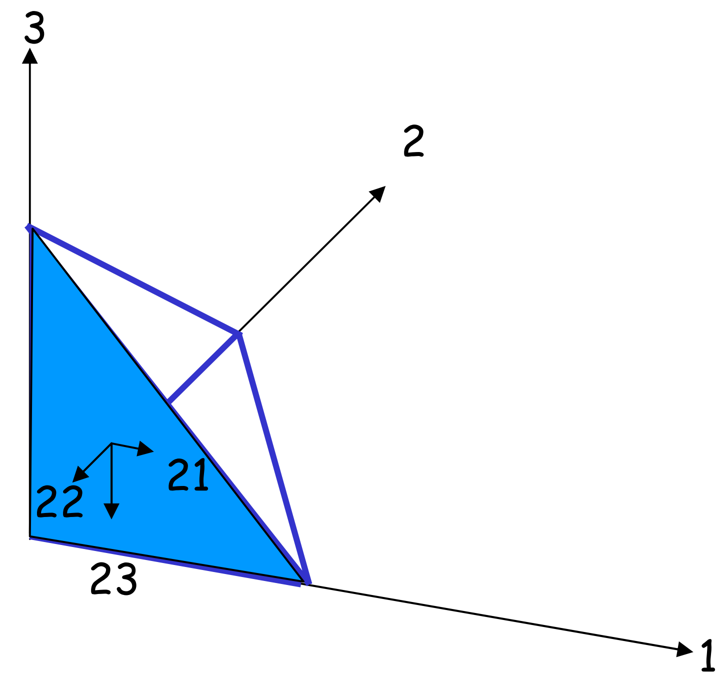
... in components we can write this as

$$t_i = \sigma_{ij} n_j$$

where σ_{ij} ist the stress tensor and n_j is a surface normal.
The stress tensor describes the forces acting on planes within a body.
Due to the symmetry condition

$$\sigma_{ij} = \sigma_{ji}$$

there are only six independent elements.



σ_{ij}

i: The vector normal to the corresponding surface

j: The direction of the force vector acting on that surface

Stress equilibrium

If a body is in equilibrium the internal forces and the forces acting on its surface have to vanish

$$\int_V f_i dV + \oint_F t_i dF = 0$$

as well as the sum over the angular momentum

$$\int_V x_i \times f_j dV + \oint_F x_i \times t_j dF = 0$$

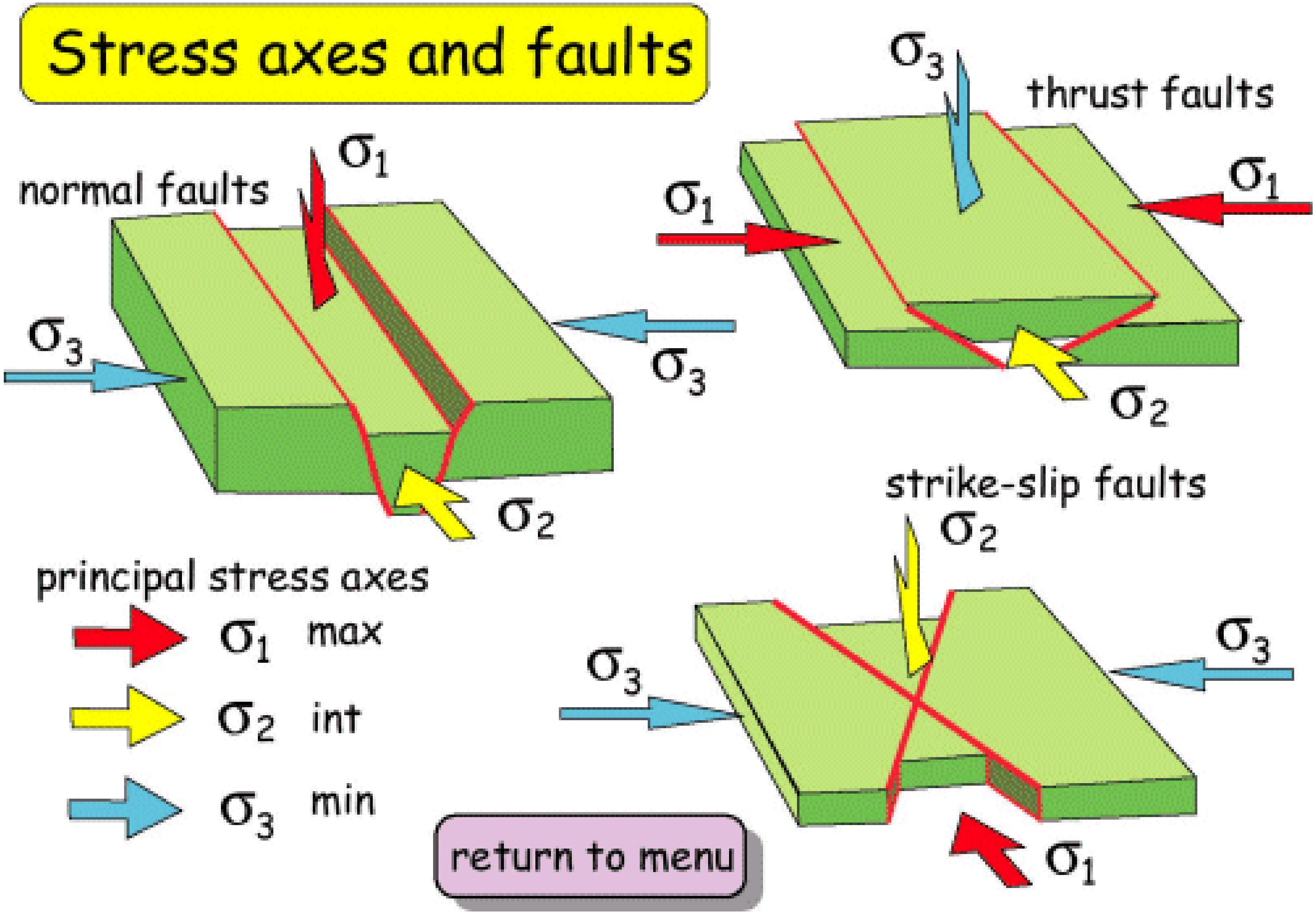
From the second equation the symmetry of the stress tensor can be derived. Using Gauss' law the first equation yields

$$f_i + \frac{\partial \sigma_{ij}}{\partial x_j} = 0$$

Stress - Glossary

Stress units	bars (10^6dyn/cm^2) 10^6Pa=1MPa=10bars 1 Pa=1 N/m^2 At sea level p=1bar At depth 3km p=1kbar
maximum compressive stress	the direction perpendicular to the minimum compressive stress, near the surface mostly in horizontal direction, linked to tectonic processes.
principle stress axes	the direction of the eigenvectors of the stress tensor

Stresses and faults



Stress-strain relation

The relation between stress and strain in general is described by the tensor of elastic constants c_{ijkl}

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}$$

Generalised Hooke's Law

From the symmetry of the stress and strain tensor and a thermodynamic condition if follows that the maximum number if independent constants of c_{ijkl} is 21. In an isotropic body, where the properties do not depend on direction the relation reduces to

$$\sigma_{ij} = \lambda\Theta\delta_{ij} + 2\mu\varepsilon_{ij}$$

Hooke's Law

where λ and μ are the Lame parameters, θ is the dilatation and δ_{ij} is the Kronecker delta.

$$\Theta\delta_{ij} = \varepsilon_{kk}\delta_{ij} = \left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right)\delta_{ij}$$

Stress-strain relation

The complete stress tensor looks like

$$\sigma_{ij} = \begin{pmatrix} (\lambda+2\mu)\varepsilon_{xx} + \lambda(\varepsilon_{yy}+\varepsilon_{zz}) & 2\mu\varepsilon_{xy} & 2\mu\varepsilon_{xz} \\ 2\mu\varepsilon_{yx} & (\lambda+2\mu)\varepsilon_{yy} + \lambda(\varepsilon_{xx}+\varepsilon_{zz}) & 2\mu\varepsilon_{yz} \\ 2\mu\varepsilon_{zx} & 2\mu\varepsilon_{zy} & (\lambda+2\mu)\varepsilon_{zz} + \lambda(\varepsilon_{xx}+\varepsilon_{yy}) \end{pmatrix}$$

There are several other possibilities to describe elasticity:
E elasticity, s Poisson's ratio, k bulk modulus

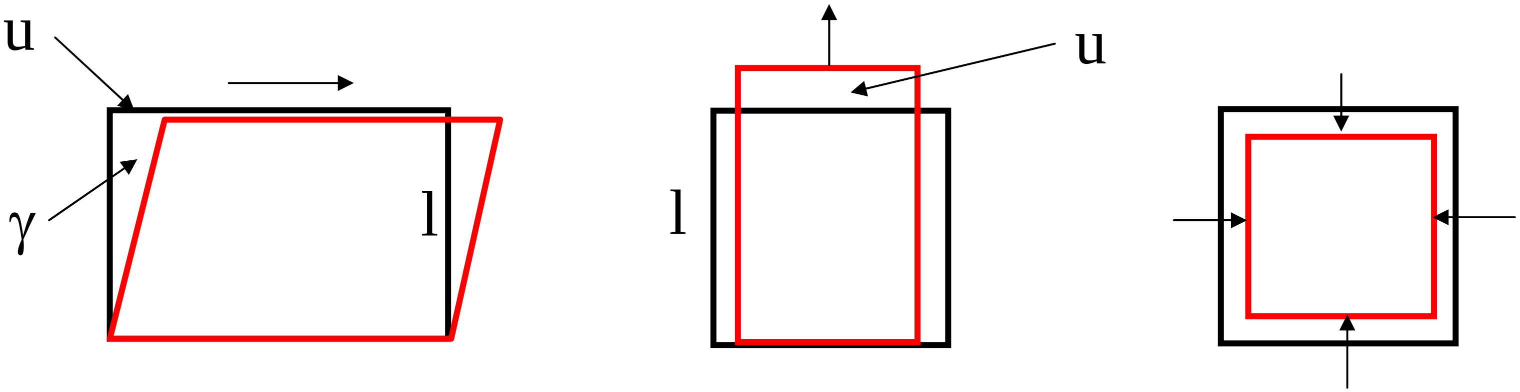
$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \qquad \sigma = \frac{\lambda}{2(\lambda+\mu)} \qquad k = \lambda + \frac{2}{3}\mu$$

$$\lambda = \frac{\sigma E}{(1+\sigma)(1-2\sigma)} \qquad \mu = \frac{E}{2(1+\sigma)}$$

For Poisson's ratio we have $0<\sigma<0.5$. A useful approximation is $\lambda=\mu$, then $\sigma=0.25$. For fluids $\sigma=0.5$ ($\mu=0$).

Stress-strain - significance

As in the case of deformation the stress-strain relation can be interpreted in simple geometric terms:



$$\sigma_{12} = \mu\gamma \qquad \sigma_{22} = E\,\frac{u}{l} \qquad P = K\,\frac{\Delta V}{V} = K\varepsilon_{ii}$$

Remember that these relations are a generalization of Hooke's Law:

F= D s

D being the spring constant and s the elongation.

Seismic wave velocities: P-waves

Material	V_p (km/s)
Unconsolidated material	
Sand (dry)	0.2-1.0
Sand (wet)	1.5-2.0
Sediments	
Sandstones	2.0-6.0
Limestones	2.0-6.0
Igneous rocks	
Granite	5.5-6.0
Gabbro	6.5-8.5
Pore fluids	
Air	0.3
Water	1.4-1.5
Oil	1.3-1.4
Other material	
Steel	6.1
Concrete	3.6

Elastic anisotropy

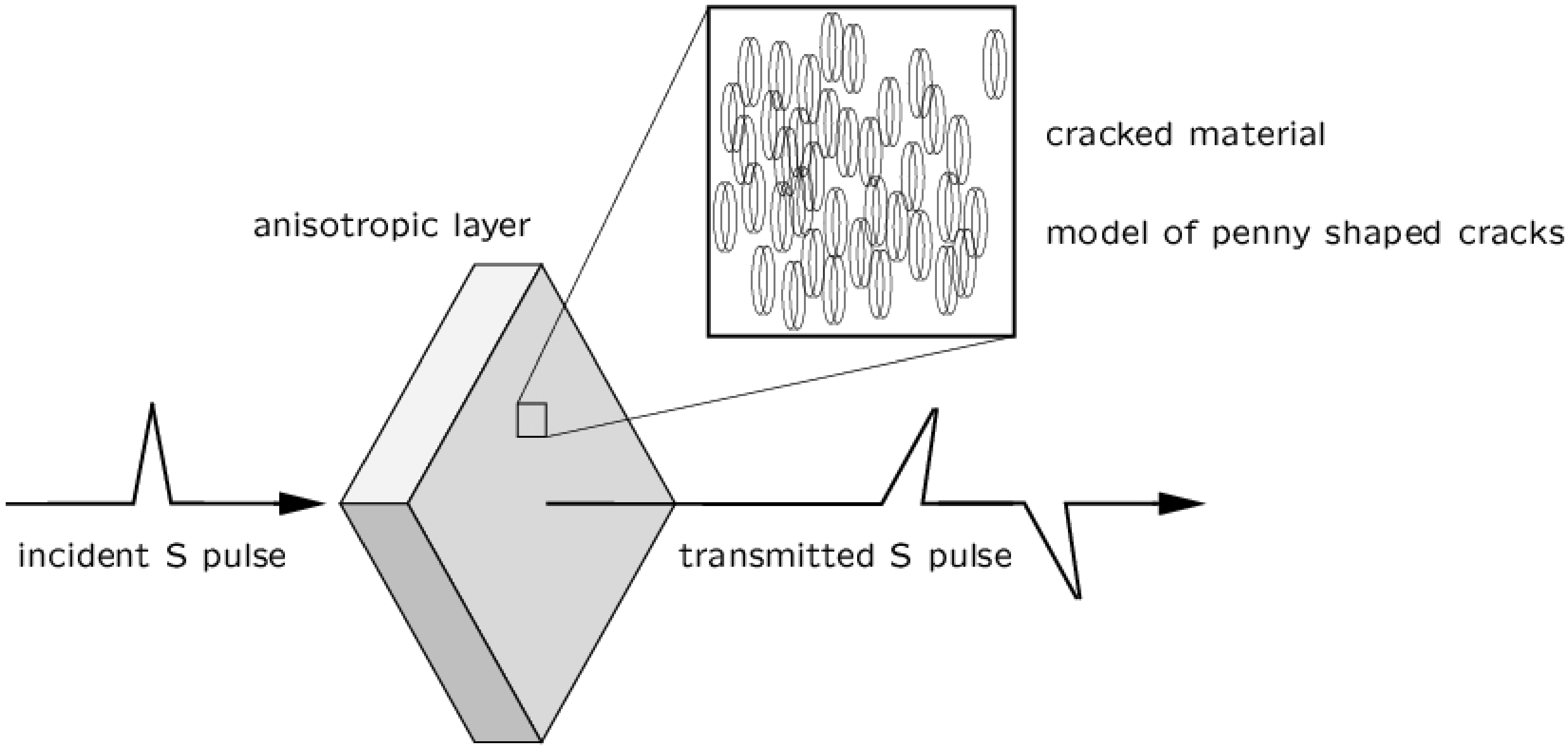
What is seismic anisotropy?

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}$$

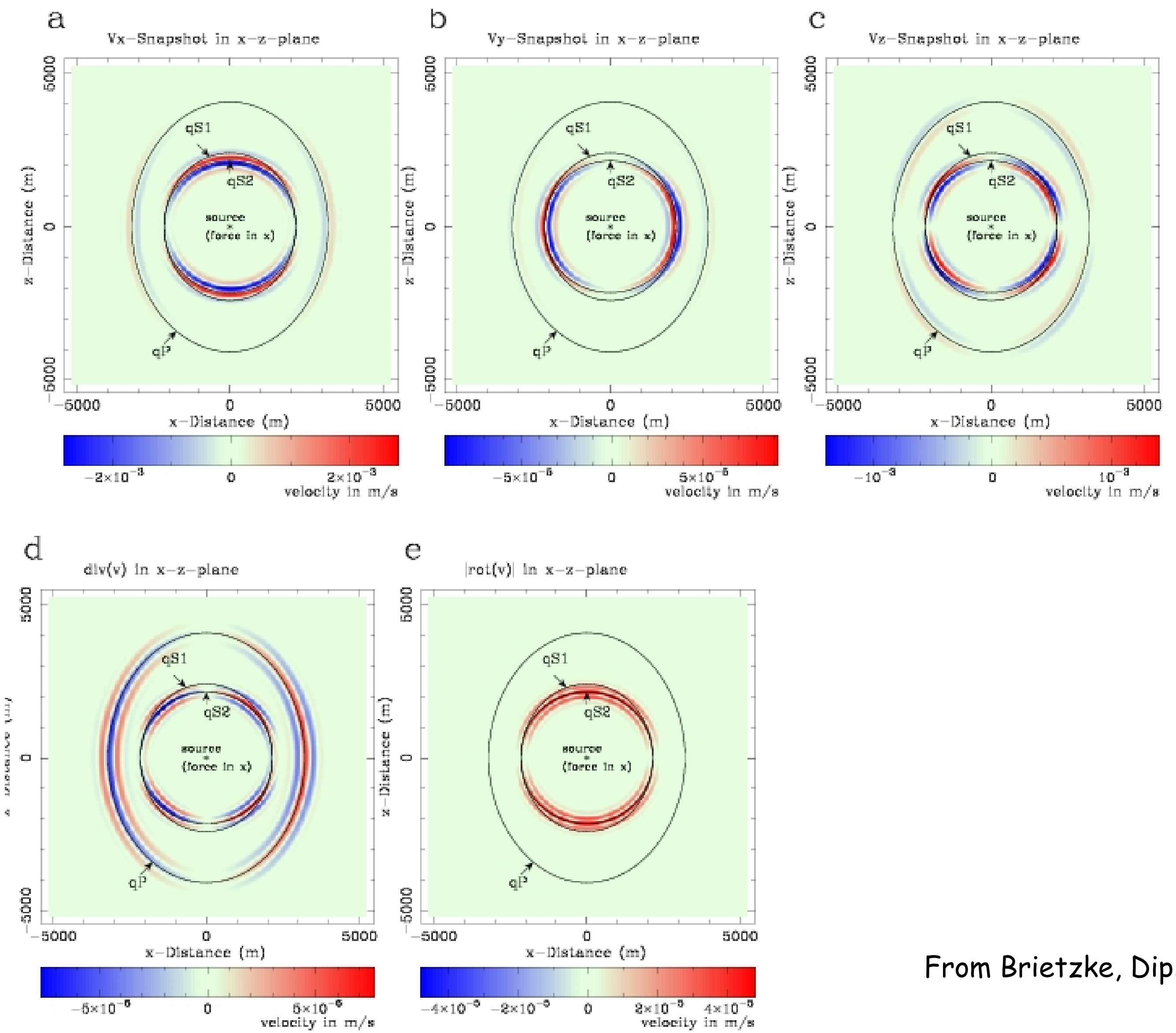
Seismic wave propagation in anisotropic media is quite different from isotropic media:

- There are in general 21 independent elastic constants (instead of 2 in the isotropic case)
- there is shear wave splitting (analogous to optical birefringence)
- waves travel at different speeds depending in the direction of propagation
- The polarization of compressional and shear waves may not be perpendicular or parallel to the wavefront, resp.

Shear-wave splitting



Anisotropic wave fronts



From Brietzke, Diplomarbeit

Elastic anisotropy - Data

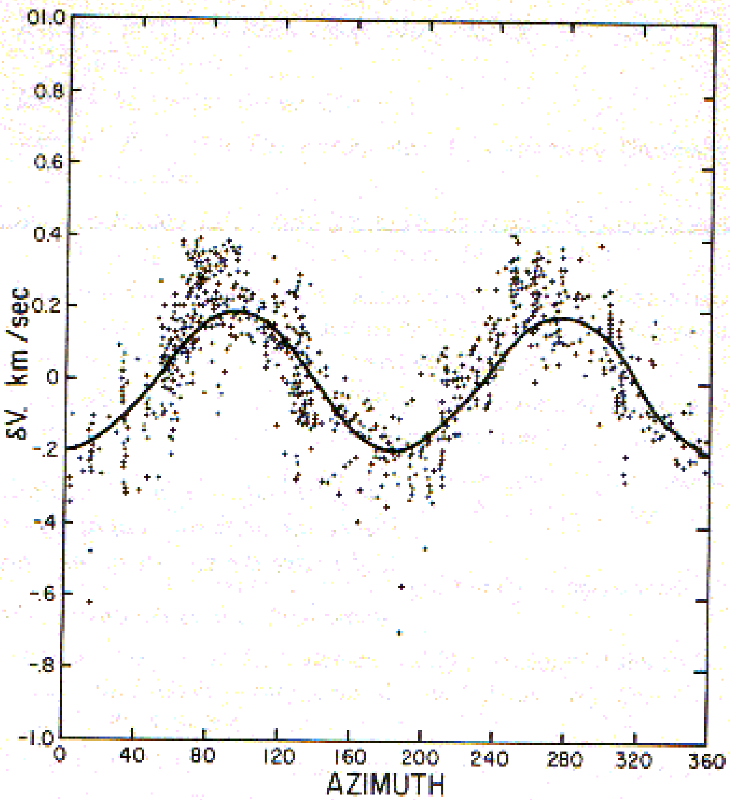


FIGURE 15-1
Azimuthal anisotropy of Pn waves in the Pacific upper mantle. Deviations are from the mean velocity of 8.159 km/s. Data points from seismic-refraction results of Morris and others (1969). The curve is the velocity measured in the laboratory for samples from the Bay of Islands ophiolite (Christensen and Salisbury, 1979).

Azimuthal variation of velocities in the upper mantle observed under the pacific ocean.

What are possible causes for this anisotropy?

- Aligned crystals
- Flow processes

Elastic anisotropy - olivine

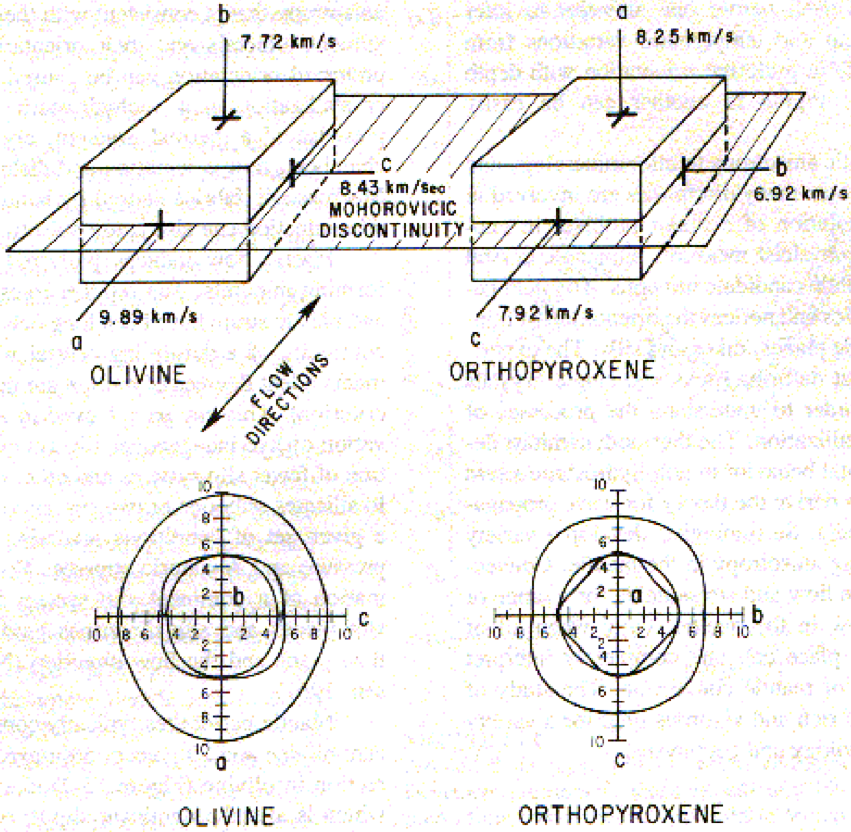


FIGURE 15-2
Olivine and orthopyroxene orientations within the upper mantle showing compressional velocities for the three crystallographic axes, and compressional and shear velocities in the olivine *a-c* plane and orthopyroxene *b-c* plane (after Christensen and Lundquist, 1982).

Explanation of observed effects with olivine crystals aligned along the direction of flow in the upper mantle

Elastic anisotropy – tensor elements

TABLE 15-5
Schematic Elastic Constant Matrices

Monoclinic

a	b	c	·	·	d
b	e	f	·	·	g
c	f	h	·	·	i
·	·	·	j	k	·
·	·	·	k	m	·
d	g	i	·	·	n

Orthorhombic

a	b	c	·	·	·
b	d	e	·	·	·
c	e	f	·	·	·
·	·	·	g	·	·
·	·	·	·	h	·
·	·	·	·	·	i

Trigonal (1)

a	b	c	d	·	·
b	a	c	$-d$	·	·
c	c	e	·	·	·
d	$-d$	·	f	·	·
·	·	·	·	f	d
·	·	·	·	d	x

Trigonal (2)

a	b	c	d	g	·
b	a	c	$-d$	$-g$	·
c	c	e	·	·	·
d	$-d$	·	f	·	$-g$
g	$-g$	·	·	f	d
·	·	·	$-g$	d	x

Tetragonal (1)

a	b	c	·	·	·
b	a	c	·	·	·
c	c	d	·	·	·
·	·	·	e	·	·
·	·	·	·	e	·
·	·	·	·	·	f

Tetragonal (2)

a	b	c	·	·	g
b	a	c	·	·	$-g$
c	c	d	·	·	·
·	·	·	e	·	·
·	·	·	·	e	·
g	$-g$	·	·	·	f

Hexagonal

a	b	c	·	·	·
b	a	c	·	·	·
c	c	d	·	·	·
·	·	·	e	·	·
·	·	·	·	e	·
·	·	·	·	·	x

Cubic

a	b	b	·	·	·
b	a	b	·	·	·
b	b	a	·	·	·
·	·	·	c	·	·
·	·	·	·	c	·
·	·	·	·	·	c

Isotropic

a	b	b	·	·	·
b	a	b	·	·	·
b	b	a	·	·	·
·	·	·	x	·	·
·	·	·	·	x	·
·	·	·	·	·	x

$x = (a - b)/2.$

Elastic anisotropy – applications

Crack-induced anisotropy

Pore space aligns itself in the stress field. Cracks are aligned perpendicular to the minimum compressive stress. The orientation of cracks is of enormous interest to reservoir engineers!

Changes in the stress field may alter the density and orientation of cracks. Could time-dependent changes allow prediction of ruptures, etc. ?

SKS - Splitting

Could anisotropy help in understanding mantle flow processes?

Equations of motion

We now have a complete description of the forces acting within an elastic body. Adding the inertia forces with opposite sign leads us from

$$f_i + \frac{\partial \sigma_{ij}}{\partial x_j} = 0$$

to

$$\rho \frac{\partial^2 u_i}{\partial t^2} = f_i + \frac{\partial \sigma_{ij}}{\partial x_j}$$

the equations of motion for dynamic elasticity

Summary: Elasticity - Stress

Seismic wave propagation can in most cases be described by **linear elasticity.**

The deformation of a medium is described by the symmetric **elasticity tensor**.

The internal forces acting on virtual planes within a medium are described by the symmetric **stress tensor**.

The stress and strain are linked by the material parameters (like spring constants) through the **generalised Hooke's Law**.

In isotropic media there are only two elastic constants, the **Lame parameters**.

In **anisotropic** media the wave speeds depend on direction and there are a maximum of 21 independant elastic constants.

The most common anisotropic symmetry systems are **hexagonal** (5) and **orthorhombic** (9 independent constants).