Finite Elements: 1D acoustic wave equation

- Helmholtz (wave) equation (time-dependent)
  - Regular grid
  - Irregular grid
- Explicit time integration
- Implicit time integraton
- Numerical Examples

Scope: Understand the basic concept of the finite element method applied to the 1D acoustic wave equation.

Acoustic wave equation in 1D

How do we solve a time-dependent problem such as the acoustic wave equation?

$$\boxed{\partial_t^2 u - v^2 \Delta u = f}$$

where *v* is the wave speed.
using the same ideas as before we multiply this equation with an arbitrary function and integrate over the whole domain, e.g. [0,1], and after partial integration

$$\boxed{\int_0^1 \partial_t^2 u \varphi_j dx - v^2 \int_0^1 \nabla u \nabla \varphi_j dx = \int_0^1 f \varphi_j dx}$$

.. we now introduce an approximation for u using our previous basis functions...

Weak form of wave equation

$$\boxed{u \approx \tilde{u} = \sum_{i=1}^{N} c_i(t)\varphi_i(x)}$$

note that now our coefficients are time-dependent!

... and ...

$$\partial_t^2 u \approx \partial_t^2 \tilde{u} = \partial_t^2 \sum_{i=1}^{N} c_i(t)\varphi_i(x)$$

together we obtain

$$\boxed{\left[\sum_i \partial_t^2 c_i \int_0^1 \varphi_i \varphi_j dx\right] + v^2 \left[\sum_i c_i \int_0^1 \nabla\varphi_i \nabla\varphi_j dx\right] = \int_0^1 f\varphi_j}$$

which we can write as ...

Time extrapolation

$$\left[\sum_i \partial_t^2 c_i \int_0^1 \varphi_i \varphi_j dx\right] + v^2 \left[\sum_i c_i \int_0^1 \nabla \varphi_i \nabla \varphi_j dx\right] = \int_0^1 f \varphi_j$$

M
mass matrix

A
stiffness matrix

b

... in Matrix form ...

$$\boxed{M^T \ddot{c} + v^2 A^T c = g}$$

... remember the coefficients c correspond to the actual values of u at the grid points for the right choice of basis functions ...

How can we solve this time-dependent problem?

Time extrapolation

$$\boxed{M^T \ddot{c} + v^2 A^T c = g}$$

... let us use a finite-difference approximation for the time derivative ...

$$M^T \left(\frac{c_{k+1} - 2c + c_{k-1}}{dt^2} \right) + v^2 A^T c_k = g$$

... leading to the solution at time t_{k+1}:

$$\boxed{c_{k+1} = \left[(M^T)^{-1} (g - v^2 A^T c_k) \right] dt^2 + 2c_k - c_{k-1}}$$

we already know how to calculate the matrix A but how can we calculate matrix M?

Mass matrix

$$\boxed{\left[\sum_i \partial_t^2 c_i \boxed{\int_0^1 \varphi_i \varphi_j dx}\right] + v^2 \left[\sum_i c_i \int_0^1 \nabla \varphi_i \nabla \varphi_j dx\right] = \int_0^1 f \varphi_j}$$

... let's recall the definition of our basis functions ...

$$M_{ij} = \int_0^1 \varphi_i \varphi_j dx$$

$$\varphi_i(\tilde{x}) = \begin{cases} \dfrac{\tilde{x}}{h_{i-1}} + 1 & for \quad -h_{i-1} < \tilde{x} \le 0 \\ 1 - \dfrac{\tilde{x}}{h_i} & for \quad 0 < \tilde{x} < h_i \\ 0 & elsewhere \end{cases} \quad , \tilde{x} = x - x_i$$

i=1	2	3	4	5	6	7
+	+	+	+	+	+	+
	h_1	h_2	h_3	h_4	h_5	h_6

... let us calculate some element of M ...

Mass matrix – some elements

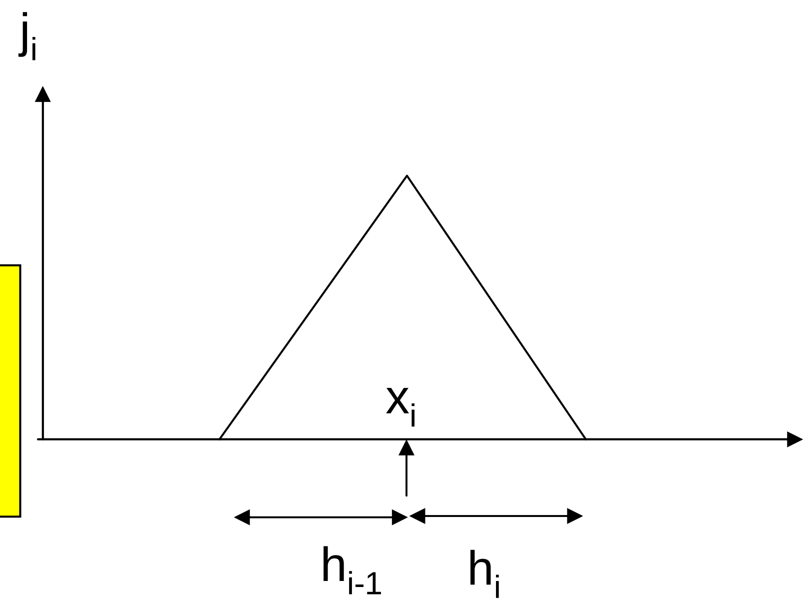
Diagonal elements: M_{ii}, i=2,n-1

$$\varphi_i(\tilde{x}) = \begin{cases} \dfrac{\tilde{x}}{h_{i-1}} + 1 & for \quad -h_{i-1} < \tilde{x} \le 0 \\ 1 - \dfrac{\tilde{x}}{h_i} & for \quad 0 < \tilde{x} < h_i \\ 0 & elsewhere \end{cases}$$

$$M_{ii} = \int_0^1 \varphi_i \varphi_i dx = \int_0^{h_{i-1}} \left(\frac{x}{h_{i-1}} \right)^2 dx + \int_0^{h_i} \left(1 - \frac{x}{h_i} \right)^2 dx$$

$$= \boxed{\frac{h_{i-1}}{3} + \frac{h_i}{3}}$$

i=1		2		3		4		5		6		7	
+		+		+		+		+		+		+	
	h_1		h_2		h_3		h_4		h_5		h_6		



Matrix assembly

M_{ij}

```
% assemble matrix Mij

M=zeros(nx);

for i=2:nx-1,
   for j=2:nx-1,
      if i==j,
         M(i,j)=h(i-1)/3+h(i)/3;
      elseif j==i+1
         M(i,j)=h(i)/6;
      elseif j==i-1
         M(i,j)=h(i)/6;
      else
         M(i,j)=0;
      end
   end
end
```

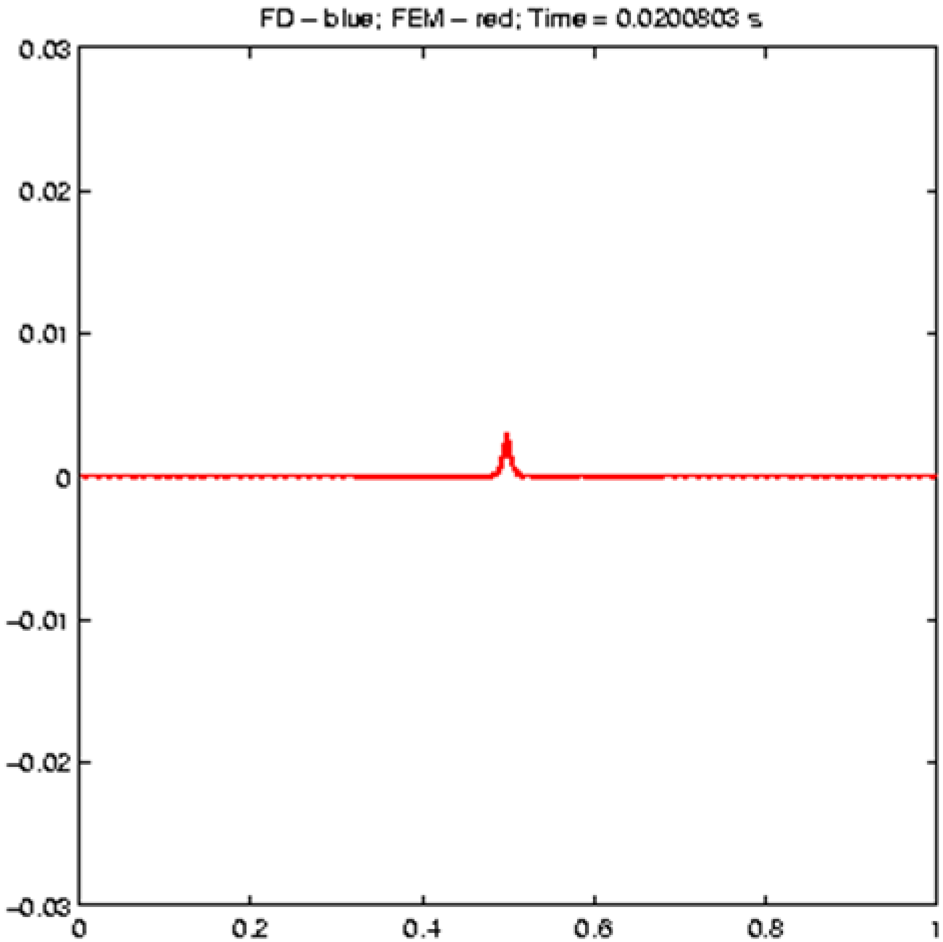
A_{ij}

```
% assemble matrix Aij

A=zeros(nx);

for i=2:nx-1,
   for j=2:nx-1,
      if i==j,
         A(i,j)=1/h(i-1)+1/h(i);
      elseif i==j+1
         A(i,j)=-1/h(i-1);
      elseif i+1==j
         A(i,j)=-1/h(i);
      else
         A(i,j)=0;
      end
   end
end
```

Numerical example



Implicit time integration

$$\boxed{M^T \ddot{c} + v^2 A^T c = g}$$

... let us use an implicit finite-difference approximation for the time derivative ...

$$M^T \left(\frac{c_{k+1} - 2c + c_{k-1}}{dt^2} \right) + v^2 A^T c_{\boxed{k+1}} = g$$

... leading to the solution at time t_{k+1}:

$$\boxed{c_{k+1} = \left[M^T + v^2 dt^2 A^T \right]^{-1} \left(g dt^2 + M^T \left(2c - c_{k-1} \right) \right)}$$

How do the numerical solutions compare?

Summary

The time-dependent problem (wave equation) leads to the introduction of the mass matrix.

The numerical solution requires the inversion of a system matrix (it may be sparse).

Both explicit or implicit formulations of the time-dependent part are possible.