Fourier Transform: Applications

• Seismograms
• Eigenmodes of the Earth
• Time derivatives of seismograms
• The pseudo-spectral method for acoustic wave propagation
### Fourier: Space and Time

#### Space
- $x$: space variable
- $L$: spatial wavelength
- $k = \frac{2\pi}{\lambda}$: spatial wavenumber
- $F(k)$: wavenumber spectrum

#### Time
- $t$: Time variable
- $T$: period
- $f$: frequency
- $\omega = 2\pi f$: angular frequency

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**Fourier integrals**

With the complex representation of sinusoidal functions $e^{ikx}$ (or $(e^{iwt})$ the Fourier transformation can be written as:

\[
\begin{align*}
    f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{-ikx} \, dx \\
    F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ikx} \, dx
\end{align*}
\]
The Fourier Transform
discrete vs. continuous

Whatever we do on the computer with data will be based on the discrete Fourier transform.

**Continuous**

\[
f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)e^{-ikx} \, dx
\]

\[
F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{ikx} \, dx
\]

**Discrete**

\[
F_k = \frac{1}{N} \sum_{j=0}^{N-1} f_j e^{-2\pi ikj / N}, \quad k = 0, 1, \ldots, N - 1
\]

\[
f_k = \sum_{j=0}^{N-1} F_j e^{2\pi ikj / N}, \quad k = 0, 1, \ldots, N - 1
\]
The Fast Fourier Transform

... the latter approach became interesting with the introduction of the Fast Fourier Transform (FFT). What’s so fast about it?

The FFT originates from a paper by Cooley and Tukey (1965, Math. Comp. vol 19 297-301) which revolutionised all fields where Fourier transforms where essential to progress.

The discrete Fourier Transform can be written as

\[
\hat{u}_k = \frac{1}{N} \sum_{j=0}^{N-1} u_j e^{-2\pi i k j / N}, k = 0,1,\ldots, N-1
\]

\[
u_k = \sum_{j=0}^{N-1} \hat{u}_j e^{2\pi i k j / N}, k = 0,1,\ldots, N-1
\]
The Fast Fourier Transform

... this can be written as matrix-vector products ...
for example the inverse transform yields ...

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & \ldots & 1 \\
1 & \omega & \omega^2 & \omega^3 & \ldots & \omega^{N-1} \\
1 & \omega^2 & \omega^4 & \omega^6 & \ldots & \omega^{2N-2} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^{N-1} & \ldots & \ldots & \ldots & \omega^{(N-1)^2}
\end{bmatrix}
\begin{bmatrix}
\hat{u}_0 \\
\hat{u}_1 \\
\hat{u}_2 \\
\vdots \\
\hat{u}_{N-1}
\end{bmatrix}
= 
\begin{bmatrix}
u_0 \\
u_1 \\
u_2 \\
\vdots \\
u_{N-1}
\end{bmatrix}
\]

... where ...

\[
\omega = e^{2\pi i / N}
\]
The Fast Fourier Transform

... the FAST bit is recognising that the full matrix - vector multiplication can be written as a few sparse matrix - vector multiplications (for details see for example Bracewell, the Fourier Transform and its applications, MacGraw-Hill) with the effect that:

Number of multiplications

<table>
<thead>
<tr>
<th></th>
<th>full matrix</th>
<th>FFT</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$N^2$</td>
<td>$2N\log_2 N$</td>
</tr>
</tbody>
</table>

this has enormous implications for large scale problems. Note: the factorisation becomes particularly simple and effective when $N$ is a highly composite number (power of 2).
The Fast Fourier Transform

<table>
<thead>
<tr>
<th>Problem</th>
<th>full matrix</th>
<th>FFT</th>
<th>Ratio full/FFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1D (nx=512)</td>
<td>2.6x10^5</td>
<td>9.2x10^3</td>
<td>28.4</td>
</tr>
<tr>
<td>1D (nx=2096)</td>
<td></td>
<td></td>
<td>94.98</td>
</tr>
<tr>
<td>1D (nx=8384)</td>
<td></td>
<td></td>
<td>312.6</td>
</tr>
</tbody>
</table>

.. the right column can be regarded as the speedup of an algorithm when the FFT is used instead of the full system.
Spectral synthesis

The red trace is the sum of all blue traces!
Phase and amplitude spectrum

The spectrum consists of two real-valued functions of angular frequency, the amplitude spectrum mod \( F(\omega) \) and the phase spectrum \( \phi(\omega) \)

\[
F(\omega) = |F(\omega)|e^{i\Phi(\omega)}
\]

In many cases the amplitude spectrum is the most important part to be considered. However there are cases where the phase spectrum plays an important role (-> resonance, seismometer)
... remember ...

\[ z^* = a - ib = r (\cos \phi - i \sin \phi) \]

\[ = r \cos \phi - ri \sin(-\phi) = r^{\frac{-i\phi}{}} \]

\[ |z^2| = zz^* = (a + ib)(a - ib) = r^2 \]
The spectrum

Amplitude spectrum

Phase spectrum

Fourier space

Physical space

Fourier: Applications

Modern Seismology — Data processing and inversion
The Fast Fourier Transform (FFT)

Most processing tools (e.g. octave, Matlab, Mathematica, Fortran, etc) have intrinsic functions for FFTs.

Matlab FFT

>> help fft

FFT Discrete Fourier transform.
FFT(X) is the discrete Fourier transform (DFT) of vector X. For matrices, the FFT operation is applied to each column. For N-D arrays, the FFT operation operates on the first non-singleton dimension.

FFT(X,N) is the N-point FFT, padded with zeros if X has less than N points and truncated if it has more.

FFT(X,[],DIM) or FFT(X,N,DIM) applies the FFT operation across the dimension DIM.

For length N input vector x, the DFT is a length N vector X, with elements

\[
X(k) = \sum_{n=1}^{N} x(n) \cdot \exp(-j \cdot 2 \cdot \pi \cdot (k-1) \cdot (n-1)/N), \quad 1 \leq k \leq N.
\]

The inverse DFT (computed by IFFT) is given by

\[
x(n) = \frac{1}{N} \sum_{k=1}^{N} X(k) \cdot \exp(j \cdot 2 \cdot \pi \cdot (k-1) \cdot (n-1)/N), \quad 1 \leq n \leq N.
\]

See also IFFT, FFT2, IFFT2, FFTSHIFT.
Frequencies in seismograms

Figure 6.2-3: Amplitude spectra for the body and surface wave segments from a large earthquake.
Amplitude spectrum
Eigenfrequencies

Figure 6.2-4: Amplitude spectra of a vertical-component seismogram from the great 1994 Bolivian earthquake.
Sound of an instrument

\[ a' \text{ - 440Hz} \]
Instrument Earth

26.-29.12.2004 (FFB)

$0S_2$ – Earth’s gravest tone

$T = 3233.5 \text{s} = 53.9 \text{min}$
Random signals may contain all frequencies. A spectrum with constant contribution of all frequencies is called a white spectrum.
Fourier Spectra: Main Cases

Gaussian signals

The spectrum of a Gaussian function will itself be a Gaussian function. How does the spectrum change, if I make the Gaussian narrower and narrower?
A transient waveform is a waveform limited in time (or space) in comparison with a harmonic waveform that is infinite.
Puls-width and Frequency Bandwidth

Narrowing physical signal

Widening frequency band

Fourier: Applications

Modern Seismology – Data processing and inversion
Spectral analysis: an Example

24 hour ground motion, do you see any signal?
Seismo-Weather

Running spectrum of the same data
Some properties of FT

- FT is linear
  signals can be treated as the sum of several signals, the transform will be the sum of their transforms

- FT of a real signals
  has symmetry properties

\[ F(-\omega) = F^* (\omega) \]

the negative frequencies can be obtained from symmetry properties

- Shifting corresponds to changing the phase (shift theorem)

\[ f(t - a) \rightarrow e^{-i\omega a} F(\omega) \]

\[ F(\omega - a) \rightarrow e^{-i\omega a} f(t) \]

- Derivative

\[ \frac{d^n}{dt^n} f(t) \rightarrow (i\omega)^n F(\omega) \]
Fourier Derivatives

.. let us recall the definition of the derivative using Fourier integrals ...

\[ \partial_x f(x) = \partial_x \left( \int_{-\infty}^{\infty} F(k)e^{-ikx} \, dk \right) \]

\[ = -\int_{-\infty}^{\infty} ikF(k)e^{-ikx} \, dk \]

... we could either ...

1) perform this calculation in the space domain by convolution

2) actually transform the function f(x) in the k-domain and back
Acoustic Wave Equation - Fourier Method

let us take the acoustic wave equation with variable density

\[
\frac{1}{\rho c^2} \partial_t^2 p = \partial_x \left( \frac{1}{\rho} \partial_x p \right)
\]

the left hand side will be expressed with our standard centered finite-difference approach

\[
\frac{1}{\rho c^2 dt^2} \left[ p(t + dt) - 2p(t) + p(t - dt) \right] = \partial_x \left( \frac{1}{\rho} \partial_x p \right)
\]

... leading to the extrapolation scheme ...
Acoustic Wave Equation - Fourier Method

\[ p(t + dt) = \rho c^2 dt \left( \frac{1}{\rho} \frac{\partial}{\partial x} p \right) + 2 p(t) - p(t - dt) \]

where the space derivatives will be calculated using the Fourier Method.

The highlighted term will be calculated as follows:

\[ P_j^n \rightarrow FFT \rightarrow \hat{P}_v^n \rightarrow ik_v \hat{P}_v^n \rightarrow FFT^{-1} \rightarrow \frac{\partial}{\partial x} P_j^n \]

multiply by \(1/\rho\)

\[ \frac{1}{\rho} \frac{\partial}{\partial x} P_j^n \rightarrow FFT \rightarrow \left( \frac{1}{\rho} \frac{\partial}{\partial x} \hat{P} \right)_v^n \rightarrow ik_v \left( \frac{1}{\rho} \frac{\partial}{\partial x} \hat{P} \right)_v^n \rightarrow FFT^{-1} \rightarrow \frac{\partial}{\partial x} \left( \frac{1}{\rho} \frac{\partial}{\partial x} P_j^n \right) \]

... then extrapolate ...
... and the first derivative using FFTs ...

function df=sder1d(f,dx)
% SDER1D(f,dx) spectral derivative of vector
nx=max(size(f));

% initialize k
kmax=pi/dx;
dk=kmax/(nx/2);
for i=1:nx/2, k(i)=(i)*dk; k(nx/2+i)=-kmax+(i)*dk; end
k=sqrt(-1)*k;

% FFT and IFFT
ff=fft(f); ff=k.*ff; df=real(ifft(ff));

.. simple and elegant ..
Fourier Method - Comparison with FD - Table

Difference (%) between numerical and analytical solution as a function of propagating frequency

Simulation time
- 3 point: 5.4s
- 5 point: 7.8s
- Fourier: 33.0s

Frequency
- 5 Hz
- 10 Hz
- 20 Hz

Difference (%) as a function of propagating frequency.
Numerical solutions and Green’s Functions

Impulse response (analytical) convolved with source
Impulse response (numerical convolved with source

Frequency increases

3 point operator
4 point operator
5 point operator
Fourier Method

0 500 1000 1500
0 500 1000 1500
0 500 1000

0 1 2 3 4 5 6 7 8 9 10
0 1 2 3 4 5 6 7 8 9 10
0 1 2 3 4 5 6 7 8 9 10