Function approximation: Fourier, Chebyshev, Lagrange

- Orthogonal functions
- Fourier Series
- Discrete Fourier Series
- Fourier Transform: properties
- Chebyshev polynomials
- Convolution
- DFT and FFT

Scope: Understanding where the Fourier Transform comes from. Moving from the continuous to the discrete world. The concepts are the basis for pseudospectral methods and the spectral element approach.

Fourier Series: one way to derive them

The Problem

we are trying to approximate a function f(x) by another function $g_n(x)$ which consists of a sum over N *orthogonal* functions $\Phi(x)$ weighted by some coefficients a_n.

$$f(x) \approx g_N(x) = \sum_{i=0}^{N} a_i \Phi_i(x)$$

The Problem

... and we are looking for optimal functions in a least squares (l_2) sense ...

$$\left\| f(x) - g_N(x) \right\|_{l_2} = \left[\int_a^b \left\{ f(x) - g_N(x) \right\}^2 dx \right]^{1/2} = \mathrm{Min!}$$

... a good choice for the basis functions Φ(x) are *orthogonal* functions. *What are orthogonal functions?* Two functions f and g are said to be orthogonal in the interval [a,b] if

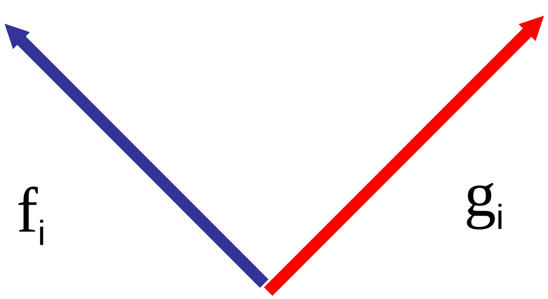
$$\int_a^b f(x) g(x) dx = 0$$

How is this related to the more conceivable concept of orthogonal vectors? Let us look at the original definition of integrals:

Orthogonal Functions

$$\int_a^b f(x)g(x)dx = \lim_{N\to\infty}\left(\sum_{i=1}^{N} f_i(x)g_i(x)\Delta x\right)$$

... where $x_0=a$ and $x_N=b$, and $x_i-x_{i-1}=\Delta x$ ...
If we interpret $f(x_i)$ and $g(x_i)$ as the ith components of an N component vector, then this sum corresponds directly to a scalar product of vectors. The vanishing of the scalar product is the condition for *orthogonality* of vectors (or functions).

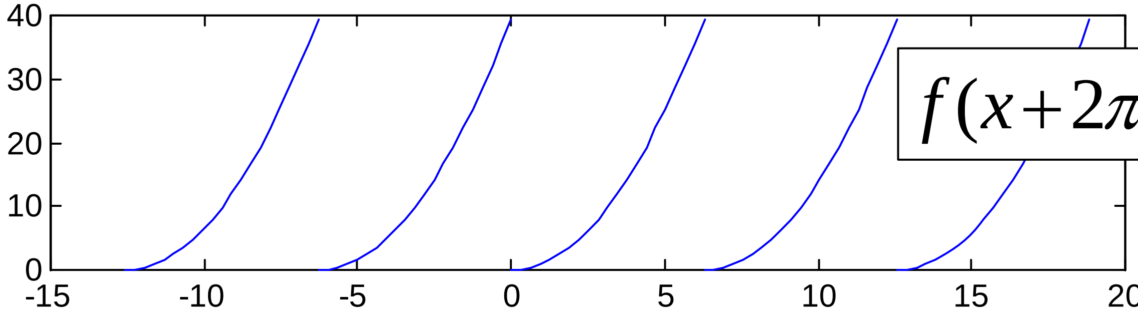


$$f_i \bullet g_i = \sum_i f_i g_i = 0$$

Periodic functions

Let us assume we have a piecewise continuous function of the form

$$f(x+2\pi)=f(x)$$



$$\boxed{f(x+2\pi)=f(x)=x^2}$$

... we want to approximate this function with a linear combination of 2π periodic functions:

$$1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots, \cos(nx), \sin(nx)$$

$$\boxed{\Rightarrow f(x)\approx g_N(x)=\frac{1}{2}a_0+\sum_{k=1}^{N}\left\{a_k\cos(kx)+b_k\sin(kx)\right\}}$$

Orthogonality

... are these functions orthogonal ?

$$\int_{-\pi}^{\pi} \cos(jx)\cos(kx)dx = \begin{cases} 0 & j \neq k \\ 2\pi & j = k = 0 \\ \pi & j = k > 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(jx)\sin(kx)dx = \begin{cases} 0 & j \neq k,\ j,k > 0 \\ \\ \pi & j = k > 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(jx)\sin(kx)dx = 0 \qquad j \geq 0, k > 0$$

... YES, and these relations are valid for any interval of length 2π.
Now we know that this is an orthogonal basis, but how can we obtain the coefficients for the basis functions?

from minimising f(x)-g(x)

Fourier coefficients

optimal functions g(x) are given if

$$\left\| g_n(x) - f(x) \right\|_2 = \mathrm{Min}! \qquad or \qquad \frac{\partial}{\partial a_k}\left\{ \left\| g_n(x) - f(x) \right\|_2 \right\} = 0$$

... with the definition of g(x) we get ...

$$\frac{\partial}{\partial a_k}\left\| g_n(x) - f(x) \right\|_2^2 = \frac{\partial}{\partial a_k}\left[\int_{-\pi}^{\pi} \left[\frac{1}{2} a_0 + \sum_{k=1}^{N} \left\{ a_k \cos(kx) + b_k \sin(kx) \right\} - f(x) \right]^2 dx \right]$$

leading to

$$g_N(x) = \frac{1}{2} a_0 + \sum_{k=1}^{N} \left\{ a_k \cos(kx) + b_k \sin(kx) \right\} \qquad \text{with}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx, \qquad k = 0,1,\ldots, N$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx, \qquad k = 1,2,\ldots, N$$

Fourier approximation of |x|

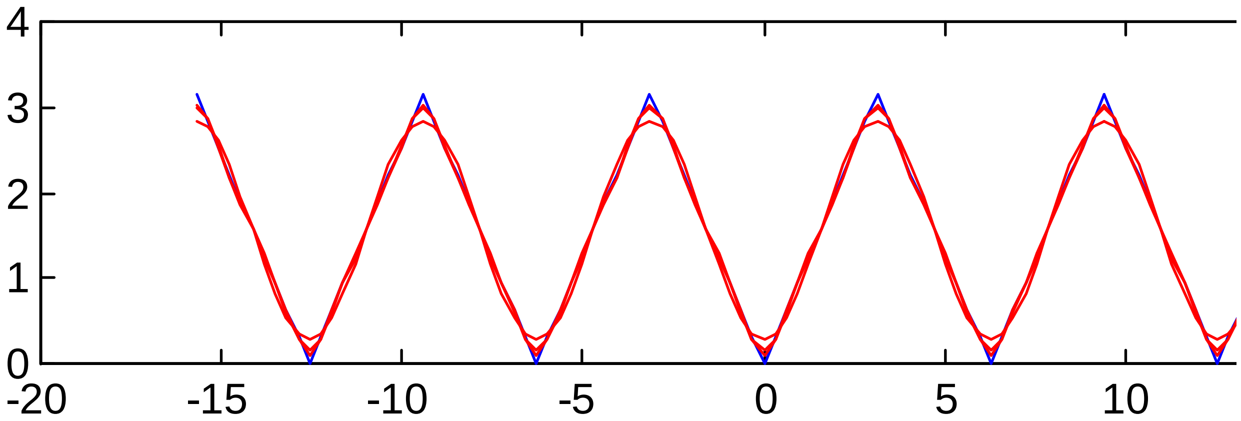
... Example ...

$$f(x) = |x|, \qquad -\pi \le x \le \pi$$

leads to the Fourier Serie

$$g(x) = \frac{1}{2}\pi - \frac{4}{\pi}\left\{ \frac{\cos(x)}{1^2} + \frac{\cos(3x)}{3^2} + \frac{\cos(5x)}{5^2} + \dots \right\}$$

.. and for n<4 g(x) looks like



Fourier approximation of x^2

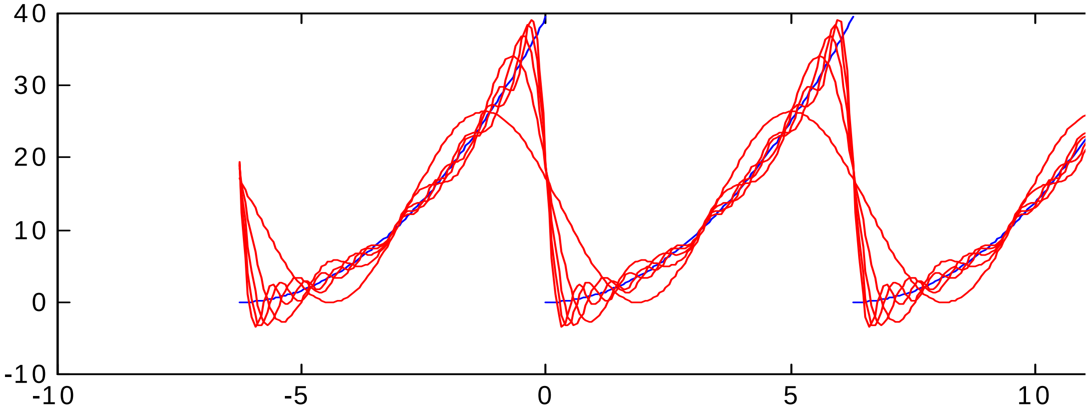
... another Example ...

$$f(x) = x^2, \qquad 0 < x < 2\pi$$

leads to the Fourier Serie

$$g_N(x) = \frac{4\pi^2}{3} + \sum_{k=1}^{N} \left\{ \frac{4}{k^2}\cos(kx) - \frac{4\pi}{k}\sin(kx) \right\}$$

.. and for N<11, g(x) looks like



Fourier - discrete functions

... what happens if we know our function f(x) only at the points

$$x_i = \frac{2\pi}{N} i$$

it turns out that in this *particular* case the coefficients are given by

$$a_k^* = \frac{2}{N} \sum_{j=1}^{N} f(x_j) \cos(kx_j), \qquad k = 0,1,2,...$$

$$b_k^* = \frac{2}{N} \sum_{j=1}^{N} f(x_j) \sin(kx_j), \qquad k = 1,2,3,...$$

.. the so-defined Fourier polynomial is the unique interpolating function to the function $f(x_j)$ with N=2m

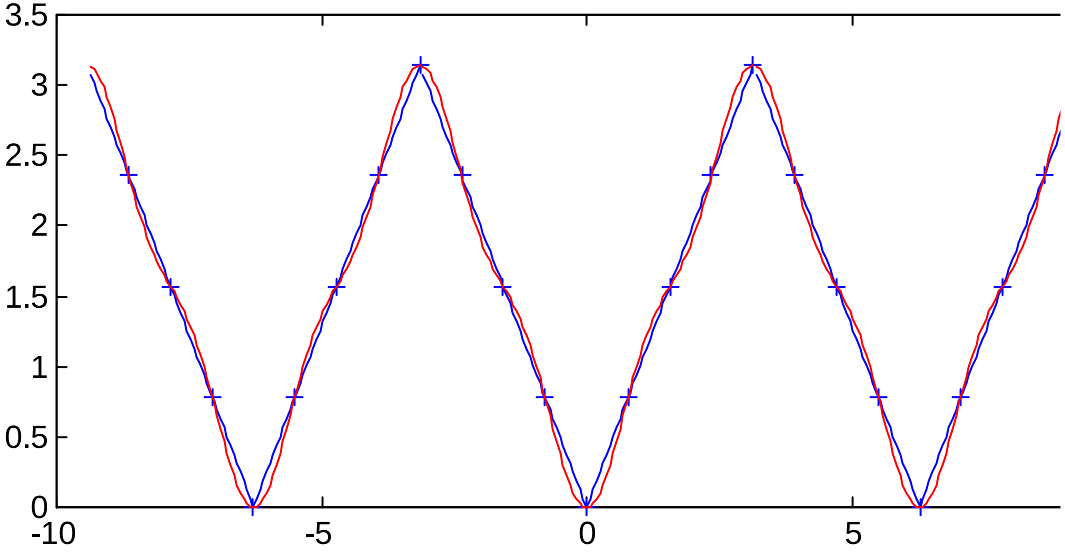
$$g_m^*(x) = \frac{1}{2} a_0^* + \sum_{k=1}^{m-1} \left\{ a_k^* \cos(kx) + b_k^* \sin(kx) \right\} + \frac{1}{2} a_m^* \cos(kx)$$

Fourier - collocation points

... with the important property that ...

$$\boxed{g_m^*(x_i) = f(x_i)}$$

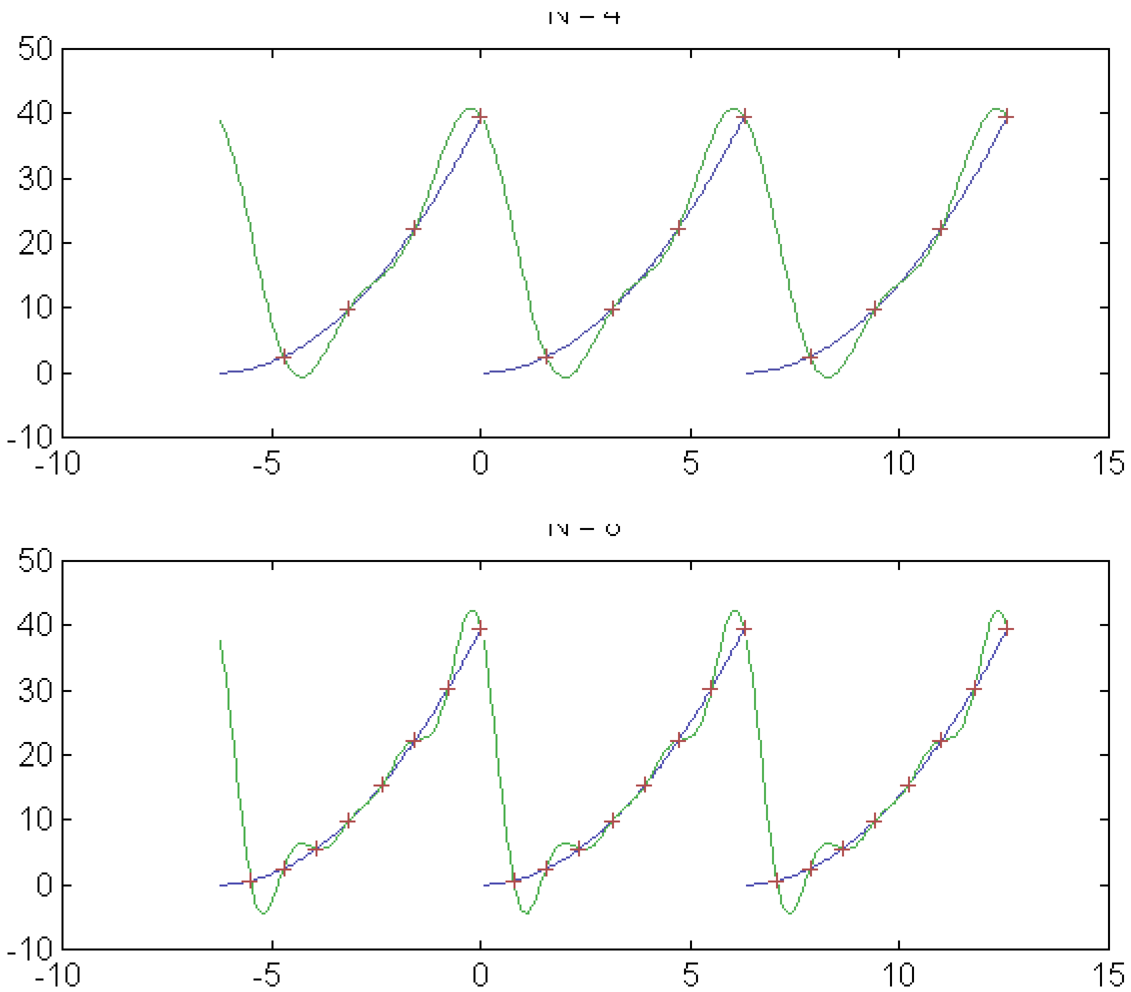
... in our previous examples ...



f(x)=|x| => f(x) - blue ; g(x) - red; x_i - '+'

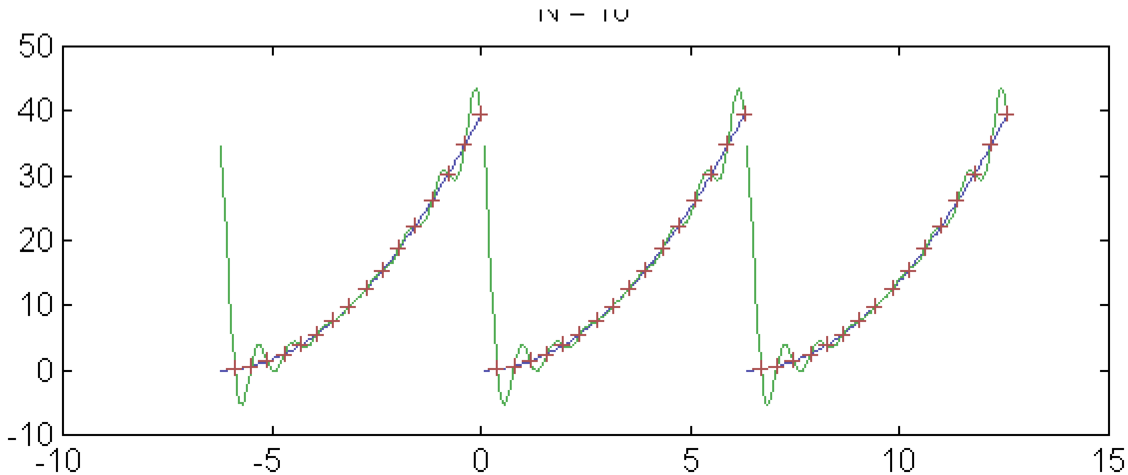
Fourier series - convergence

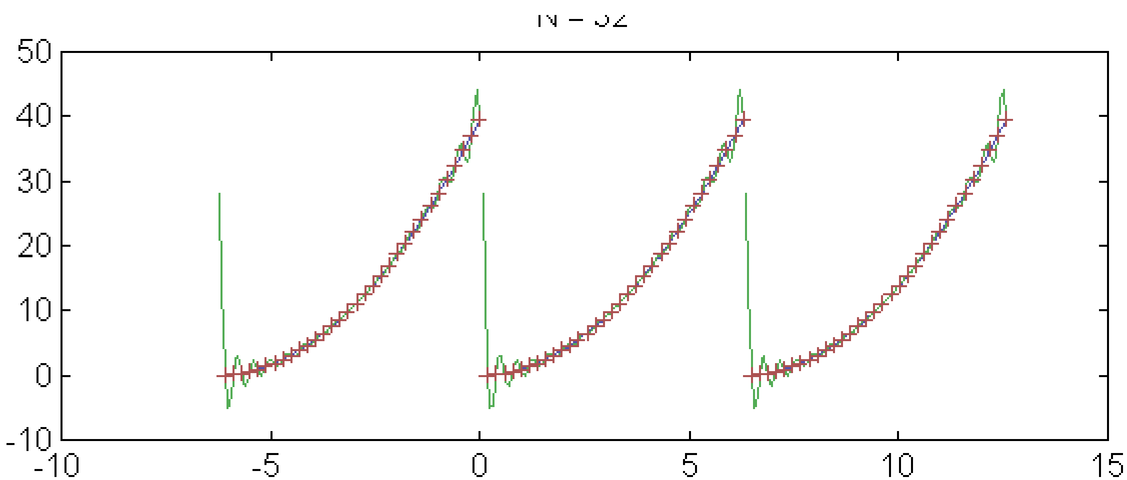
$f(x)=x^2$ => $f(x)$ - blue ; $g(x)$ - red; x_i - ‘+’



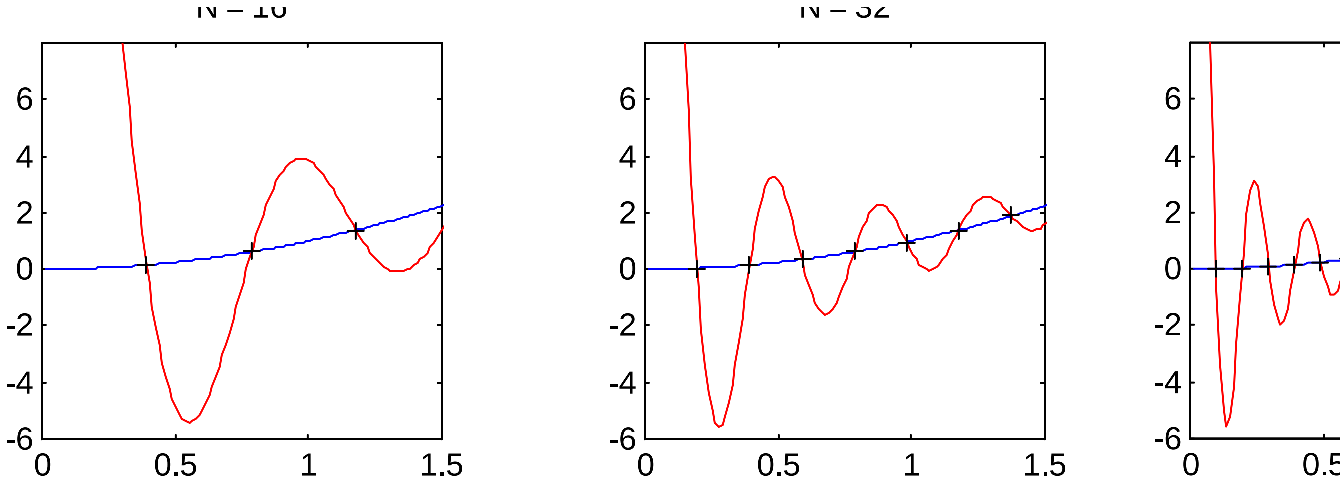
Fourier series - convergence

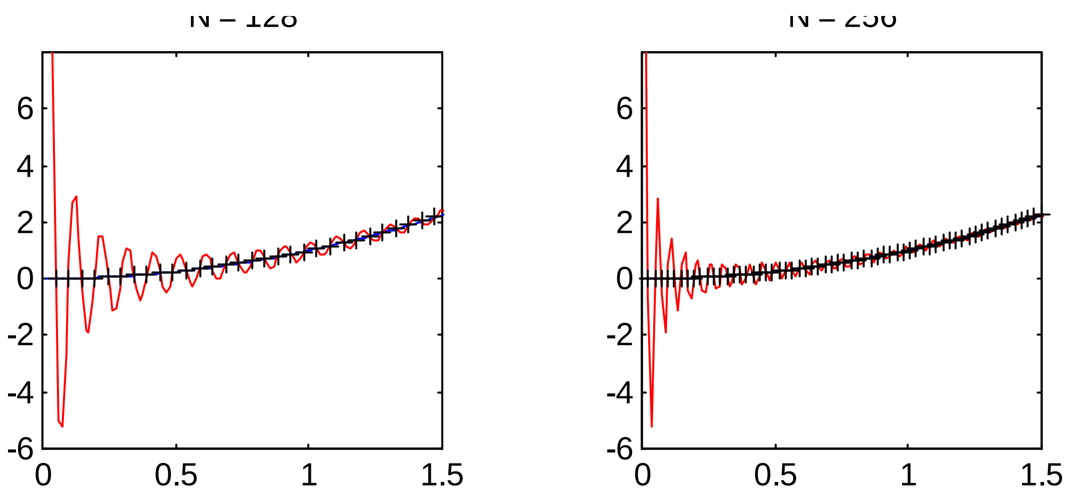
$f(x)=x^2$ => $f(x)$ - blue ; $g(x)$ - red; x_i - '+'





$f(x)=x^2$ => $f(x)$ - blue ; $g(x)$ - red; x_i - '+'





The overshoot for equi-spaced Fourier interpolations is ≈14% of the step height.

Chebyshev polynomials

We have seen that Fourier series are excellent for interpolating (and differentiating) periodic functions defined on a regularly spaced grid. In many circumstances physical phenomena which are not periodic (in space) and occur in a *limited area.* This quest leads to the use of **Chebyshev polynomials**.

We depart by observing that cos(nφ) can be expressed by a polynomial in cos(φ):

$$\cos(2\varphi) = 2\cos^2\varphi - 1$$

$$\cos(3\varphi) = 4\cos^3\varphi - 3\cos\varphi$$

$$\cos(4\varphi) = 8\cos^4\varphi - 8\cos^2\varphi + 1$$

... which leads us to the definition:

Chebyshev polynomials - definition

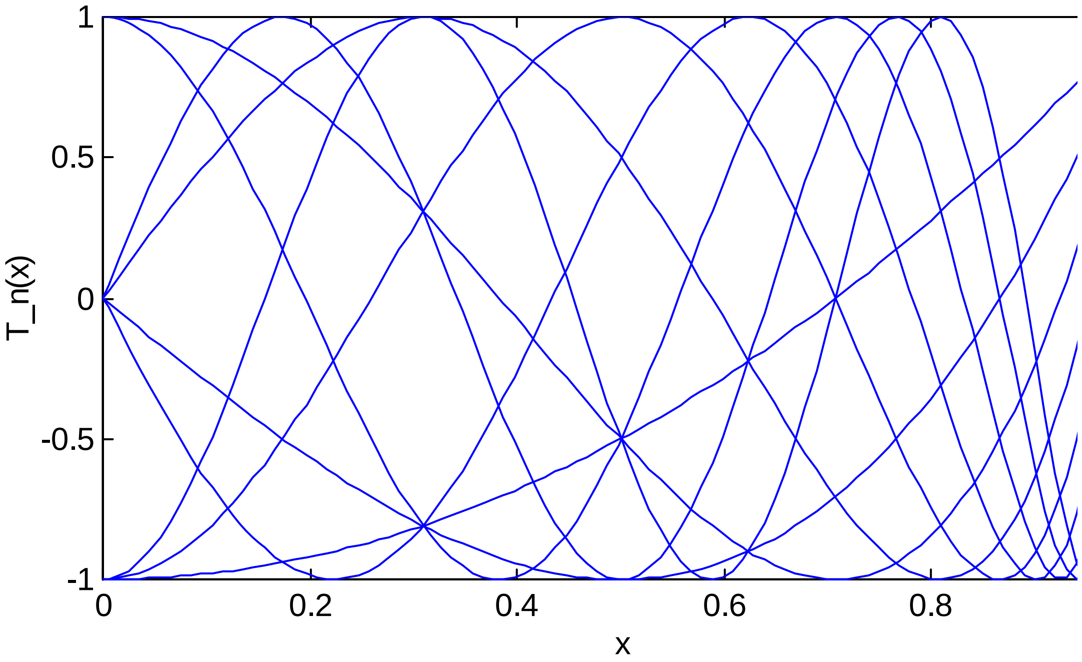
$$\cos(n\varphi) = T_n(\cos(\varphi)) = T_n(x), \qquad x = \cos(\varphi), \qquad x \in [-1,1], \qquad n \in N$$

... for the Chebyshev polynomials $T_n(x)$. Note that because of $x=\cos(\varphi)$ they are defined in the interval [-1,1] (which - however - can be extended to $\Re$). The first polynomials are

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$
$$T_4(x) = 8x^4 - 8x^2 + 1 \qquad \text{where}$$
$$|T_n(x)| \le 1 \quad \text{for} \quad x \in [-1,1] \quad \text{and} \quad n \in N_0$$

Chebyshev polynomials - Graphical

The first ten polynomials look like [0, -1]

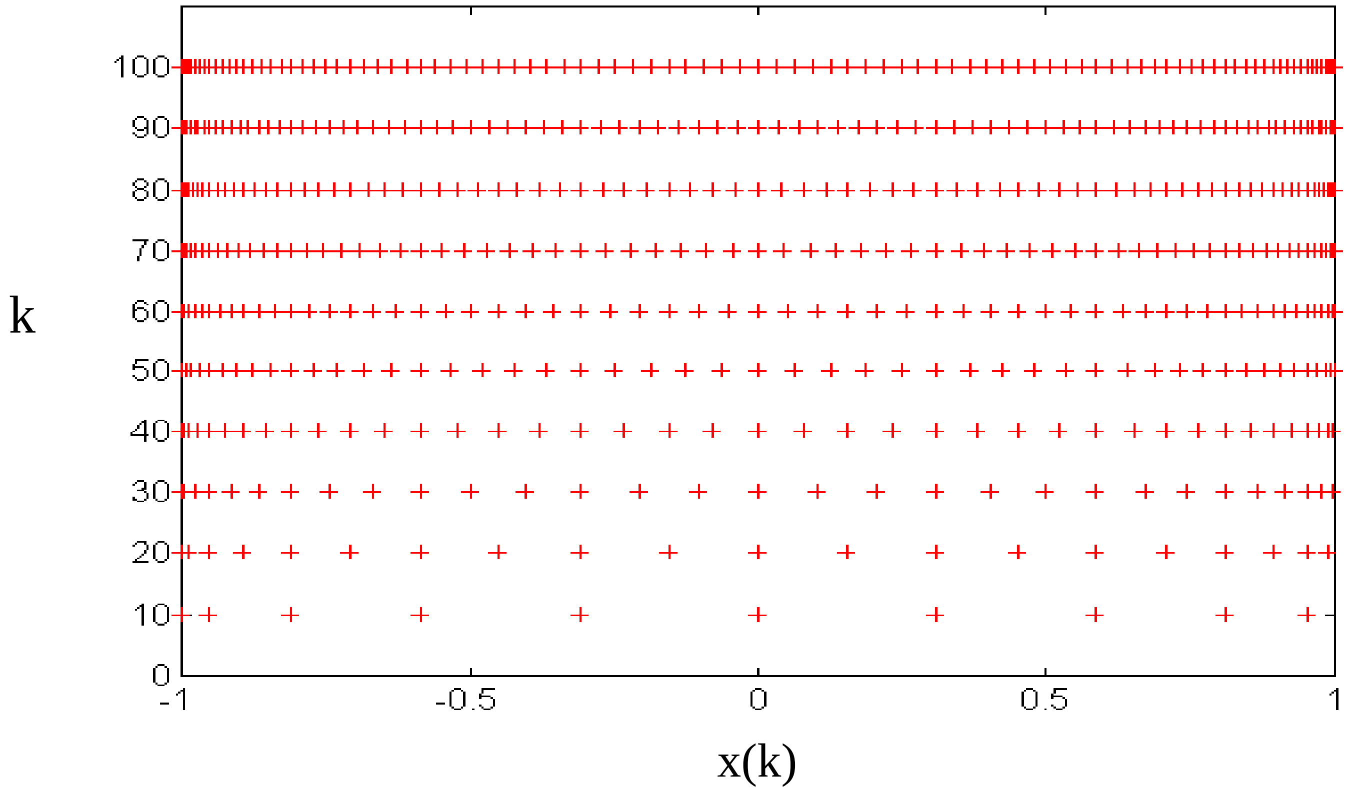


The n-th polynomial has extrema with values 1 or -1 at

$$x_k^{(ext)} = \cos(\frac{k\pi}{n}), \qquad k = 0,1,2,3,..., n$$

Chebyshev collocation points

These extrema are not equidistant (like the Fourier extrema)



$$x_k^{(ext)} = \cos\left(\frac{k\pi}{n}\right), \qquad k = 0,1,2,3,\ldots, n$$

Chebyshev polynomials - orthogonality

... are the Chebyshev polynomials orthogonal?

Chebyshev polynomials are an orthogonal set of functions in the interval [-1,1] with respect to the weight function $1/\sqrt{1-x^2}$ *such that*

$$\int_{-1}^{1} T_k(x) T_j(x) \frac{dx}{\sqrt{1-x^2}} = \left\{ \begin{matrix} 0 & for & k \neq j \\ \pi/2 & for & k = j > 0 \\ \pi & for & k = j = 0 \end{matrix} \right\}, \qquad k, j \in N_0$$

... this can be easily verified noting that

$$x = \cos\varphi, \qquad dx = -\sin\varphi d\varphi$$

$$T_k(x) = \cos(k\varphi), \quad T_j(x) = \cos(j\varphi)$$

Chebyshev polynomials - interpolation

... we are now faced with the same problem as with the Fourier series. We want to approximate a function f(x), this time not a periodical function but a function which is defined between [-1,1]. We are looking for $g_n(x)$

$$f(x) \approx g_n(x) = \frac{1}{2} c_0 T_0(x) + \sum_{k=1}^{n} c_k T_k(x)$$

... and we are faced with the problem, how we can determine the coefficients c_k. Again we obtain this by finding the extremum (minimum)

$$\frac{\partial}{\partial c_k}\left[\int_{-1}^{1} \left\{g_n(x) - f(x)\right\}^2 \frac{dx}{\sqrt{1-x^2}}\right] = 0$$

Chebyshev polynomials - interpolation

... to obtain ...

$$c_k = \frac{2}{\pi}\int_{-1}^{1} f(x)T_k(x)\frac{dx}{\sqrt{1-x^2}}, \qquad k = 0,1,2,\ldots, n$$

... surprisingly these coefficients can be calculated with FFT techniques, noting that

$$c_k = \frac{2}{\pi}\int_{0}^{\pi} f(\cos\varphi)\cos k\varphi d\varphi, \qquad k = 0,1,2,\ldots, n$$

... and the fact that f(cosφ) is a 2π-periodic function ...

$$c_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\cos\varphi)\cos k\varphi d\varphi, \qquad k = 0,1,2,\ldots, n$$

... which means that the coefficients c_k are the Fourier coefficients a_k of the periodic function F(φ)=f(cos φ)!

Chebyshev - discrete functions

... what happens if we know our function f(x) only at the points

$$x_i = \cos \frac{\pi}{N} i$$

in this *particular* case the coefficients are given by

$$c_k^* = \frac{2}{N} \sum_{j=1}^{N} f(\cos \varphi_j) \cos(k \varphi_j), \qquad k = 0,1,2,\ldots N/2$$

... leading to the polynomial ...

$$g_m^*(x) = \frac{1}{2} c_0^* T_0 + \sum_{k=1}^{m} c_k^* T_k(x)$$

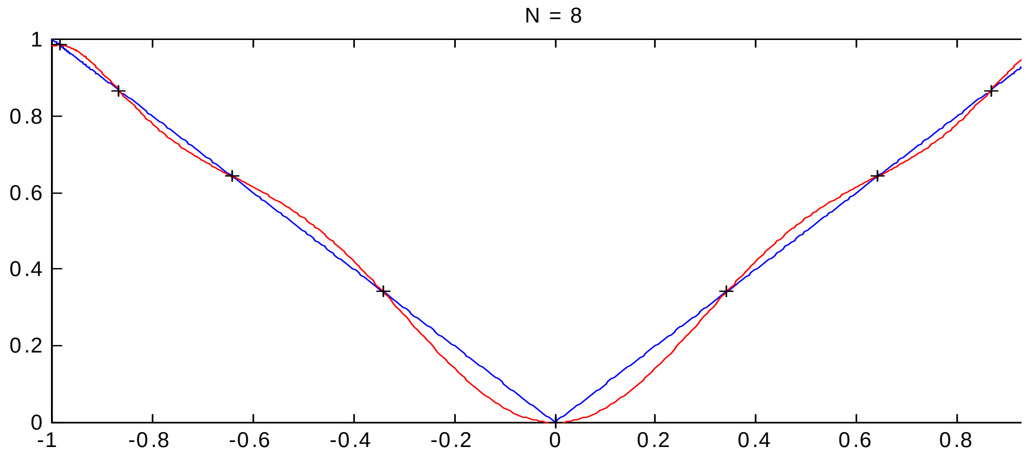
... with the property

$$g_m^*(x) = f(x) \quad \text{at} \quad x_j = \cos(\pi j/N) \qquad j = 0,1,2,\ldots, \quad N$$

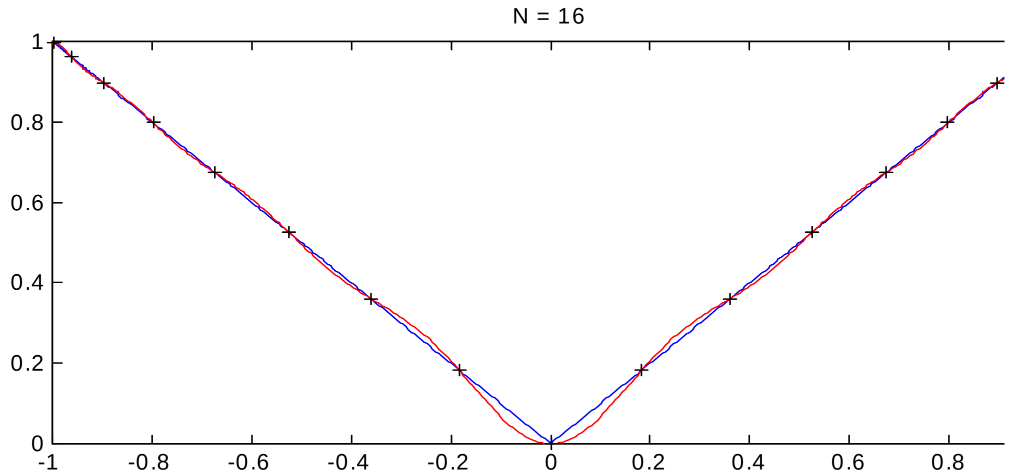
Chebyshev - collocation points - |x|

f(x)=|x| => f(x) - blue ; $g_n(x)$ - red; x_i - '+'

8 points



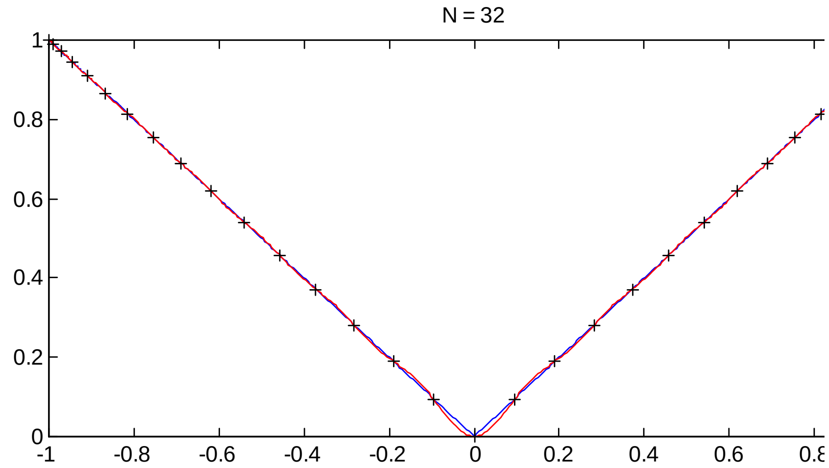
16 points



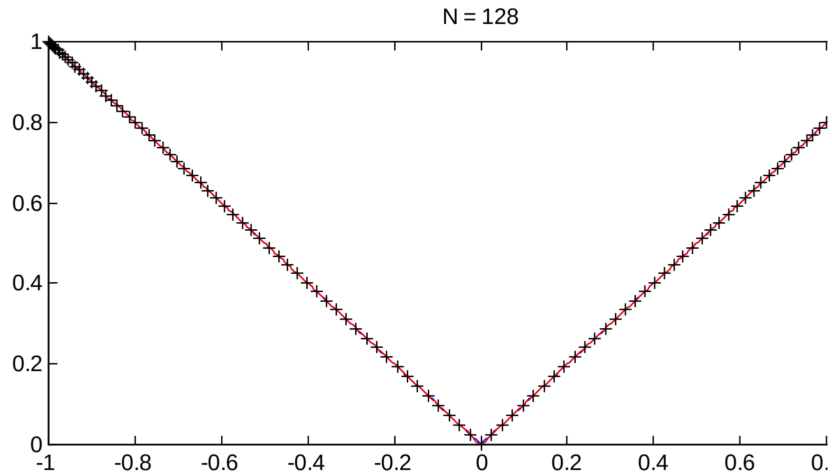
Chebyshev - collocation points - |x|

$f(x)=|x| \Rightarrow f(x)$ - blue ; $g_n(x)$ - red; x_i - '+'

32 points



128 points

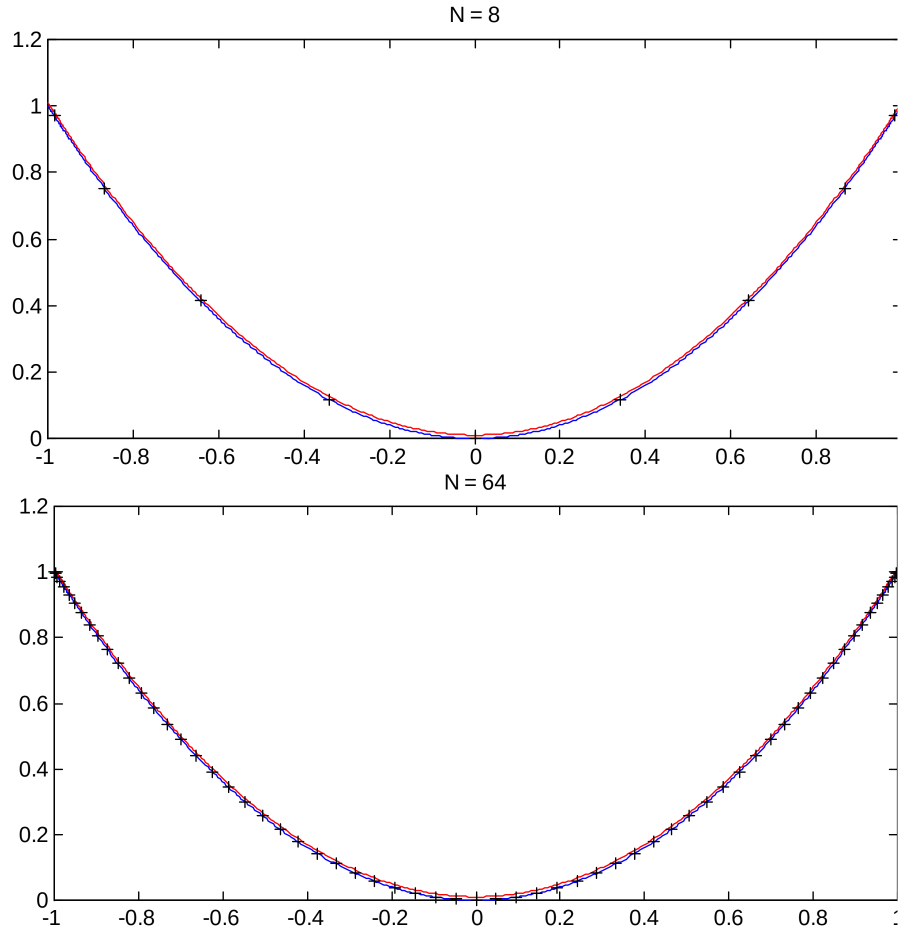


Chebyshev - collocation points - x^2

$f(x)=x^2$ => f(x) - blue ; $g_n(x)$ - red; x_i - '+'

8 points

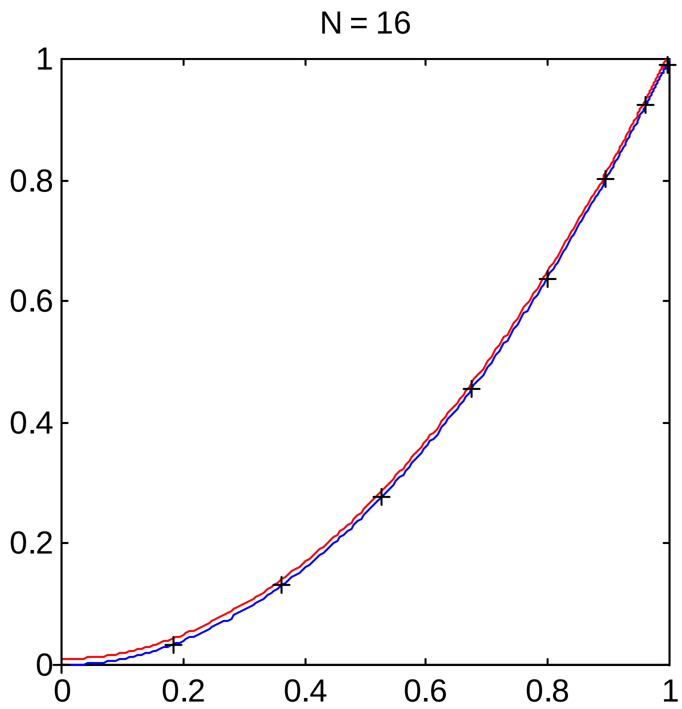
64 points



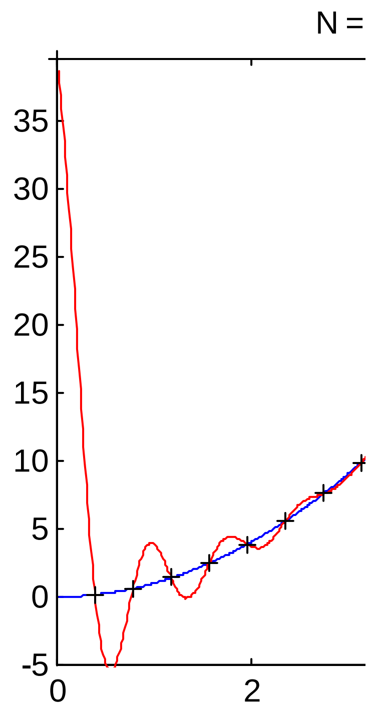
The interpolating function $g_n(x)$ was shifted by a small amount to be visible at all!

Chebyshev vs. Fourier - numerical

Chebyshev



Fourier

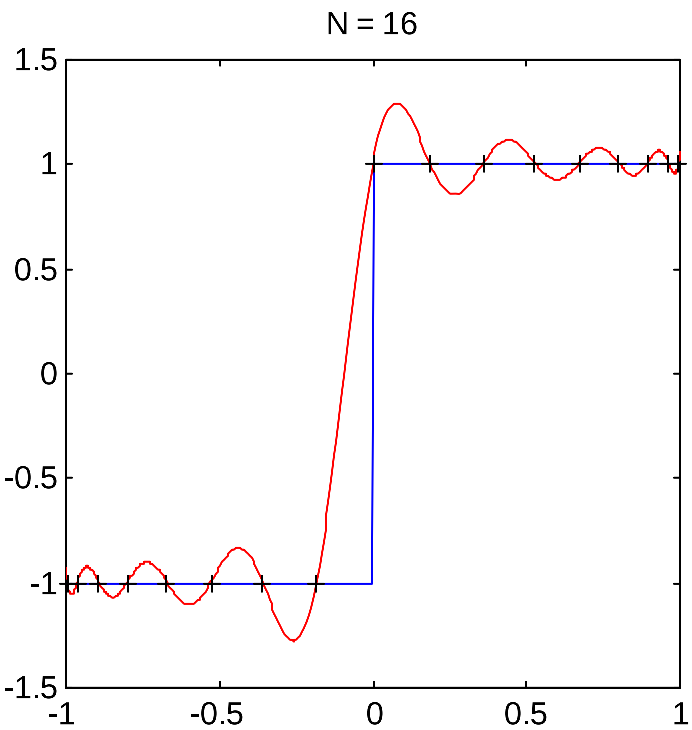


$f(x)=x^2$ => f(x) - blue ; $g_N(x)$ - red; x_i - '+'

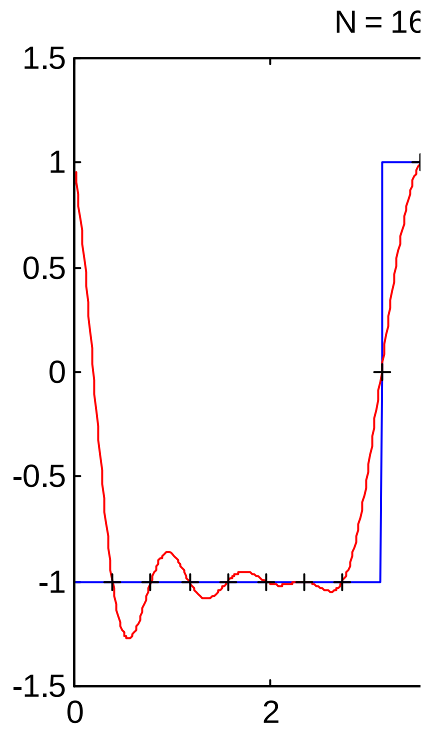
This graph speaks for itself ! Gibb's phenomenon with Chebyshev?

Chebyshev vs. Fourier - Gibb's

Chebyshev



Fourier

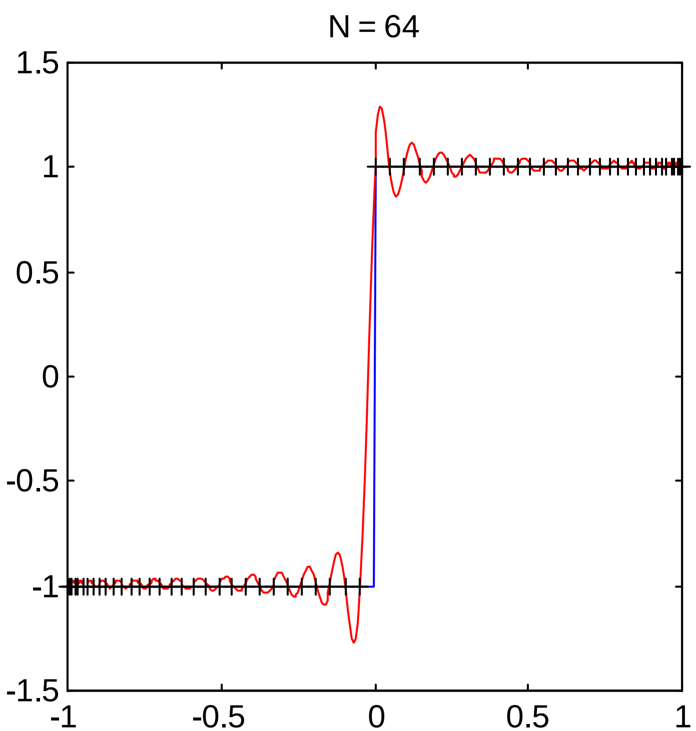


$f(x)=\mathrm{sign}(x-\pi)$ => f(x) - blue ; $g_N(x)$ - red; x_i - '+'

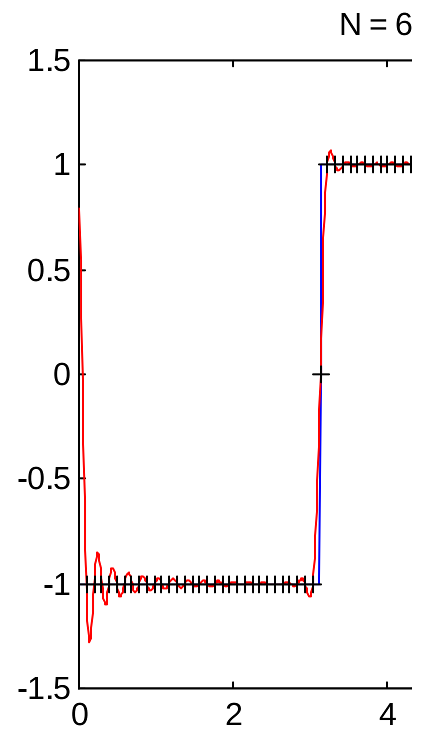
Gibb's phenomenon with Chebyshev? YES!

Chebyshev vs. Fourier - Gibb's

Chebyshev



Fourier



f(x)=sign(x-π) => f(x) - blue ; g_N(x) - red; x_i - '+'

Fourier vs. Chebyshev

Fourier		Chebyshev
$x_i = \frac{2\pi}{N} i$	*collocation points*	$x_i = \cos \frac{\pi}{N} i$
periodic functions	*domain*	limited area [-1,1]
$\cos(nx), \sin(nx)$	*basis functions*	$T_n(x) = \cos(n\varphi),$ $x = \cos\varphi$
$g_m^*(x) = \frac{1}{2}a_0^* + \sum_{k=1}^{m-1}\{a_k^*\cos(kx) + b_k^*\sin(kx)\} + \frac{1}{2}a_m^*\cos(kx)$	*interpolating function*	$g_m^*(x) = \frac{1}{2}c_0^* T_0 + \sum_{k=1}^{m} c_k^* T_k(x)$

Fourier vs. Chebyshev (cont'd)

Fourier

$$a_k^* = \frac{2}{N}\sum_{j=1}^{N} f(x_j)\cos(kx_j)$$

$$b_k^* = \frac{2}{N}\sum_{j=1}^{N} f(x_j)\sin(kx_j)$$

• Gibb's phenomenon for discontinuous functions

• Efficient calculation via FFT

• infinite domain through periodicity

coefficients

some properties

Chebyshev

$$c_k^* = \frac{2}{N}\sum_{j=1}^{N} f(\cos\varphi_j)\cos(k\varphi_j)$$

• limited area calculations

• grid densification at boundaries

• coefficients via FFT

• excellent convergence at boundaries

• Gibb's phenomenon

The Fourier Transform Pair

$$F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$$

Forward transform

$$f(t)=\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}d\omega$$

Inverse transform

Note the conventions concerning the sign of the exponents and the factor.

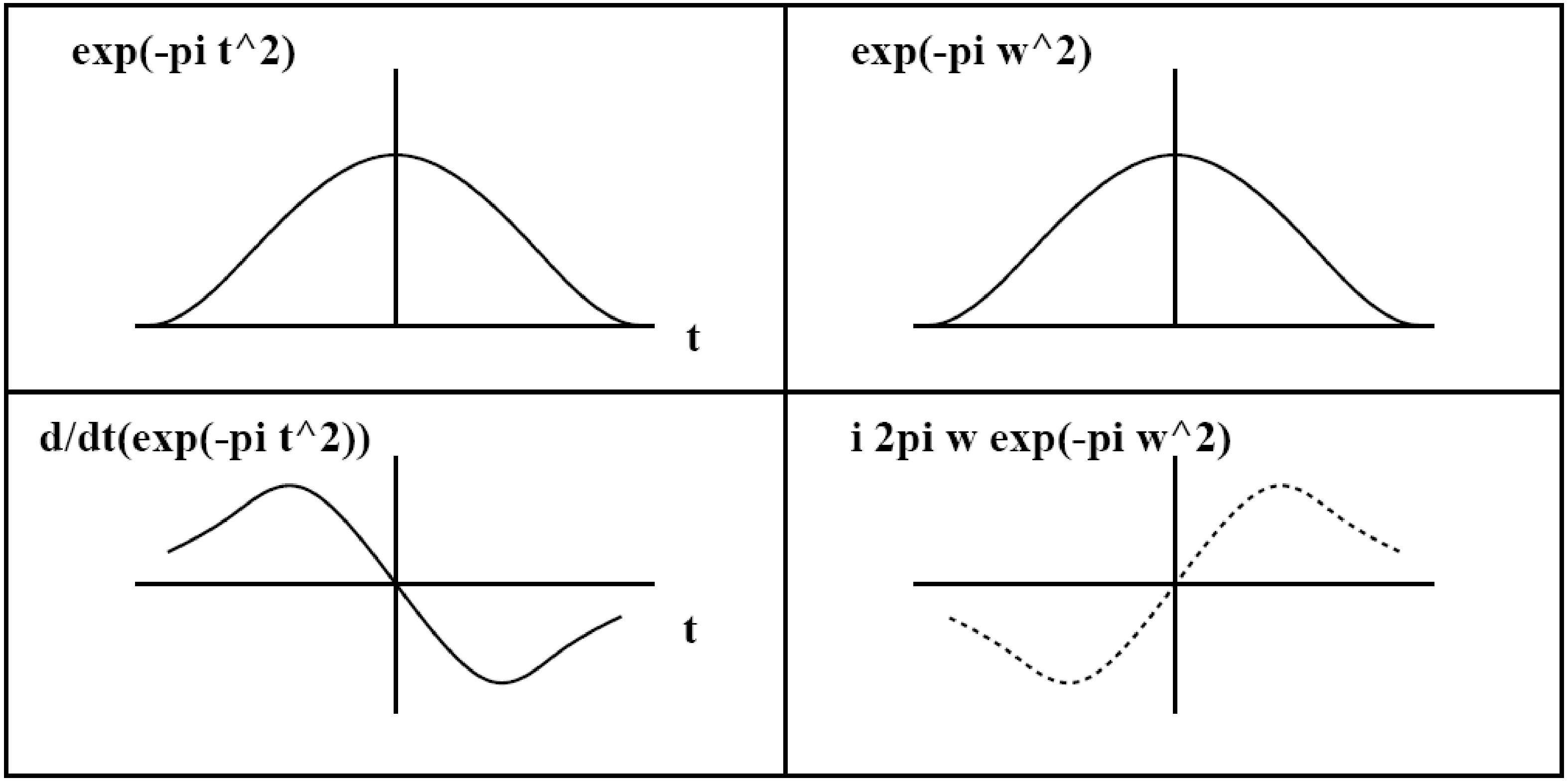
Some properties of the Fourier Transform

Defining as the FT: $$f(t) \Rightarrow F(\omega)$$

- Linearity $$af_1(t) + bf_2(t) \Rightarrow aF_1(\omega) + bF_2(\omega)$$
- Symmetry $$f(-t) \Rightarrow 2\pi F(-\omega)$$
- Time shifting $$f(t + \Delta t) \Rightarrow e^{i\omega\Delta t} F(\omega)$$
- Time differentiation $$\frac{\partial^n f(t)}{\partial t^n} \Rightarrow (-i\omega)^n F(\omega)$$

Differentiation theorem

- Time differentiation $\dfrac{\partial^n f(t)}{\partial t^n} \Rightarrow (-i\omega)^n F(\omega)$



Convolution

The **convolution** operation is at the heart of **linear systems.**

Definition:

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(t')g(t-t')dt' = \int_{-\infty}^{\infty} f(t-t')g(t')dt'$$

Properties:

$$f(t) * g(t) = g(t) * f(t)$$

$$f(t) * \partial(t) = f(t)$$

$$f(t) * H(t) = \int f(t)dt$$

H(t) is the Heaviside function:

The convolution theorem

A convolution in the time domain corresponds to a multiplication in the frequency domain.

… and vice versa …

a convolution in the frequency domain corresponds to a multiplication in the time domain

$$f(t) * g(t) \Rightarrow F(\omega)G(\omega)$$

$$f(t)g(t) \Rightarrow F(\omega) * G(\omega)$$

The first relation is of tremendous practical implication!

Summary

- The Fourier Transform can be derived from the problem of approximating an arbitrary function.
- A regular set of points allows exact interpolation (or derivation) of arbitrary functions
- There are other basis functions (e.g., Chebyshev polynomials, Legendre polynomials) with similar properties
- These properties are the basis for the success of the spectral element method