Elasticity and Seismic Waves

- Some mathematical basics

- Strain-displacement relation
  - Linear elasticity
  - Strain tensor - meaning of its elements

- Stress-strain relation (Hooke’s Law)
  - Stress tensor
  - Symmetry
  - Elasticity tensor
  - Lame’s parameters

- Equation of Motion
  - P and S waves
  - Plane wave solutions
• Linear elasticity (teleseismic waves)
• rupture, breaking
• stable slip (aseismic)
• stick-slip (with sudden ruptures)
Linear and non-linear stress and strain

Stress vs. strain for a loading cycle with rock that breaks. For wave propagation problems assuming linear elasticity is usually sufficient.
Horizontal stresses are influenced by tectonic forces (regional and local). This implies that usually there are two uneven horizontal principal stress directions.

Example: Cologne Basin

When all three orthogonal principal stresses are equal we speak of hydrostatic stress.
A time-dependent perturbation of an elastic medium (e.g. a rupture, an earthquake, a meteorite impact, a nuclear explosion etc.) generates elastic waves emanating from the source region. These disturbances produce local changes in stress and strain.

To understand the propagation of elastic waves we need to describe kinematically the deformation of our medium and the resulting forces (stress). The relation between deformation and stress is governed by elastic constants.

The time-dependence of these disturbances will lead us to the elastic wave equation as a consequence of conservation of energy and momentum.
The mathematical description of deformation processes heavily relies on vector analysis. We therefore review the fundamental concepts of vectors and tensors.

Usually vectors are written in boldface type, \( \mathbf{x} \) is a scalar but \( \mathbf{y} \) is a vector, \( y_i \) are the scalar components of a vector

\[
\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \quad a\mathbf{y} = \begin{pmatrix} ay_1 \\ ay_2 \\ ay_3 \end{pmatrix} \quad a\mathbf{y} + b\mathbf{x} = \begin{pmatrix} ay_1 + bx_1 \\ ay_2 + bx_2 \\ ay_3 + bx_3 \end{pmatrix}
\]

Scalar or Dot Product

\[
\mathbf{a} \cdot \mathbf{b} = (a_1b_1 + a_2b_2 + a_3b_3) = |a| |b| \cos \theta
\]

\[
|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}
\]
The vector cross product is defined as:

\[
a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}
\]

The triple scalar product is defined as

\[
a \cdot (b \times c)
\]

which is a scalar and represents the volume of the parallelepiped defined by \(a, b,\) and \(c.\) It is also calculated like a determinant:

\[
a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}
\]
Assume that we have a scalar field \( \Phi(x) \), we want to know how it changes with respect to the coordinate axes, this leads to a vector called the **gradient of \( \Phi \)**

\[
\nabla \Phi = \begin{pmatrix}
\frac{\partial}{\partial x} \Phi \\
\frac{\partial}{\partial y} \Phi \\
\frac{\partial}{\partial z} \Phi
\end{pmatrix}
\]

With the **nabla operator** \( \nabla = \begin{pmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{pmatrix} \) and \( \frac{\partial}{\partial x} = \frac{\partial}{\partial x} \)

The gradient is a vector that points in the direction of maximum rate of change of the scalar function \( \Phi(x) \).

**What happens if we have a vector field?**
The **divergence** is the scalar product of the nabla operator with a vector field \( \mathbf{V}(\mathbf{x}) \). The divergence of a vector field is a scalar!

\[
\nabla \cdot \mathbf{V} = \partial_x V_x + \partial_y V_y + \partial_z V_z
\]

Physically the divergence can be interpreted as the net flow out of a volume (or change in volume). E.g. the divergence of the seismic wavefield corresponds to compressional waves.

The **curl** is the vector product of the nabla operator with a vector field \( \mathbf{V}(\mathbf{x}) \). The curl of a vector field is a vector!

\[
\nabla \times \mathbf{V} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\partial_x & \partial_y & \partial_z \\
V_x & V_y & V_z
\end{vmatrix} = \begin{pmatrix}
\partial_y V_z - \partial_z V_y \\
\partial_z V_x - \partial_x V_z \\
\partial_x V_y - \partial_y V_x
\end{pmatrix}
\]

The curl of a vector field represents the rotational part of that field (e.g. shear waves in a seismic wavefield)
Gauss' theorem is a relation between a volume integral over the divergence of a vector field $\mathbf{F}$ and a surface integral over the values of the field at its surface $S$:

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{F} dV$$

... it is one of the most widely used relations in mathematical physics. The physical interpretation is again that the value of this integral can be considered the net flow out of volume $V$. 
Let us consider a point $P_0$ at position $r$ relative to some fixed origin and a second point $Q_0$ displaced from $P_0$ by $dx$.

**Unstrained state:**
Relative position of point $P_0$ w.r.t. $Q_0$ is $\delta x$.

**Strained state:**
Relative position of point $P_0$ has been displaced a distance $u$ to $P_1$ and point $Q_0$ a distance $v$ to $Q_1$.

Relative position of point $P_1$ w.r.t. $Q_1$ is $\delta y = \delta x + \delta u$. The change in relative position between $Q$ and $P$ is just $\delta u$. 

The change in relative position between $Q$ and $P$ is just $\delta u$. 

$\delta x$ $\delta y$ $\delta u$ $\delta v$
The relative displacement in the unstrained state is \( u(r) \). The relative displacement in the strained state is \( v = u(r + \delta x) \).

So finally we arrive at expressing the relative displacement due to strain:

\[ \delta u = u(r + \delta x) - u(r) \]

We now apply Taylor’s theorem in 3-D to arrive at:

\[ \delta u_i = \frac{\partial u_i}{\partial x_k} \delta x_k \]

What does this equation mean?
The partial derivatives of the vector components

\[ \frac{\partial u_i}{\partial x_k} \]

represent a second-rank tensor which can be resolved into a symmetric and anti-symmetric part:

\[ \delta u_i = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) \delta x_k - \frac{1}{2} \left( \frac{\partial u_k}{\partial x_i} - \frac{\partial u_i}{\partial x_k} \right) \delta x_k \]

- symmetric
- deformation

- antisymmetric
- pure rotation
Linear Elasticity - deformation tensor

The symmetric part is called the deformation tensor:

\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]

and describes the relation between deformation and displacement in linear elasticity. In 2-D this tensor looks like:

\[
\varepsilon_{ij} = \begin{bmatrix}
\frac{\partial u_x}{\partial x} & \frac{1}{2} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \\
\frac{1}{2} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) & \frac{\partial u_y}{\partial y}
\end{bmatrix}
\]
Deformation tensor - its elements

Through eigenvector analysis the meaning of the elements of the deformation tensor can be clarified:

The deformation tensor assigns each point - represented by position vector $\mathbf{y}$ a new position with vector $\mathbf{u}$ (summation over repeated indices applies):

$$u_i = \varepsilon_{ij} y_j$$

The eigenvectors of the deformation tensor are those $y$'s for which the tensor is a scalar, the eigenvalues $\lambda$:

$$u_i = \lambda y_i$$

The eigenvalues $\lambda$ can be obtained solving the system:

$$|\varepsilon_{ij} - \lambda \delta_{ij}| = 0$$
Thus

\[ u_1 = \lambda_1 y_1 \]
\[ u_2 = \lambda_2 y_2 \]
\[ u_3 = \lambda_3 y_3 \]

... in other words ...

the eigenvalues are the relative change of length along the three coordinate axes

\[ \lambda_1 = \frac{u_1}{y_1} \]

shear angle

In arbitrary coordinates the diagonal elements are the relative change of length along the coordinate axes and the off-diagonal elements are the infinitesimal shear angles.
The trace of a tensor is defined as the sum over the diagonal elements. Thus:

$$\varepsilon_{ii} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

This trace is linked to the volumetric change after deformation. Before deformation the volume was $V_0$. Because the diagonal elements are the relative change of lengths along each direction, the new volume after deformation is

$$V = (1 + \varepsilon_{xx})(1 + \varepsilon_{yy})(1 + \varepsilon_{zz})$$

... and neglecting higher-order terms ...

$$V = 1 + \varepsilon_{ii} = V_0 + \varepsilon_{ii}$$

$$\Theta = \frac{\Delta V}{V_0} = \varepsilon_{ii} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \text{div} u = \nabla \cdot u$$
The fact that we have linearised the strain-displacement relation is quite severe. It means that the elements of the strain tensor should be $\ll 1$. Is this the case in seismology?

Let’s consider an example. The 1999 Taiwan earthquake ($M=7.6$) was recorded in FFB. The maximum ground displacement was 1.5mm measured for surface waves of approx. 30s period. Let us assume a phase velocity of 5km/s. How big is the strain at the Earth’s surface, give an estimate!

The answer is that $\varepsilon$ would be on the order of $10^{-7} \ll 1$. This is typical for global seismology if we are far away from the source, so that the assumption of infinitesimal displacements is acceptable.

For displacements closer to the source this assumption is not valid. There we need a finite strain theory. Strong motion seismology is an own field in seismology concentrating on effects close to the seismic source.
Strainmeter
FRACTURE PROPAGATION AROUND A COMPRESSED BOREHOLE

Loading

56 MPa

28 MPa

Pre-existing fractures around a borehole

Fractures start to propagate in shear mode

Fracture opening

Fracture propagation mainly in shear

Fracture propagation mainly in shear

Source: www.fracom.fi
In an elastic body there are restoring forces if deformation takes place. These forces can be seen as acting on planes inside the body. **Forces divided by an areas are called stresses.**

In order for the deformed body to remain deformed these forces have to compensate each other. We will see that the relationship between the stress and the deformation (strain) is linear and can be described by tensors.

The tractions $t_k$ along axis $k$ are

\[
\begin{pmatrix}
  t_{k1} \\
  t_{k2} \\
  t_{k3}
\end{pmatrix}
\]

... and along an arbitrary direction

\[
t = t_i n_i
\]

... which - using the summation convention yields ..

\[
t = t_1 n_1 + t_2 n_2 + t_3 n_3
\]
... in components we can write this as

\[ t_i = \sigma_{ij} n_j \]

where \( \sigma_{ij} \) is the stress tensor and \( n_j \) is a surface normal. The stress tensor describes the forces acting on planes within a body. Due to the symmetry condition

\[ \sigma_{ij} = \sigma_{ji} \]

there are only six independent elements.

\( \sigma_{ij} \)
The vector normal to the corresponding surface

The direction of the force vector acting on that surface
Stress equilibrium

If a body is in equilibrium the internal forces and the forces acting on its surface have to vanish

\[ \int_{V} f_i dV + \oint_{F} t_i dF = 0 \]

as well as the sum over the angular momentum

\[ \int_{V} x_i \times f_j dV + \oint_{F} x_i \times t_j dF = 0 \]

From the second equation the symmetry of the stress tensor can be derived. Using Gauss’ law the first equation yields

\[ f_i + \frac{\partial \sigma_{ij}}{\partial x_j} = 0 \]
### Stress - Glossary

| Stress units | bars \((10^6 \text{dyn/cm}^2)\)  
| | \(10^6 \text{Pa} = 1 \text{MPa} = 10 \text{bars}\)  
| | \(1 \text{ Pa} = 1 \text{ N/m}^2\)  
| | At sea level \(p = 1 \text{ bar}\)  
| | At depth 3 km \(p = 1 \text{ kbar}\)  
| maximum compressive stress | the direction perpendicular to the minimum compressive stress, near the surface mostly in horizontal direction, linked to tectonic processes.  
| principle stress axes | the direction of the eigenvectors of the stress tensor |
The relation between stress and strain in general is described by the tensor of elastic constants $c_{ijkl}$

$$\sigma_{ij} = c_{ijkl} \varepsilon_{kl}$$

Generalised Hooke's Law

From the symmetry of the stress and strain tensor and a thermodynamic condition it follows that the maximum number of independent constants of $c_{ijkl}$ is 21. In an isotropic body, where the properties do not depend on direction the relation reduces to

$$\sigma_{ij} = \lambda \Theta \delta_{ij} + 2 \mu \varepsilon_{ij}$$

Hooke's Law

where $\lambda$ and $\mu$ are the Lame parameters, $\Theta$ is the dilatation and $\delta_{ij}$ is the Kronecker delta.

$$\Theta \delta_{ij} = \varepsilon_{kk} \delta_{ij} = (\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) \delta_{ij}$$
The complete stress tensor looks like

\[
\sigma_{ij} = \begin{pmatrix}
(\lambda + 2\mu)\varepsilon_{xx} + \lambda(\varepsilon_{yy} + \varepsilon_{zz}) & 2\mu\varepsilon_{xy} & 2\mu\varepsilon_{xz} \\
2\mu\varepsilon_{yx} & (\lambda + 2\mu)\varepsilon_{yy} + \lambda(\varepsilon_{xx} + \varepsilon_{zz}) & 2\mu\varepsilon_{yz} \\
2\mu\varepsilon_{zx} & 2\mu\varepsilon_{zy} & (\lambda + 2\mu)\varepsilon_{zz} + \lambda(\varepsilon_{xx} + \varepsilon_{yy})
\end{pmatrix}
\]

There are several other possibilities to describe elasticity:
- $E$ elasticity, $\sigma$ Poisson's ratio, $k$ bulk modulus

\[
E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \quad \sigma = \frac{\lambda}{2(\lambda + \mu)} \quad k = \lambda + \frac{2}{3}\mu
\]

\[
\lambda = \frac{\sigma E}{(1 + \sigma)(1 - 2\sigma)} \quad \mu = \frac{E}{2(1 + \sigma)}
\]

For Poisson's ratio we have $0 < \sigma < 0.5$. A useful approximation is $\lambda = \mu$, then $\sigma = 0.25$. For fluids $\sigma = 0.5$ ($\mu = 0$).
As in the case of deformation the stress-strain relation can be interpreted in simple geometric terms:

\[ \sigma_{12} = \mu \gamma \]

\[ \sigma_{22} = E \frac{u}{l} \]

\[ P = K \frac{\Delta V}{V} = K \varepsilon_{ii} \]

Remember that these relations are a generalization of Hooke’s Law:

\[ F = D s \]

D being the spring constant and s the elongation.
## Seismic wave velocities: $P$-waves

<table>
<thead>
<tr>
<th>Material</th>
<th>$V_p$ (km/s)</th>
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<tbody>
<tr>
<td><strong>Unconsolidated material</strong></td>
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<tr>
<td>Sand (dry)</td>
<td>0.2-1.0</td>
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<td>Sand (wet)</td>
<td>1.5-2.0</td>
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<td><strong>Sediments</strong></td>
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<td>Sandstones</td>
<td>2.0-6.0</td>
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<td>Limestones</td>
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<td><strong>Igneous rocks</strong></td>
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<td>Granite</td>
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<td><strong>Other material</strong></td>
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<td>Steel</td>
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Elastic anisotropy

What is seismic anisotropy?

\[ \sigma_{ij} = c_{ijkl} \varepsilon_{kl} \]

Seismic wave propagation in anisotropic media is quite different from isotropic media:

- There are in general 21 independent elastic constants (instead of 2 in the isotropic case)
- there is shear wave splitting (analogous to optical birefringence)
- waves travel at different speeds depending in the direction of propagation
- The polarization of compressional and shear waves may not be perpendicular or parallel to the wavefront, resp.
Shear-wave splitting

incident S pulse

anisotropic layer

transmitted S pulse

cracked material

model of penny shaped cracks
Anisotropic wave fronts

From Brietzke, Diplomarbeit

Seismology and the Earth’s Deep Interior

Elasticity and Seismic Waves
Azimuthal variation of velocities in the upper mantle observed under the pacific ocean.

What are possible causes for this anisotropy?

- Aligned crystals
- Flow processes
Elastic anisotropy - olivine

Explanation of observed effects with olivine crystals aligned along the direction of flow in the upper mantle.
### Elastic anisotropy - tensor elements

#### TABLE 15-5
Schematic Elastic Constant Matrices

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<tr>
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<th>Orthorhombic</th>
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$x = (a - b)/2.$
Elastic anisotropy - applications

Crack-induced anisotropy

Pore space aligns itself in the stress field. Cracks are aligned perpendicular to the minimum compressive stress. The orientation of cracks is of enormous interest to reservoir engineers!

Changes in the stress field may alter the density and orientation of cracks. Could time-dependent changes allow prediction of ruptures, etc.?

SKS - Splitting

Could anisotropy help in understanding mantle flow processes?
We now have a complete description of the forces acting within an elastic body. Adding the inertia forces with opposite sign leads us from

\[ f_i + \frac{\partial \sigma_{ij}}{\partial x_j} = 0 \]

to

\[ \rho \frac{\partial^2 u_i}{\partial t^2} = f_i + \frac{\partial \sigma_{ij}}{\partial x_j} \]

the equations of motion for dynamic elasticity
Seismic wave propagation can in most cases be described by linear elasticity.

The deformation of a medium is described by the symmetric elasticity tensor.

The internal forces acting on virtual planes within a medium are described by the symmetric stress tensor.

The stress and strain are linked by the material parameters (like spring constants) through the generalised Hooke’s Law.

In isotropic media there are only two elastic constants, the Lame parameters.

In anisotropic media the wave speeds depend on direction and there are a maximum of 21 independent elastic constants.

The most common anisotropic symmetry systems are hexagonal (5) and orthorhombic (9 independent constants).