

Fourier Series and Transforms

- Orthogonal functions
- Fourier Series
- Discrete Fourier Series
- Fourier Transform
- Chebyshev polynomials

Scope: we are trying to approximate an arbitrary function and obtain basis functions with appropriate coefficients.

Fourier Series

The Problem

we are trying to approximate a function $f(x)$ by another function $g_n(x)$ which consists of a sum over N *orthogonal* functions $\Phi(x)$ weighted by some coefficients a_n .

$$f(x) \approx g_N(x) = \sum_{i=0}^N a_i \Phi_i(x)$$

The Problem

... and we are looking for optimal functions in a least squares (L_2) sense ...

$$\|f(x) - g_N(x)\|_{L_2} = \left[\int_a^b \{f(x) - g_N(x)\}^2 dx \right]^{1/2} = \text{Min!}$$

... a good choice for the basis functions $\Phi(x)$ are *orthogonal* functions.
What are orthogonal functions? Two functions f and g are said to be orthogonal in the interval $[a,b]$ if

$$\int_a^b f(x)g(x)dx = 0$$

How is this related to the more conceivable concept of orthogonal vectors? Let us look at the original definition of integrals:

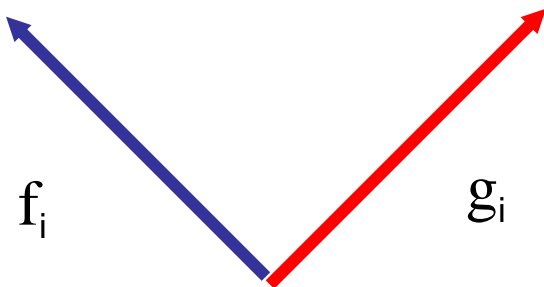
Orthogonal Functions

$$\int_a^b f(x)g(x)dx = \lim_{N \rightarrow \infty} \left(\sum_{i=1}^N f_i(x)g_i(x)\Delta x \right)$$

... where $x_0=a$ and $x_N=b$, and $x_i-x_{i-1}=\Delta x$...

If we interpret $f(x_i)$ and $g(x_i)$ as the i th components of an N component vector, then this sum corresponds directly to a scalar product of vectors.

The vanishing of the scalar product is the condition for *orthogonality* of vectors (or functions).

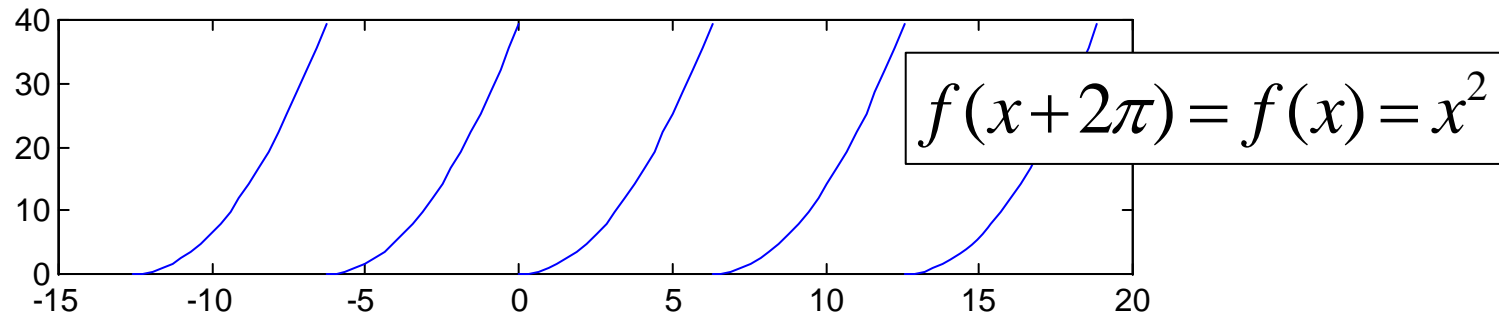


$$f_i \bullet g_i = \sum_i f_i g_i = 0$$

Periodic functions

Let us assume we have a piecewise continuous function of the form

$$f(x+2\pi) = f(x)$$



... we want to approximate this function with a linear combination of 2π periodic functions:

$$1, \cos(x), \sin(x), \cos(2x), \sin(2x), \dots, \cos(nx), \sin(nx)$$

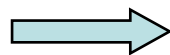
$$\Rightarrow f(x) \approx g_N(x) = \frac{1}{2}a_0 + \sum_{k=1}^N \{a_k \cos(kx) + b_k \sin(kx)\}$$

Orthogonality

... are these functions orthogonal ?

$$\begin{aligned} \int_{-\pi}^{\pi} \cos(jx) \cos(kx) dx &= \begin{cases} 0 & j \neq k \\ 2\pi & j = k = 0 \\ \pi & j = k > 0 \end{cases} \\ \int_{-\pi}^{\pi} \sin(jx) \sin(kx) dx &= \begin{cases} 0 & j \neq k, j, k > 0 \\ \pi & j = k > 0 \end{cases} \\ \int_{-\pi}^{\pi} \cos(jx) \sin(kx) dx &= 0 \quad j \geq 0, k > 0 \end{aligned}$$

... YES, and these relations are valid for any interval of length 2π .
Now we know that this is an orthogonal basis, but how can we obtain the coefficients for the basis functions?



from minimising $f(x)-g(x)$

Fourier coefficients

optimal functions $g(x)$ are given if

$$\|g_n(x) - f(x)\|_2 = \text{Min !} \quad \text{or} \quad \frac{\partial}{\partial a_k} \left\{ \|g_n(x) - f(x)\|_2 \right\} = 0$$

... with the definition of $g(x)$ we get ...

$$\frac{\partial}{\partial a_k} \|g_n(x) - f(x)\|_2^2 = \frac{\partial}{\partial a_k} \left[\int_{-\pi}^{\pi} \left[\frac{1}{2} a_0 + \sum_{k=1}^N \{a_k \cos(kx) + b_k \sin(kx)\} - f(x) \right]^2 dx \right]$$

leading to

$$\begin{aligned} g_N(x) &= \frac{1}{2} a_0 + \sum_{k=1}^N \{a_k \cos(kx) + b_k \sin(kx)\} \quad \text{with} \\ a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx, \quad k = 0, 1, \dots, N \\ b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx, \quad k = 1, 2, \dots, N \end{aligned}$$

Fourier approximation of $|x|$

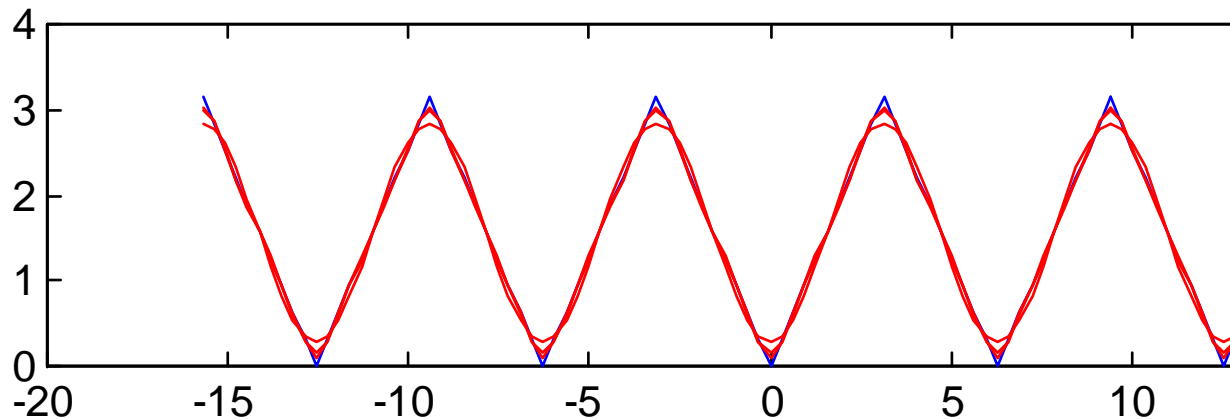
... Example ...

$$f(x) = |x|, \quad -\pi \leq x \leq \pi$$

leads to the Fourier Serie

$$g(x) = \frac{1}{2}\pi - \frac{4}{\pi} \left\{ \frac{\cos(x)}{1^2} + \frac{\cos(3x)}{3^2} + \frac{\cos(5x)}{5^2} + \dots \right\}$$

.. and for $n < 4$ $g(x)$ looks like



Fourier approximation of x^2

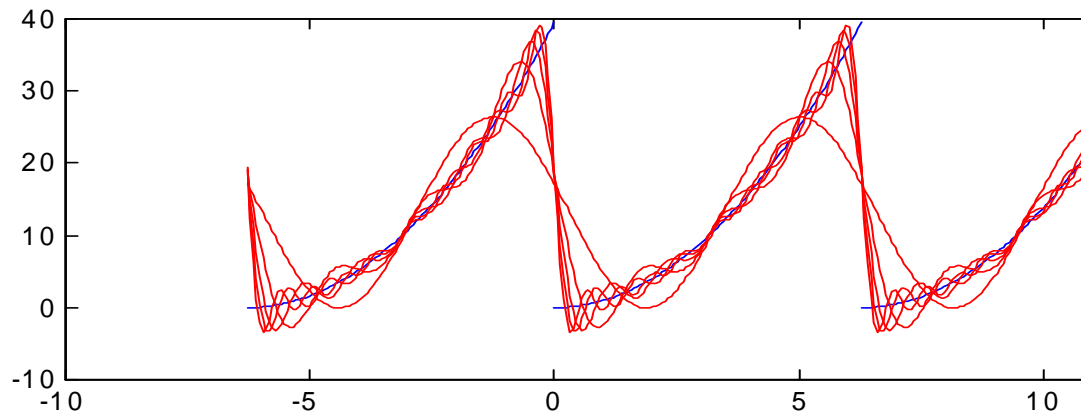
... another Example ...

$$f(x) = x^2, \quad 0 < x < 2\pi$$

leads to the Fourier Serie

$$g_N(x) = \frac{4\pi^2}{3} + \sum_{k=1}^N \left\{ \frac{4}{k^2} \cos(kx) - \frac{4\pi}{k} \sin(kx) \right\}$$

.. and for $N < 11$, $g(x)$ looks like



Fourier - discrete functions

... what happens if we know our function $f(x)$ only at the points

$$x_i = \frac{2\pi}{N} i$$

it turns out that in this *particular* case the coefficients are given by

$$\begin{aligned} a_k^* &= \frac{2}{N} \sum_{j=1}^N f(x_j) \cos(kx_j), & k &= 0, 1, 2, \dots \\ b_k^* &= \frac{2}{N} \sum_{j=1}^N f(x_j) \sin(kx_j), & k &= 1, 2, 3, \dots \end{aligned}$$

.. the so-defined Fourier polynomial is the unique interpolating function to the function $f(x_j)$ with $N=2m$

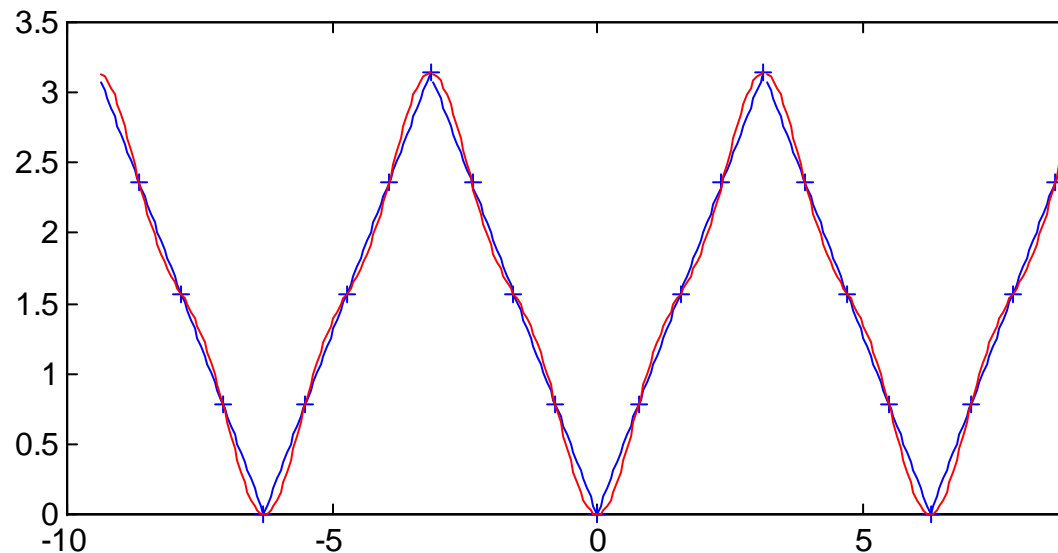
$$g_m^*(x) = \frac{1}{2} a_0^* + \sum_{k=1}^{m-1} \{a_k^* \cos(kx) + b_k^* \sin(kx)\} + \frac{1}{2} a_m^* \cos(kx)$$

Fourier - collocation points

... with the important property that ...

$$g_m^*(x_i) = f(x_i)$$

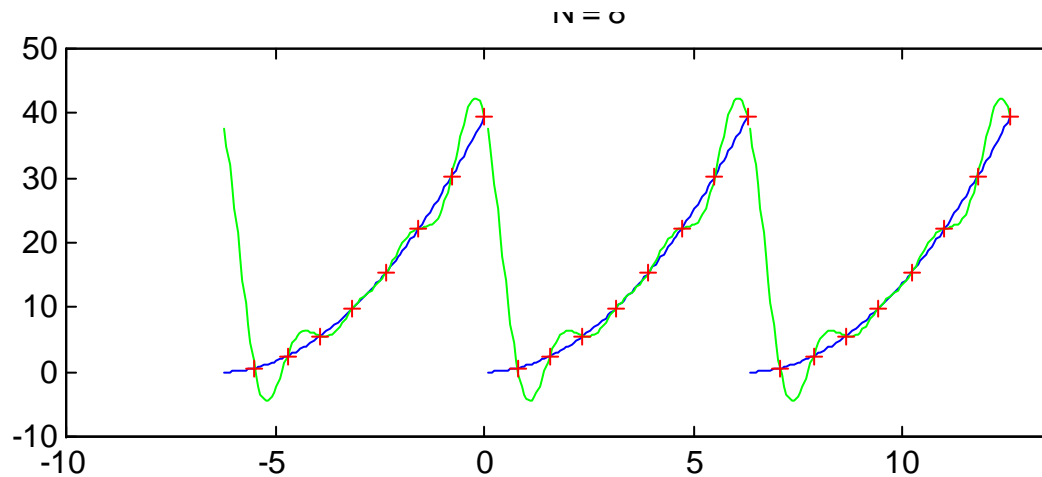
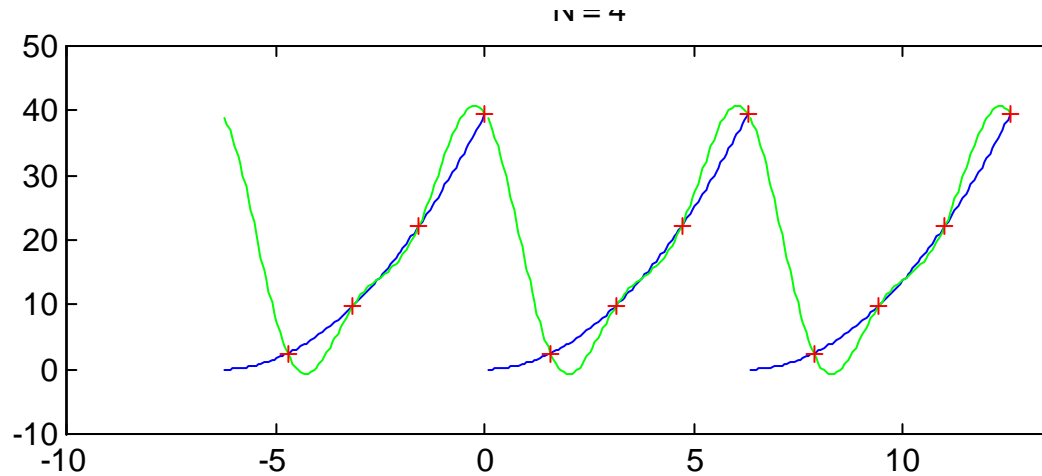
... in our previous examples ...



$f(x)=|x| \Rightarrow f(x)$ - blue ; $g(x)$ - red; x_i - '+'

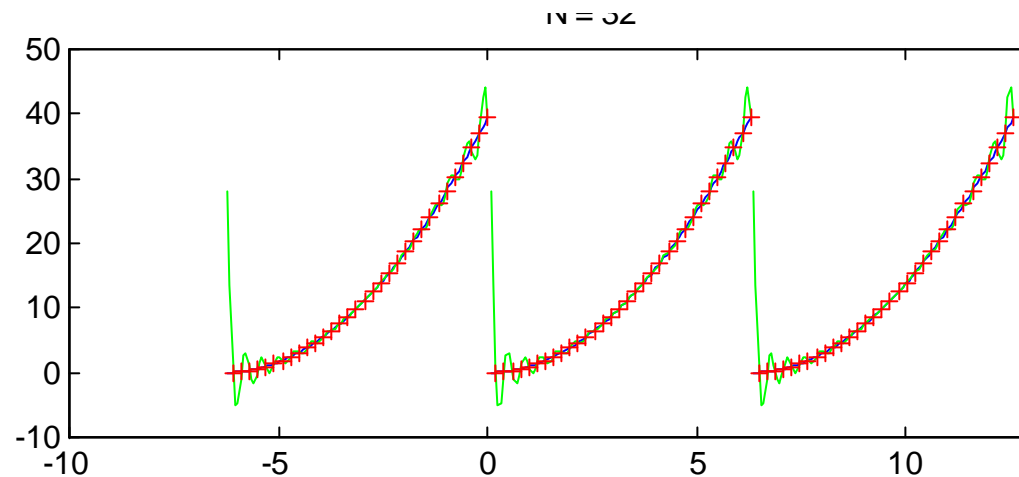
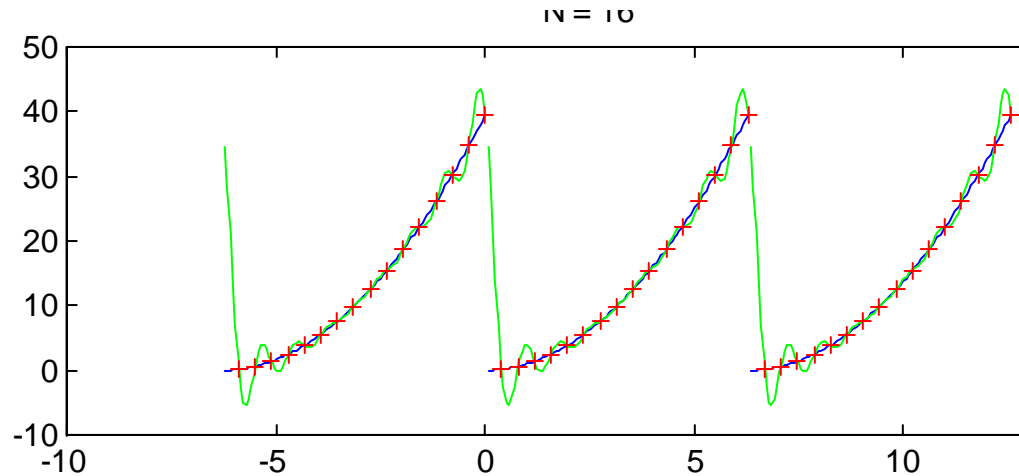
Fourier series - convergence

$f(x)=x^2 \Rightarrow f(x)$ - blue ; $g(x)$ - red; x_i - '+'



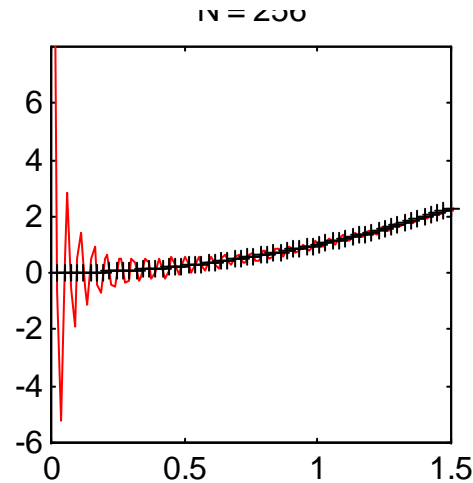
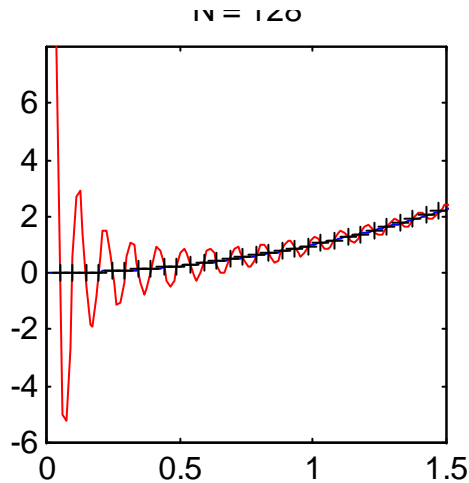
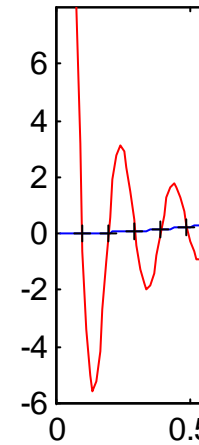
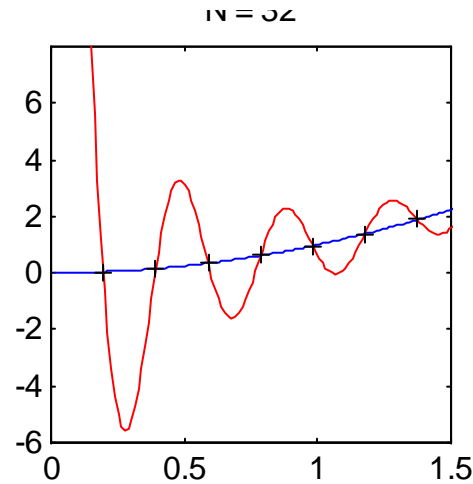
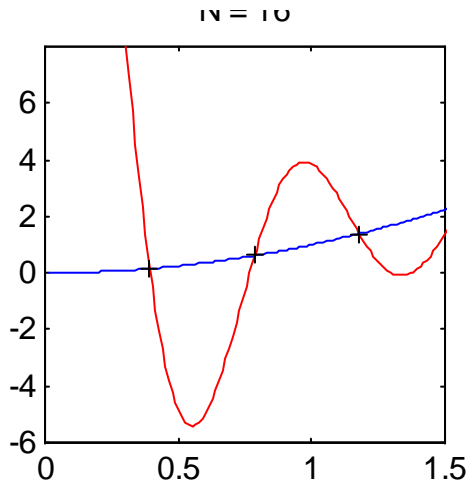
Fourier series - convergence

$f(x)=x^2 \Rightarrow f(x)$ - blue ; $g(x)$ - red; x_i - '+'



Gibb's phenomenon

$f(x)=x^2 \Rightarrow f(x)$ - blue ; $g(x)$ - red; x_i - '+'



The overshoot for equi-spaced Fourier interpolations is $\approx 14\%$ of the step height.

Chebyshev polynomials

We have seen that Fourier series are excellent for interpolating (and differentiating) periodic functions defined on a regularly spaced grid. In many circumstances physical phenomena which are not periodic (in space) and occur in a *limited area*. This quest leads to the use of **Chebyshev polynomials**.

We depart by observing that $\cos(n\varphi)$ can be expressed by a polynomial in $\cos(\varphi)$:

$$\cos(2\varphi) = 2 \cos^2 \varphi - 1$$

$$\cos(3\varphi) = 4 \cos^3 \varphi - 3 \cos \varphi$$

$$\cos(4\varphi) = 8 \cos^4 \varphi - 8 \cos^2 \varphi + 1$$

... which leads us to the definition:

Chebyshev polynomials - definition

$$\cos(n\varphi) = T_n(\cos(\varphi)) = T_n(x), \quad x = \cos(\varphi), \quad x \in [-1,1], \quad n \in N$$

... for the Chebyshev polynomials $T_n(x)$. Note that because of $x=\cos(\varphi)$ they are defined in the interval $[-1,1]$ (which - however - can be extended to \mathfrak{R}). The first polynomials are

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

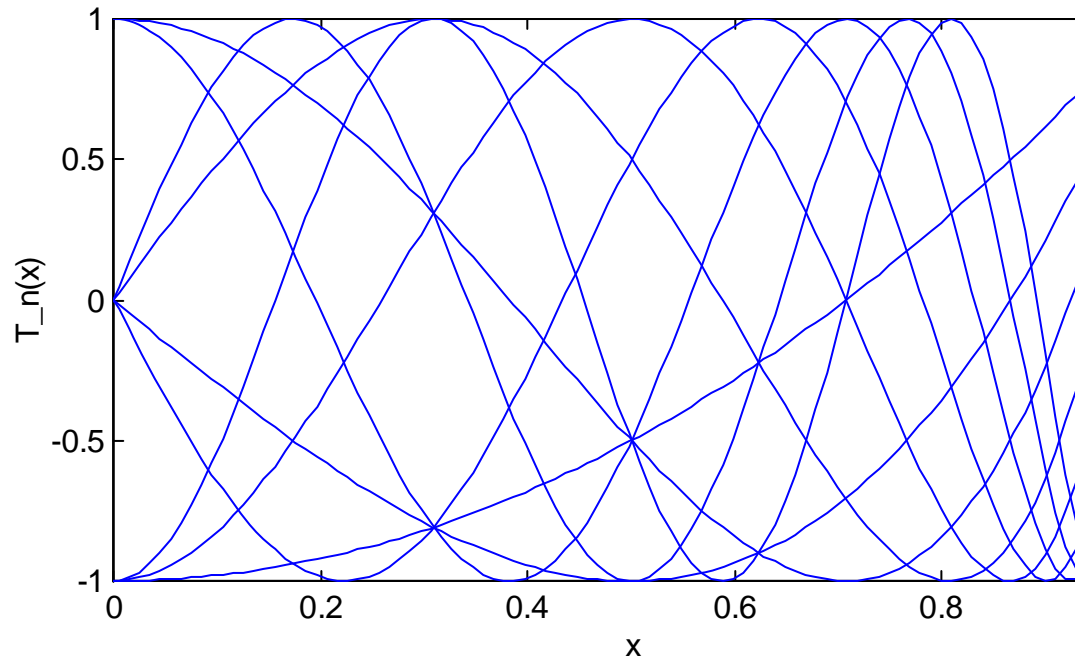
$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1 \quad \text{where}$$

$$|T_n(x)| \leq 1 \quad \text{for } x \in [-1,1] \quad \text{and } n \in N_0$$

Chebyshev polynomials - Graphical

The first ten polynomials look like [0, -1]

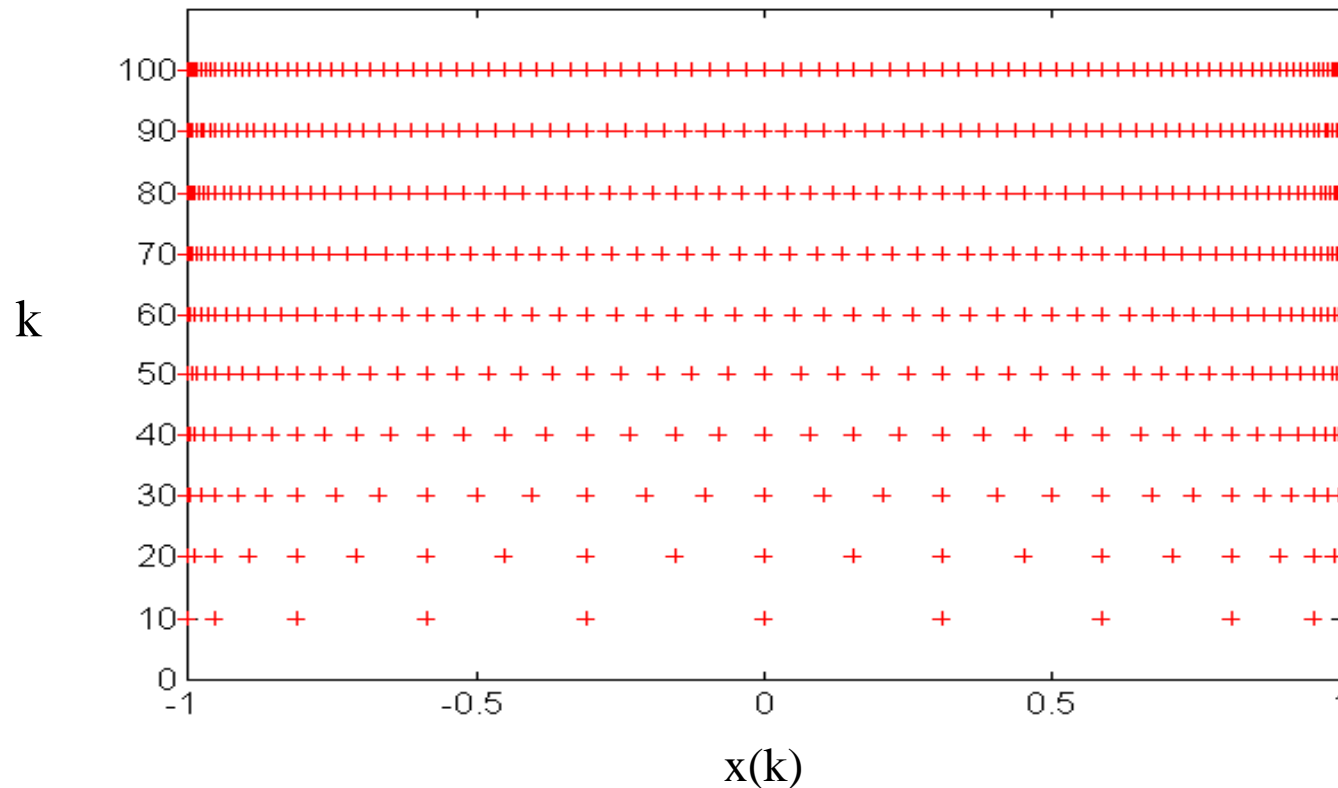


The n-th polynomial has extrema with values 1 or -1 at

$$x_k^{(ext)} = \cos\left(\frac{k\pi}{n}\right), \quad k = 0, 1, 2, 3, \dots, n$$

Chebyshev collocation points

These extrema are not equidistant (like the Fourier extrema)



$$x_k^{(ext)} = \cos\left(\frac{k\pi}{n}\right), \quad k = 0, 1, 2, 3, \dots, n$$

Chebyshev polynomials - orthogonality

... are the Chebyshev polynomials orthogonal?

Chebyshev polynomials are an orthogonal set of functions in the interval $[-1, 1]$ with respect to the weight function $1/\sqrt{1-x^2}$ such that

$$\int_{-1}^1 T_k(x) T_j(x) \frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0 & \text{for } k \neq j \\ \pi / 2 & \text{for } k = j > 0 \\ \pi & \text{for } k = j = 0 \end{cases}, \quad k, j \in N_0$$

... this can be easily verified noting that

$$x = \cos \varphi, \quad dx = -\sin \varphi d\varphi$$

$$T_k(x) = \cos(k\varphi), \quad T_j(x) = \cos(j\varphi)$$

Chebyshev polynomials - interpolation

... we are now faced with the same problem as with the Fourier series. We want to approximate a function $f(x)$, this time not a periodical function but a function which is defined between $[-1,1]$.

We are looking for $g_n(x)$

$$f(x) \approx g_n(x) = \frac{1}{2}c_0T_0(x) + \sum_{k=1}^n c_kT_k(x)$$

... and we are faced with the problem, how we can determine the coefficients c_k . Again we obtain this by finding the extremum (minimum)

$$\frac{\partial}{\partial c_k} \left[\int_{-1}^1 \{g_n(x) - f(x)\}^2 \frac{dx}{\sqrt{1-x^2}} \right] = 0$$

Chebyshev polynomials - interpolation

... to obtain ...

$$c_k = \frac{2}{\pi} \int_{-1}^1 f(x) T_k(x) \frac{dx}{\sqrt{1-x^2}}, \quad k = 0, 1, 2, \dots, n$$

... surprisingly these coefficients can be calculated with FFT techniques, noting that

$$c_k = \frac{2}{\pi} \int_0^\pi f(\cos \varphi) \cos k\varphi d\varphi, \quad k = 0, 1, 2, \dots, n$$

... and the fact that $f(\cos \varphi)$ is a 2π -periodic function ...

$$c_k = \frac{1}{\pi} \int_{-\pi}^\pi f(\cos \varphi) \cos k\varphi d\varphi, \quad k = 0, 1, 2, \dots, n$$

... which means that the coefficients c_k are the Fourier coefficients a_k of the periodic function $F(\varphi) = f(\cos \varphi)$!

Chebyshev - discrete functions

... what happens if we know our function $f(x)$ only at the points

$$x_i = \cos \frac{\pi}{N} i$$

in this *particular* case the coefficients are given by

$$c_k^* = \frac{2}{N} \sum_{j=1}^N f(\cos \varphi_j) \cos(k \varphi_j), \quad k = 0, 1, 2, \dots, N/2$$

... leading to the polynomial ...

$$g_m^*(x) = \frac{1}{2} c_0^* T_0 + \sum_{k=1}^m c_k^* T_k(x)$$

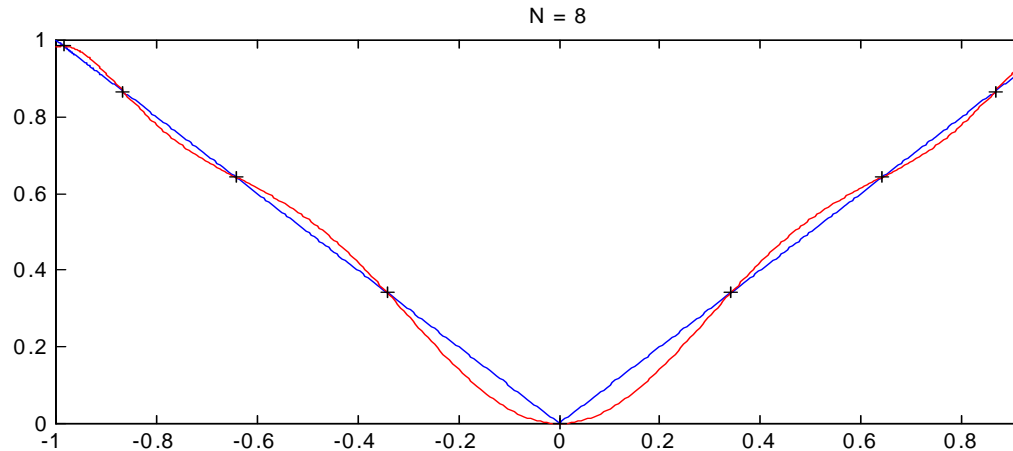
... with the property

$$g_m^*(x) = f(x) \quad \text{at} \quad x_j = \cos(\pi j/N) \quad j = 0, 1, 2, \dots, N$$

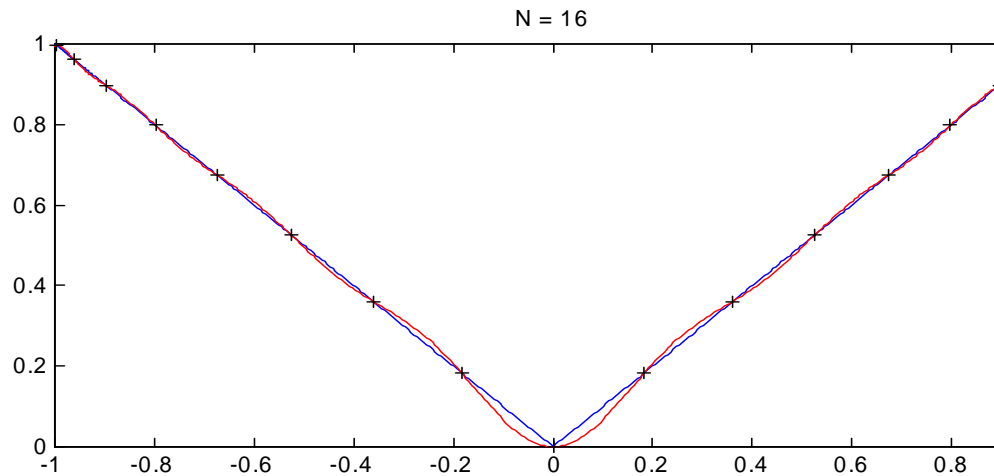
Chebyshev - collocation points - $|x|$

$f(x)=|x| \Rightarrow f(x)$ - blue ; $g_n(x)$ - red; x_i - '+'

8 points



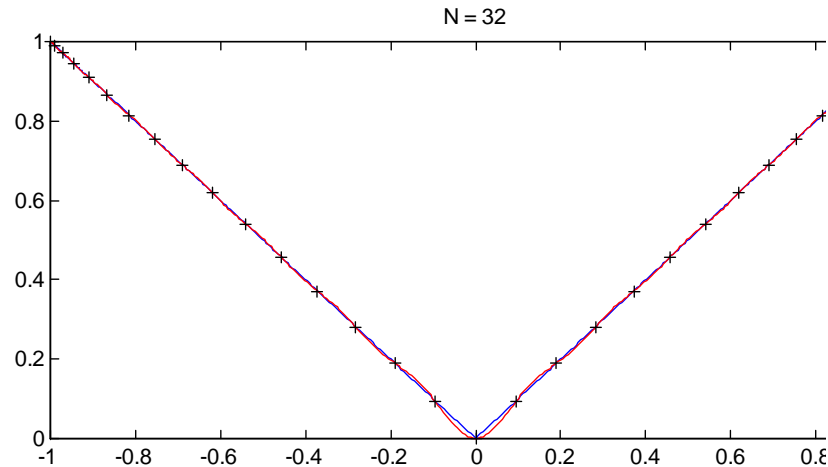
16 points



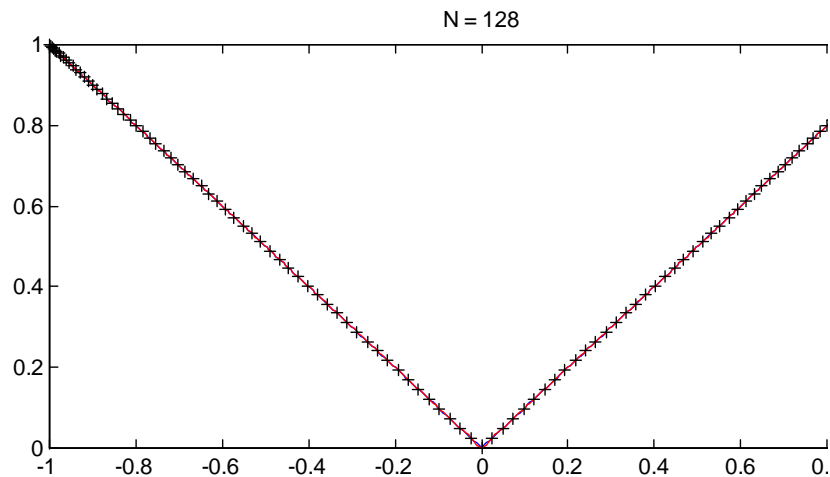
Chebyshev - collocation points - $|x|$

$f(x)=|x| \Rightarrow f(x)$ - blue ; $g_n(x)$ - red; x_i - '+'

32 points



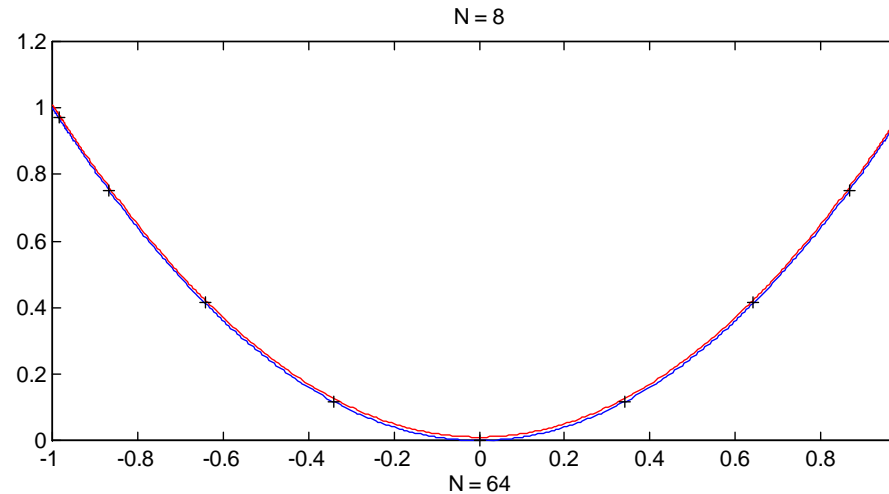
128 points



Chebyshev - collocation points - x^2

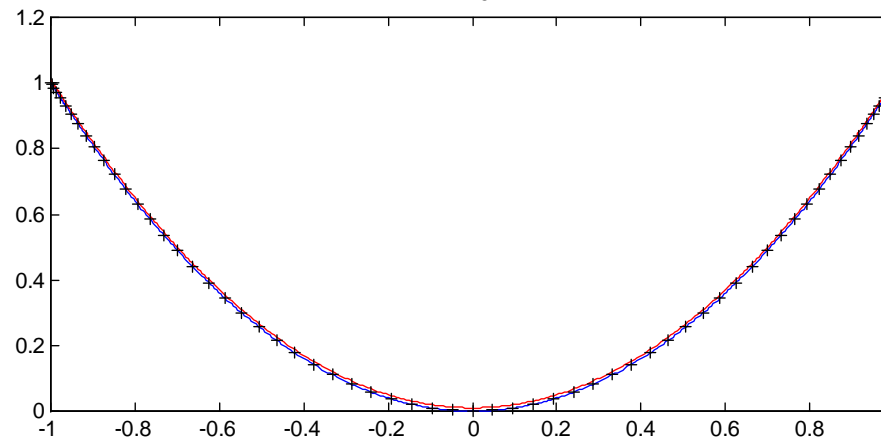
$f(x)=x^2 \Rightarrow f(x)$ - blue ; $g_n(x)$ - red; x_i - '+'

8 points



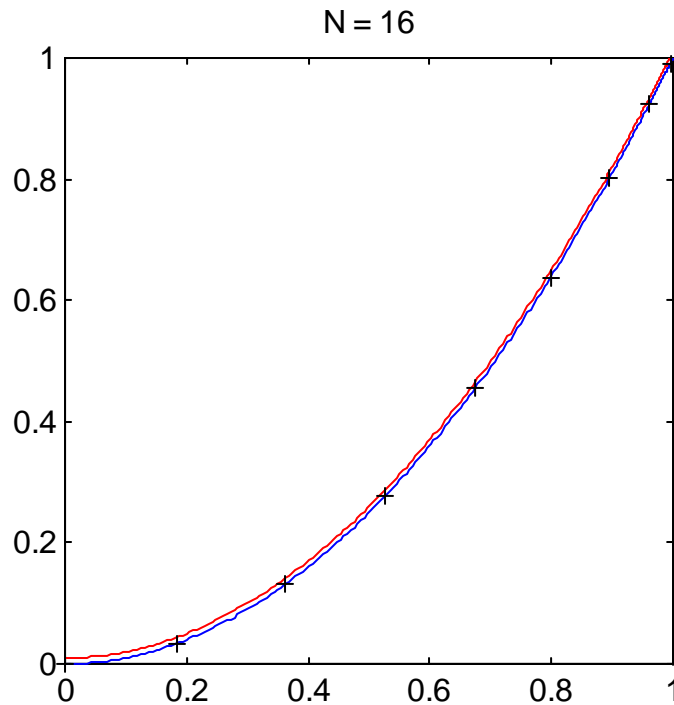
The interpolating function $g_n(x)$ was shifted by a small amount to be visible at all!

64 points

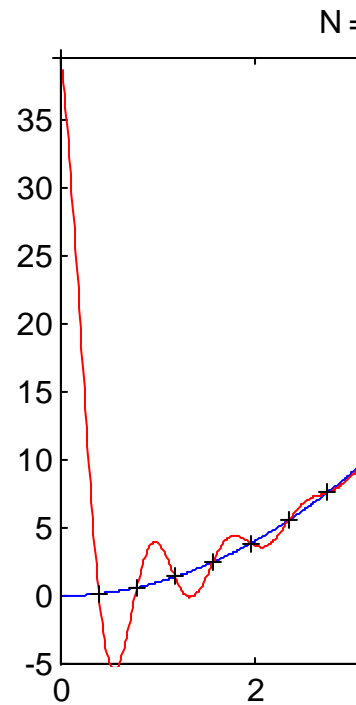


Chebyshev vs. Fourier - numerical

Chebyshev



Fourier

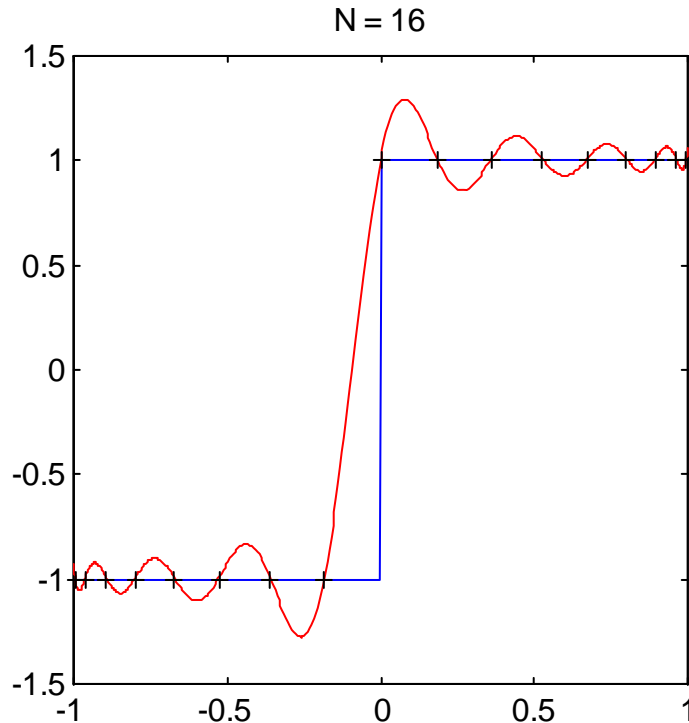


$f(x)=x^2 \Rightarrow f(x)$ - blue ; $g_N(x)$ - red; x_i - '+'

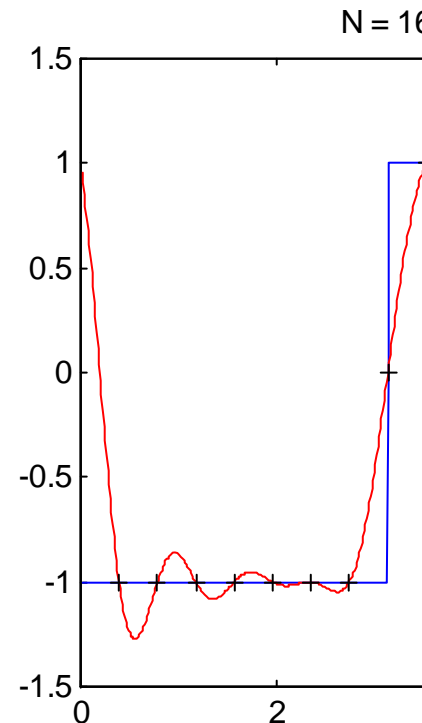
This graph speaks for itself ! Gibb's phenomenon with Chebyshev?

Chebyshev vs. Fourier - Gibb's

Chebyshev



Fourier

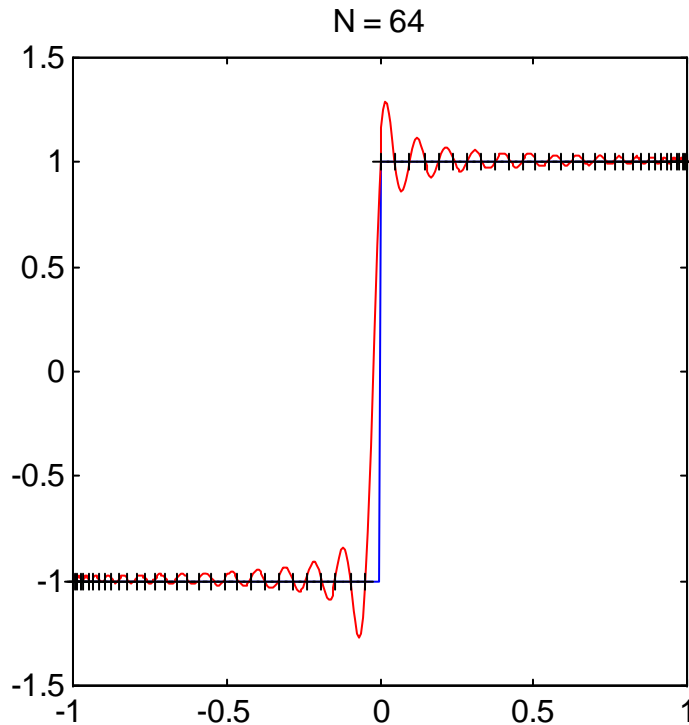


$f(x) = \text{sign}(x - \pi) \Rightarrow f(x)$ - blue ; $g_N(x)$ - red; x_i - '+'

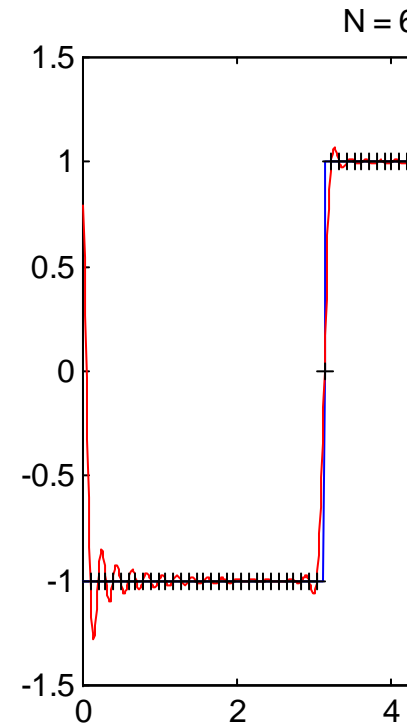
Gibb's phenomenon with Chebyshev? YES!

Chebyshev vs. Fourier - Gibb's

Chebyshev



Fourier



$f(x) = \text{sign}(x - \pi) \Rightarrow f(x)$ - blue ; $g_N(x)$ - red; x_i - '+'

Fourier vs. Chebyshev

Fourier

$$x_i = \frac{2\pi}{N} i$$

periodic functions

$$\cos(nx), \sin(nx)$$

$$\begin{aligned} g_m^*(x) &= \frac{1}{2} a_0^* \\ &+ \sum_{k=1}^{m-1} \{ a_k^* \cos(kx) + b_k^* \sin(kx) \} \\ &+ \frac{1}{2} a_m^* \cos(kx) \end{aligned}$$

collocation points

domain

basis functions

*interpolating
function*

Chebyshev

$$x_i = \cos \frac{\pi}{N} i$$

limited area [-1,1]

$$\begin{aligned} T_n(x) &= \cos(n \varphi), \\ x &= \cos \varphi \end{aligned}$$

$$g_m^*(x) = \frac{1}{2} c_0^* T_0 + \sum_{k=1}^m c_k^* T_k(x)$$

Fourier vs. Chebyshev (cont'd)

Fourier

$$a_k^* = \frac{2}{N} \sum_{j=1}^N f(x_j) \cos(kx_j)$$

$$b_k^* = \frac{2}{N} \sum_{j=1}^N f(x_j) \sin(kx_j)$$

- Gibb's phenomenon for discontinuous functions
- Efficient calculation via FFT
- infinite domain through periodicity

coefficients

some properties

Chebyshev

$$c_k^* = \frac{2}{N} \sum_{j=1}^N f(\cos \varphi_j) \cos(k\varphi_j)$$

- limited area calculations
- grid densification at boundaries
 - coefficients via FFT
- excellent convergence at boundaries
- Gibb's phenomenon