

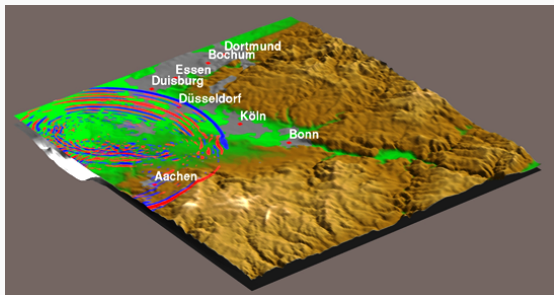
The Finite Difference Method

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Introduction

Motivation



- Simple concept
- Robust
- Easy to parallelize
- Regular grids
- Explicit method

History

- Several Pioneers of solving PDEs with finite-difference method (Lewis Fry Richardson, Richard Southwell, Richard Courant, Kurt Friedrichs, Hans Lewy, Peter Lax and John von Neumann)
- First application to elastic wave propagation (Alterman and Karal, 1968)
- Simulating Love waves and was the first showing snapshots of seismic wave fields (Boore, 1970)
- Concept of staggered-grids by solving the problem of rupture propagation (Madariaga, 1976 and Virieux and Madariaga, 1982)

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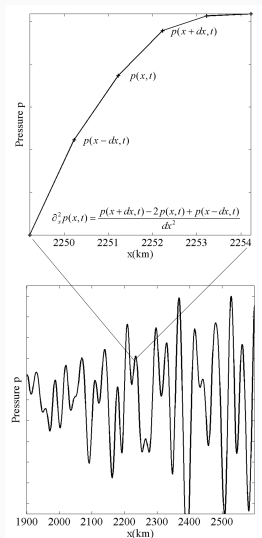
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- Extension to 3D because of parallel computations (Frankel and Vidale, 1992; Olsen and Archuleta, 1996; etc.)
- Application to spherical geometry by Igel and Weber, 1995; Chaljub and Tarantola, 1997 and 3D spherical sections by Igel et al., 2002
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Finite Differences in a Nutshell



- Snapshot in space of the pressure field p
- Zoom into the wave field with grid points indicated by +
- Exact interpolate using Taylor series

Scalar wave equation

1D acoustic wave equation

$$\ddot{p}(x, t) = c(x)^2 \partial_x^2 p(x, t) + s(x, t)$$

- p pressure
- c acoustic velocity
- s source term

Approximation with a difference formula

$$\ddot{p}(x, t) \approx \frac{p(x, t + dt) - 2p(x, t) + p(x, t - dt)}{dt^2}$$

Finite Differences and Taylor Series

Finite Differences

Forward derivative

$$d_x f(x) = \lim_{dx \rightarrow 0} \frac{f(x + dx) - f(x)}{dx}$$

Centered derivative

$$d_x f(x) = \lim_{dx \rightarrow 0} \frac{f(x + dx) - f(x - dx)}{2dx}$$

Backward derivative

$$d_x f(x) = \lim_{dx \rightarrow 0} \frac{f(x) - f(x - dx)}{dx}$$

Finite Differences

Forward derivative

$$d_x f^+ \approx \frac{f(x + dx) - f(x)}{dx}$$

Centered derivative

$$d_x f^c \approx \frac{f(x + dx) - f(x - dx)}{2dx}$$

Backward derivative

$$d_x f^- \approx \frac{f(x) - f(x - dx)}{dx}$$

Finite Differences and Taylor Series

The approximate sign is important here as the derivatives at point x are not exact. Understanding the accuracy by looking at the definition of Taylor Series:

$$f(x + dx) = f(x) + f'(x) dx + \frac{1}{2!} f''(x) dx^2 + O(dx^3)$$

Subtraction with $f(x)$ and division by dx leads to the definition of the forward derivative:

$$\frac{f(x+dx)-f(x)}{dx} = f'(x) + \frac{1}{2!} f''(x) dx + O(dx^2)$$

Finite Differences and Taylor Series

Using the same approach - adding the Taylor Series for $f(x + dx)$ and $f(x - dx)$ and dividing by $2dx$ leads to:

$$\frac{f(x+dx) - f(x-dx)}{2dx} = f'(x) + O(dx^2)$$

This implies a centered finite-difference scheme more rapidly converges to the correct derivative on a regular grid

⇒ It matters which of the approximate formula one chooses

⇒ It does not imply that one or the other finite-difference approximation is always the better one

Higher Derivatives

The partial differential equations have often 2nd (seldom higher) derivatives
Developing from first derivatives by mixing a forward and a backward definition yields

$$\partial_x^2 f \approx \frac{\frac{f(x+dx)-f(x)}{dx} - \frac{f(x)-f(x-dx)}{dx}}{dx} = \frac{f(x+dx) - 2f(x) + f(x-dx)}{dx^2}$$

Higher Derivatives - Alternative derivation

Determining the weights with which the function values have to be multiplied to obtain derivative approximations ...

$$a f(x + dx) = a \left[f(x) + f'(x) dx + \frac{1}{2!} f''(x) dx^2 + \dots \right]$$

$$b f(x) = b [f(x)]$$

$$c f(x - dx) = c \left[f(x) - f'(x) dx + \frac{1}{2!} f''(x) dx^2 - \dots \right]$$

Higher Derivatives - Alternative derivation

How to determine a, b, and c?

$$\begin{aligned} &af(x+dx) + bf(x) + cf(x-dx) \approx \\ &\quad f(x)[a + b + c] \\ &\quad + dx f' [a - c] \\ &\quad + \frac{1}{2!} dx^2 f'' [a + c] \end{aligned}$$

Higher Derivatives - Alternative derivation

To obtain a 2nd derivative we require

$$a + b + c = 0$$

$$a \quad - c = 0$$

$$a \quad + c = \frac{2!}{dx^2}$$

Higher Derivatives - Alternative derivation

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{2!}{dx} \end{pmatrix}$$

A **w** = **s**

with solution

$$\mathbf{w} = \mathbf{A}^{-1}\mathbf{s}$$

$$a = \frac{1}{dx^2}$$

$$b = -\frac{2}{dx^2}$$

$$c = \frac{1}{dx^2}$$

High-Order Operators

What happens if we extend the *domain of influence* for the derivative(s) of our function $f(x)$?

Let us search for a 5-point operator for the second derivative

$$f'' \approx af(x + 2dx) + bf(x + dx) + cf(x) + df(x - dx) + ef(x - 2dx)$$

$$a + b + c + d + e = 0$$

$$2a + b - d - 2e = 0$$

$$4a + b + d + 4e = \frac{1}{2dx^2}$$

$$8a + b - d - 8e = 0$$

$$16a + b + d + 16e = 0$$

High-Order Operators

Using matrix inversion we obtain a unique solution

$$a = -\frac{1}{12dx^2}$$

$$b = \frac{4}{3dx^2}$$

$$c = -\frac{5}{2dx^2}$$

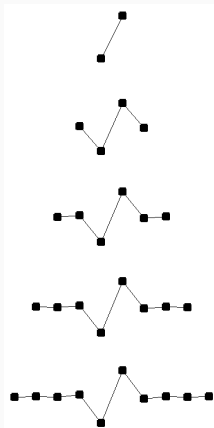
$$d = \frac{4}{3dx^2}$$

$$e = -\frac{1}{12dx^2}.$$

with a leading error term for the 2nd derivative is $O(dx^4)$

$\implies$ Accuracy improvement

High-Order Operators



Graphical illustration of the
Taylor Operators for the first
derivative for higher orders

The weights rapidly become
small for increasing distance to
central point of evaluation

Finite-Difference Approximation of Wave Equations

Acoustic waves in 1D

To solve the wave equation, we start with the simplest wave equation:

The constant density acoustic wave equation in 1D

$$\ddot{p} = c^2 \partial_x^2 p + s$$

imposing pressure-free conditions at the two boundaries as

$$p(x) |_{x=0,L} = 0$$



Acoustic waves in 1D

The following dependencies apply:

$p \rightarrow p(\mathbf{x}, t)$	pressure
$c \rightarrow c(\mathbf{x})$	P-velocity
$s \rightarrow s(\mathbf{x}, t)$	source term

As a first step we need to discretize space and time and we do that with a constant increment that we denote dx and dt .

$$\begin{aligned}x_j &= jdx, & j &= 0, j_{max} \\t_n &= ndt, & n &= 0, n_{max}\end{aligned}$$

Acoustic waves in 1D

Starting from the continuous description of the partial differential equation to a discrete description. The upper index will correspond to the time discretization, the lower index will correspond to the spatial discretization

$$p_j^{n+1} \rightarrow p(x_j, t_n + dt)$$

$$p_j^n \rightarrow p(x_j, t_n)$$

$$p_j^{n-1} \rightarrow p(x_j, t_n - dt)$$

$$p_{j+1}^n \rightarrow p(x_j + dx, t_n)$$

$$p_j^n \rightarrow p(x_j, t_n)$$

$$p_{j-1}^n \rightarrow p(x_j - dx, t_n)$$

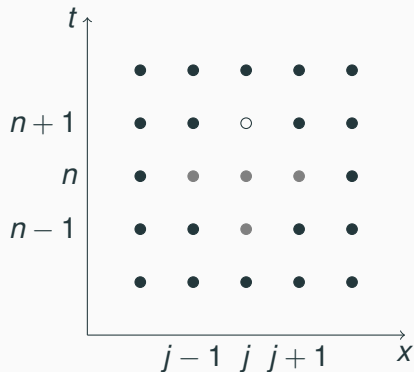
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Acoustic waves in 1D

$$\frac{p_j^{n+1} - 2p_j^n + p_j^{n-1}}{\Delta t^2} = c_j^2 \left[\frac{p_{j+1}^n - 2p_j^n + p_{j-1}^n}{\Delta x^2} \right] + s_j^n.$$

the r.h.s. is defined at same time level n

the l.h.s. requires information from three different time levels



Acoustic waves in 1D

Assuming that information at time level n (the presence) and $n - 1$ (the past) is known, we can solve for the unknown field p_j^{n+1} :

$$p_j^{n+1} = c_j^2 \frac{dt^2}{dx^2} [p_{j+1}^n - 2p_j^n + p_{j-1}^n] + 2p_j^n - p_j^{n-1} + dt^2 s_j^n$$

The initial condition of our wave simulation problem is such that everything is at rest at time $t = 0$:

$$p(x, t)|_{t=0} = 0, \quad \dot{p}(x, t)|_{t=0} = 0.$$

Acoustic waves in 1D

Waves begin to radiate as soon as the source term $s(x, t)$ starts to act

For simplicity: the source acts directly at a grid point with index j_s

Temporal behaviour of the source can be calculated by Green's function

$$s(x, t) = \delta(x - x_s) \delta(t - t_s)$$

where x_s and t_s are source location and source time and $\delta()$ corresponds to the delta function

A delta function contains all frequencies and we cannot expect that our numerical algorithm is capable of providing accurate solutions Operating with a band-limited source-time function:

$$s(x, t) = \delta(x - x_s) f(t)$$

where the temporal behaviour $f(t)$ is chosen according to our specific physical problem

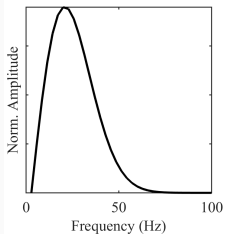
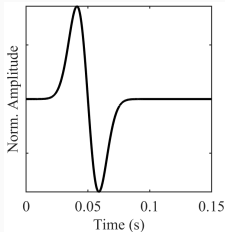
Example

Simulating acoustic wave propagation in a 10km column (e.g. the atmosphere) and assume an air sound speed of $c = 0.343m/s$. We would like to *hear* the sound wave so it would need a dominant frequency of at least 20 Hz. For the purpose of this exercise we initialize the source time function $f(t)$ using the first derivative of a Gauss function.

$$f(t) = -8 f_0 (t - t_0) e^{-\frac{1}{(4f_0)^2} (t-t_0)^2}$$

where t_0 corresponds to the time of the zero-crossing, f_0 is the dominant frequency

Example



- What is the minimum spatial wavelength that propagates inside the medium?
- What is the maximum velocity inside the medium?
- What is the propagation distance of the wavefield (e.g., in dominant wavelengths)?

Example

Sufficient to look at the relation between frequency and wavenumber:

$$c = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f$$

where c is velocity, T is period, λ is wavelength, f is frequency, and $\omega = 2\pi f$ is angular frequency

dominant wavelength of $f_0 = 20\text{Hz}$

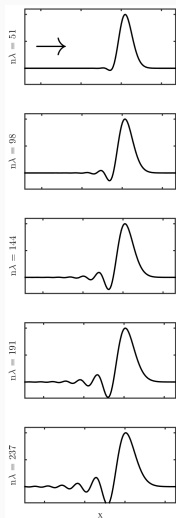
substantial amount of energy in the wavelet is at frequencies above 20 Hz

$\implies \lambda = 17\text{m}$ and $\lambda = 7\text{m}$ for frequencies 20Hz and 50Hz, respectively

Python code fragment

```
# Time extrapolation
for it in range(nt):
    # calculate partial derivatives (omit boundaries)
    for i in range(1, nx - 1):
        d2p[i] = (p[i + 1] - 2 * p[i] + p[i - 1]) / dx ** 2
    # Time extrapolation
    pnew = 2 * p - pold + dt ** 2 * c ** 2 * d2p
    # Add source term at isrc
    pnew[isrc] = pnew[isrc] + dt ** 2 * src[it] / dx
    # Remap time levels
    pold, p = p, pnew
```

Result



Choosing a grid increment of $dx = 0.5m \rightarrow$
about 24 points per spatial wavelength for the
dominant frequency

Setting time increment $dt = 0.0012 \rightarrow$ around
40 points per dominant period

Summary

- Replacing the partial derivatives by finite differences allows partial differential equations such as the wave equation to be solved directly for (in principle) arbitrarily heterogeneous media
- The accuracy of finite-difference operators can be improved by using information from more grid points (i.e., longer operators). The weights for the grid points can be obtained using Taylor series