

# Tutorial on Rotations in the Theories of Finite Deformation and Micropolar (Cosserat) Elasticity

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**Abstract** Although earthquake source studies have had a great impact on tectonics studies, there are at least two important problems for which seismology seems unable to provide answers. One of them refers to the rotation about vertical axes of crustal blocks in continental areas of diffuse deformation. The other problem is the stress rotations observed after large earthquakes. In both cases there are a number of competing explanations but none is supported by hard evidence. These problems are unlikely to be solved by conventional seismology, but the situation may be different if rotation data are acquired. In the near field of large earthquakes the linearized theory may not apply or a different theory may be needed. In this tutorial we consider rotations from two different points of view: the classical nonlinear theory and a nonclassical linear theory. In the nonlinear theory the deformation tensor can be expressed as the product of two tensors, one corresponding to a rotation and the other to strain, applied sequentially. In contrast, in the linearized theory the deformation tensor is the sum of a rotation and a strain tensor and the order of their application is immaterial. A linear theory that includes rotations not considered by classical elasticity (linear or not) is the micropolar theory, which deals with materials with microstructure. This theory assigns to each point in space six degrees of freedom, three corresponding to position and three corresponding to rotations. The specification of a linear micropolar isotropic body requires six elastic moduli, two of which are the classical Lamé's parameters. Wave propagation in a micropolar medium is more complicated than in a linear elastic medium, with two coupled wave equations. The micropolar theory has been successful with media having periodic inner structures, but there is very little experimental work on solids with more complicated structure.

## Introduction

Although seismology is an extremely successful science, most of its success comes from the analysis of far-field data or from the near field of small earthquakes. However, the nature of the near field of large earthquakes and the details of the earthquake rupture itself constitute some of the still unsolved great questions of seismology. As is well known, seismic source information is also extremely important in tectonics studies, (e.g., Jackson, 2002a), but what may not be as well known is that there are some fundamental tectonic problems still awaiting answers and that seismology seems unable to provide them. One of the tectonically important extant problems is the rotation about vertical axes of crustal blocks in continental areas of diffuse deformation. For example, the Western Transverse Ranges (WTR) in southern California have undergone a clockwise rotation of about  $90^\circ$  since the Miocene (Kamerling and Luyendyk, 1979). The WTR is a large block (about  $150 \times 80$  km) and the areas to the north and south of it appear unrotated. Several geometrical models have been proposed to explain the

kinematics of the rotation, but according to Onderdonk (2005) all of them have the problem that there are overlaps and holes at the rotation boundaries and that there is no evidence that these space problems are accommodated by internal deformation, as postulated. Another relevant question regarding these rotations is the nature of the forces that drive them. In general, there are two possible models, depending on whether the motion is driven by forces applied to either the bases or the sides or the blocks. In the first case the strength of the lithosphere resides in the mantle, and in the second case the strength resides in the upper crust. This question itself is a matter of current debate. A popular model likens the lithosphere to a jelly sandwich, with a strong upper crust, a weak lower crust, and a strong uppermost mantle (Brace and Kohlstedt, 1980). More recently, a radically different model with a strong seismogenic upper crust and a weak mantle has been introduced by Jackson (2002b). The general validity of this new model has already been criticized (Burov and Watts, 2006), while Afonso and Ranalli

(2004) concluded that neither model has general applicability and that the strength of the continental lithosphere varies laterally across the earth and depends on composition and tectonothermal history. Although it was expected that the investigation of block rotations would help settle this matter, it seems that it does not give conclusive answers to the questions of the driving force and the strength of the lithosphere (Jackson, 2002c). However, a recent analysis of the rotation of the WTR seems to favor the side forces (Onderdonk, 2007).

The question of block rotation is further complicated by the fact that it occurs within a stronger deformation field acting over a region much larger than the blocks. This regional field originates in the flow of the lower lithosphere and is usually described in terms of a velocity field. This field exerts ultimate control on the motions of the blocks, but because it changes with time, there is a conflict between the faults that bound the rotating blocks and the time-varying velocity field. The reason is that there is a very limited range of fault orientations that can be active for a given value of the velocity field (Holt and Haines, 1993). As a consequence, the faults that bound the blocks at a given time will eventually become inactive because they will become misoriented with respect to the regional stresses (assuming that the blocks are rigid). Eventually, those faults will be replaced by others with more favorable orientations, but before that happens the existing faults may adapt to the deformation field, perhaps by changing their rakes (Jackson, 2002c, 2004). For example, Jackson and McKenzie (1999) show pictures of slightly curved striations exposed on an active normal fault in Greece. This unusual exposure is about 50 m long in the direction of slip, and the observed curvature, about  $1^\circ$  from top to bottom, is compatible with a change in the slip vector as a result of a block rotation about a vertical axis over a number of earthquake cycles. These authors noted, however, that their interpretation raises some questions and difficulties. For example, it is not clear how the slip vector can change direction without deforming the corrugations over the whole fault surface. The relation between block and fault rotations was also made by Nur and Ron (2003), who framed the problem in a frictional context.

It is also worth noting that in addition to block rotations about vertical axes, rotations about horizontal axes are also possible. These types of rotations are usually associated with normal faulting in areas of continental extension (Jackson and White, 1989). For example, Cannon *et al.* (2001) investigated fault and geodetic data associated with the 1975  $M$  7.2 Kilauea volcano (Hawaii) normal-fault earthquake and concluded that the observations could not be explained unless horizontal rotation of the hanging-wall blocks was included. Another example of inferred block rotation is provided by the 1992  $M_w$  7.3 Landers, California, strike-slip earthquake (Peltzer *et al.*, 1994). In this case the horizontal rotation of an  $\approx 3 \times 5$  km crustal block was needed to explain certain patterns in synthetic aperture radar images.

Rotations are also important in seismology because of the stress rotations that have been determined after the occurrence of relatively large earthquakes, such as the 1983  $M_L$  6.4 Coalinga (Michael, 1987; Hardebeck and Michael, 2006), the 1992  $M_w$  7.3 Landers (Hauksson, 1994), and the 1994  $M_w$  6.7 Northridge (Zhao *et al.*, 1997) earthquakes in California and the 2002  $M_w$  7.9 Denali (Alaska, Ratchkovski, 2003) and 1999  $M_w$  7.4 Izmit (Turkey, Bohnhoff *et al.*, 2006) earthquakes. Although questions have been raised as to whether some of the rotations are real or processing artifacts (Townend and Zoback, 2001; Hardebeck and Michael, 2006; Townend, 2006), there seems to be a consensus that large earthquakes produce stress rotations. What is not clear, however, is what causes them. For example, Michael (1987) and Zhao *et al.* (1997) suggested the possibility of inelastic processes in the rupture zone, while Hauksson (1994) explained his results in terms of stress refraction on a weak fault zone. More recently, Smith and Dieterich (2007) found that a heterogeneous 3D fractal distribution of crustal stresses creates an apparent stress rotation after the occurrence of a major earthquake.

The questions and problems described previously have been around for a long time, and observational seismology, as practiced today, has not been able to provide the information required to answer and solve them. It may well be that what is needed is to collect new types of information, such as rotational motions in the near field of large earthquakes. For example, if the rotation of a block about an axis occurs as part of the earthquake process, it would be very helpful if the rotation could be detected by seismological means. Regarding stress rotations, the results of Smith and Dieterich (2007) are significant, but the question here is whether there is any objective way to assess the validity of their model. Again, it would be desirable to have observational evidence to help constrain the solution of what appears to be an ill-posed problem, and the investigation of near-field rotations may provide valuable new insights. As the work of Suryanto *et al.* (2006) and Igel *et al.* (2007) have shown, the rotations measured in the far field are consistent with those determined using conventional seismic data, which may mean that truly new rotational information may be available only in the near field. In addition, theories that go beyond the standard linearized theory used today may be needed. For example, the assumption of small strains in the source area may be too restrictive and may have to be relaxed. In fact, Nigbor (1994) recorded the rotations generated by a large chemical explosion and concluded that near-field terms and large strains are likely responsible for much of the observed ground rotation. As shown subsequently, finite deformations can be represented by the consecutive application of two tensors, one of which rotates the principal directions of strain. In contrast, in the infinitesimal theory the two tensors are added together, which simplifies the mathematics involved. On the other hand, the classical theory of elasticity, linear or not, is based on the assumption that the interaction between the particles in a medium takes effect via contact forces only, that is,

couples are not considered. This restriction was removed by a theory introduced by the Cosserat brothers in 1909, which allowed the rotation of the particles of the medium. This theory has expanded greatly, although its application to seismic and tectonic problems has been very limited (see subsequently). For completeness, we also note that the classical linear seismic theory also allows the presence of rotations when faulting takes place on a finite-thickness fault and there is a zone of strength weakening near the advancing crack tip (Knopoff and Chen, 2009).

In this article we will go over the basic mathematical aspects of the finite strain and Cosserat elasticity theories and will discuss a number of basic properties of rotations. Familiarity with the basic concepts of classical continuum mechanics, tensor calculus, and indicial notation is assumed. Introductory references are, for example, Atkin and Fox (1980), Mase (1970), and Pujol (2003).

### Rotations and Orthogonal Second-Order Tensors

This section is based on Chadwick (1999) and Ogden (1997) for the finite rotations and on Pujol (2003) for the infinitesimal ones. The major results are proved in the Appendix. As we will see in the next section, finite deformations can be expressed in terms of two tensors, one of which is related to finite rotations. Here we will derive an expression for that tensor by consideration of the following problem: if  $\mathbf{r}'$  is a vector obtained by rotation of a vector  $\mathbf{r}$ , find the second-order tensor  $\mathbf{Q}$  such that  $\mathbf{r}' = \mathbf{Q}\mathbf{r}$ . Rotations are defined by two elements, an axis and an angle. The axis, in turn, can be defined by a unit vector. Let  $\mathbf{a}$  and  $\phi$  be these two elements and let  $\mathbf{r}$  be an arbitrary vector. Then a rotation of angle  $\phi$  of  $\mathbf{r}$  about  $\mathbf{a}$  results in the vector

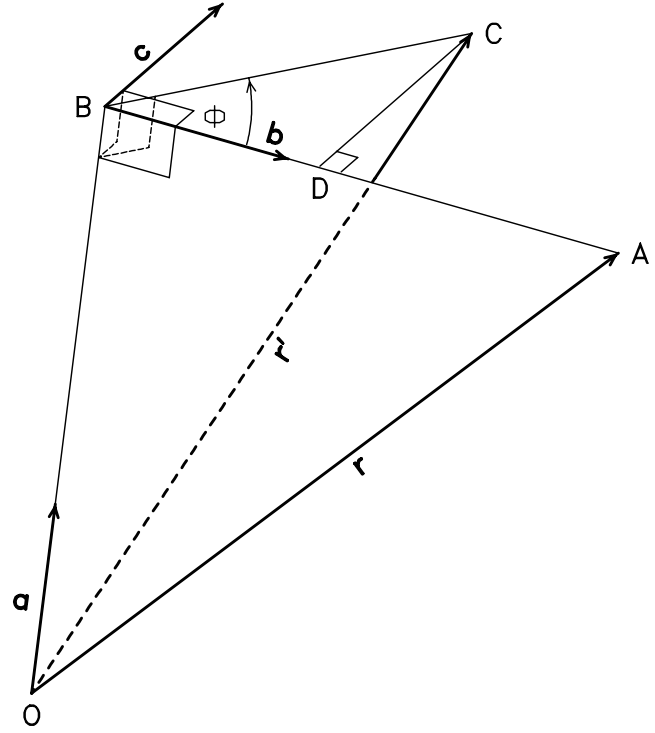
$$\mathbf{r}' = \mathbf{Q}\mathbf{r} = \cos \phi \mathbf{r} + (1 - \cos \phi)(\mathbf{r} \cdot \mathbf{a})\mathbf{a} + \sin \phi \mathbf{a} \times \mathbf{r}. \quad (1)$$

Using this result we get the following explicit expression for  $\mathbf{Q}$ :

$$\mathbf{Q} = \mathbf{a}\mathbf{a} + (\mathbf{b}\mathbf{b} + \mathbf{c}\mathbf{c}) \cos \phi - (\mathbf{b}\mathbf{c} - \mathbf{c}\mathbf{b}) \sin \phi, \quad (2)$$

where the juxtaposed vectors constitute dyads. The geometry of the problem and the vectors involved are shown in Figure 1. As can be seen from equation (1),  $\mathbf{a}$  is an eigenvector of  $\mathbf{Q}$  with an eigenvalue equal to one. Two other eigenvalues are  $\exp(\pm i\phi)$ , which means that  $\mathbf{a}$  is the only real-valued eigenvector. For this reason, in equation (2) the only constraint on  $\mathbf{b}$  and  $\mathbf{c}$  is that they, together with  $\mathbf{a}$ , form an orthonormal set. When  $\mathbf{Q}$  is applied to an arbitrary vector  $\mathbf{r}$ , there are two possibilities depending on whether  $\mathbf{b}$  and  $\mathbf{c}$  have been fixed or not. In the first case we get the following alternative expression for  $\mathbf{Q}\mathbf{r}$ :

$$\begin{aligned} \mathbf{Q}\mathbf{r} = & r_a \mathbf{a} + (r_b \cos \phi - r_c \sin \phi) \mathbf{b} \\ & + (r_b \sin \phi + r_c \cos \phi) \mathbf{c}, \end{aligned} \quad (3)$$



**Figure 1.** Geometry for the derivation of equation (1). The vector  $\mathbf{r}'$  is obtained by rotation of angle  $\phi$  of  $\mathbf{r}$  about  $\mathbf{a}$ . The vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  constitute an orthonormal set.

where

$$r_a = \mathbf{a} \cdot \mathbf{r}, \quad r_b = \mathbf{b} \cdot \mathbf{r}, \quad r_c = \mathbf{c} \cdot \mathbf{r}, \quad (4)$$

and a dot indicates the scalar (or dot) product. In the second case  $\mathbf{b}$  can be chosen as shown in Figure 1, and equation (1) is recovered.

The tensor  $\mathbf{Q}$  also has the property that

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{Q} \mathbf{Q}^T = \mathbf{I}, \quad (5)$$

where the superscript  $T$  indicates transposition and  $\mathbf{I}$  represents the identity tensor (see the Appendix). Tensors that satisfy equation (5) are known as orthogonal, by analogy with a similar definition for matrices. All the tensors introduced in this article are of the second order, and because they admit matrix representations, expressions such as  $\mathbf{Q}\mathbf{r}$  and  $\mathbf{Q}\mathbf{Q}^T$  can be interpreted in terms of products involving matrices and vectors (see the Appendix).

Immediate consequences of equation (5) are

$$|\mathbf{Q}\mathbf{u}| = |\mathbf{u}|, \quad (\mathbf{Q}\mathbf{u}_1) \cdot (\mathbf{Q}\mathbf{u}_2) = \mathbf{u}_1 \cdot \mathbf{u}_2, \quad (6)$$

where the vertical bars indicate vector length and the vectors involved are arbitrary. Therefore, rotations preserve vector length and the angle between two vectors, as expected.

It is interesting to note that equation (1) allows writing the components of  $\mathbf{Q}$  in terms of only  $\phi$  and the components of  $\mathbf{a}$ , namely,

$$Q_{ij} = (\cos \phi) \delta_{ij} + (1 - \cos \phi) a_i a_j - (\sin \phi) \epsilon_{ijk} a_k \quad (7)$$

(Jansen and Boon, 1967). However, this equation cannot be converted into a coordinate-free expression involving only  $\phi$  and  $\mathbf{a}$ . On the other hand, equation (7) can be derived from equation (2) (see the Appendix).

The preceding equations correspond to finite rotations. When  $\phi \ll 1$  the rotation is infinitesimal. In this case,  $\cos \phi \approx 1$  and  $\sin \phi \approx \phi$ , and equation (1) becomes

$$\mathbf{r}' = \mathbf{Q}\mathbf{r} = \mathbf{r} + \phi \mathbf{a} \times \mathbf{r} \quad (8)$$

within the approximations involved. This result can be derived by direct consideration of infinitesimal rotations. Now let  $\mathbf{P}$  be the tensor corresponding to an infinitesimal rotation of angle  $\theta$  about a vector  $\mathbf{t}$  and consider the effect of  $\mathbf{P}$  on  $\mathbf{Q}\mathbf{r}$ . After letting  $\mathbf{d} = \mathbf{a} \times \mathbf{r}$  and using equation (8) we get

$$\begin{aligned} \mathbf{P}\mathbf{Q}\mathbf{r} &= \mathbf{P}\mathbf{r} + \phi \mathbf{P}\mathbf{d} = \mathbf{r} + \theta \mathbf{t} \times \mathbf{r} + \phi \mathbf{d} + \theta \phi \mathbf{t} \times \mathbf{d} \\ &= \mathbf{r} + \theta \mathbf{t} \times \mathbf{r} + \phi \mathbf{a} \times \mathbf{r} = \mathbf{Q}\mathbf{P}\mathbf{r}, \end{aligned} \quad (9)$$

where the term containing  $\theta\phi$  has been neglected. This result shows that infinitesimal rotations are additive. Thus, they commute. In general, these properties do not apply to finite rotations.

In general, the components of  $\mathbf{Q}$  corresponding to an infinitesimal rotation are of the form

$$Q_{ij} = \delta_{ij} + \alpha_{ij}, \quad |\alpha_{ij}| \ll 1, \quad (10)$$

where  $\alpha_{ij}$  is an antisymmetric tensor (i.e.,  $\alpha_{ij} = -\alpha_{ji}$ ). This antisymmetry follows from equation (5). In addition, every antisymmetric tensor has an associated vector (see the Appendix), which in the case of  $\alpha_{ij}$  is the vector  $\phi \mathbf{a}$  in equation (8).

Using equation (7) with  $\phi \ll 1$  we get the following expression

$$Q_{ij} = \delta_{ij} - \phi \epsilon_{ijk} a_k. \quad (11)$$

Comparison with equation (10) shows that  $\alpha_{ij}$  is equal to minus the second term on the right.

Infinitesimal rotations allow the introduction of the concept of angular velocity in a simple way (i.e., Synge, 1960). Let  $\mathbf{r}$  be a function of time representing the position of a particle and let  $d\mathbf{r} = \mathbf{r}' - \mathbf{r}$ . To indicate the infinitesimal nature of  $\phi$  it will be replaced by  $d\phi$ . The velocity  $\mathbf{v}$  of the particle is defined by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\phi}{dt} \mathbf{a} \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{r}, \quad \boldsymbol{\omega} \equiv \frac{d\phi}{dt} \mathbf{a}, \quad (12)$$

where equation (8) was used and  $\boldsymbol{\omega}$  is the angular velocity vector.

Finally, recall that the result of the vector product of two vectors  $\mathbf{a}$  and  $\mathbf{b}$  is a pseudovector, say  $\mathbf{c}$ , whose direction is such that the triad  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  has the handedness of the coordinate system (i.e., it is right- or left-handed). Changing the handedness changes the direction of  $\mathbf{c}$ . This is what makes  $\mathbf{c}$

a pseudovector (also known as axial vector), rather than a vector. When the handedness is fixed,  $\mathbf{c}$  can be treated as a regular vector, but the distinction cannot be always ignored. For example, vectors cannot be equated to pseudovectors because they behave differently under reflections of coordinate axes (e.g., Byron and Fuller, 1992). An example of reflection is provided by a transformation whose matrix has diagonal elements equal to 1, 1,  $-1$  and all the other elements are equal to zero. The constitutive equation for micropolar solids, discussed subsequently, provides an example of how to handle pseudovectors.

## Finite Deformations

Classical continuum mechanics studies the deformation and motion of bodies ignoring the discrete nature of matter. Let  $V$  and  $\mathcal{V}$  and  $S$  and  $\mathcal{S}$  indicate the volume and surface of a body before and after deformation, respectively. The deformation will be assumed to be a function of time,  $t$ , and  $V$  and  $S$  will indicate a reference state. As a consequence of the deformation, a volume  $\Delta V$  within  $V$  will become  $\Delta v$  within  $\mathcal{V}$ . Let  $\mathbf{X} = (X_1, X_2, X_3)$  indicate the vector position of a particle in  $\Delta V$  and  $\mathbf{x} = (x_1, x_2, x_3)$  indicate the vector position in  $\Delta v$  corresponding to the particle that was initially at  $\mathbf{X}$ . Vector  $\mathbf{X}$  labels the particles of the body while  $\mathbf{x}$  describes the motion of the particles. We can write

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t), \quad \mathbf{X} = \mathbf{X}(\mathbf{x}, t), \quad (13)$$

where  $\mathbf{X}(\mathbf{x}, t)$  can be considered the inverse motion. This pair of equations can be viewed as a transformation of coordinates and requires the assumption that its Jacobian does not vanish. Deformation can be described using the Lagrangian (or material) and Eulerian (or spatial) point of views. In the first case  $\mathbf{X}$  is the independent variable and in the second case it is  $\mathbf{x}$ . With few exceptions, capital and small letters will be used to represent properties in the reference and deformed states, respectively.

Most of the following results are taken from, or based on, Ogden (1997). Let us take the differential of  $\mathbf{x}$ . From equations (13) we get

$$dx_i = \frac{\partial x_i}{\partial X_J} dX_J. \quad (14)$$

The tensor form of this equation is

$$d\mathbf{x} = \mathbf{A} d\mathbf{X}, \quad (15)$$

where  $\mathbf{A}$  is the deformation gradient, which is a tensor. The components of  $\mathbf{A}$  are  $A_{i,J} = x_{i,J}$ . Now we will investigate two geometric aspects of the deformation. First we will consider the ratio of the lengths of  $d\mathbf{x}$  and  $d\mathbf{X}$ , denoted  $ds$  and  $dS$ , respectively. Let  $\mathbf{t}$  be the unit vector along  $d\mathbf{X}$ . Then we can write

$$d\mathbf{X} = \mathbf{t} dS, \quad d\mathbf{x} = \mathbf{A} \mathbf{t} dS. \quad (16)$$

The ratio of lengths

$$\lambda(\mathbf{t}) \equiv \frac{ds}{dS} = |\mathbf{A}\mathbf{t}| = [(\mathbf{A}\mathbf{t}) \cdot (\mathbf{A}\mathbf{t})]^{1/2} = [\mathbf{t} \cdot (\mathbf{A}^T \mathbf{A} \mathbf{t})]^{1/2} \quad (17)$$

is known as the stretch in the direction  $\mathbf{t}$ . For the last equality, see equation (A29).

Now consider two line elements  $d\mathbf{X}_1$  and  $d\mathbf{X}_2$  and the corresponding  $d\mathbf{x}_1$  and  $d\mathbf{x}_2$ . Let  $\theta$  and  $\theta + \gamma$  be the angles between these two pairs of vectors. Forming the scalar products  $d\mathbf{X}_1 \cdot d\mathbf{X}_2$  and  $d\mathbf{x}_1 \cdot d\mathbf{x}_2$  and using equations (16) and (17) with appropriate subindices we get

$$\cos \theta = \mathbf{t}_1 \cdot \mathbf{t}_2, \quad \cos(\theta + \gamma) = \frac{\mathbf{t}_1 \cdot (\mathbf{A}^T \mathbf{A} \mathbf{t}_2)}{\lambda(\mathbf{t}_1)\lambda(\mathbf{t}_2)}. \quad (18)$$

These two equations allow the computation of  $\gamma$ , which is the change in angle caused by the deformation.

A different view of the deformation can be obtained when  $\mathbf{A}$  is written in terms of its polar decomposition, given by

$$\mathbf{A} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}, \quad (19)$$

where  $\mathbf{R}$  corresponds to a finite rotation (as defined in the previous section) and  $\mathbf{U}$  and  $\mathbf{V}$  are second-order positive definite symmetric tensors (see the Appendix) satisfying

$$\mathbf{A}^T \mathbf{A} = \mathbf{U}^T \mathbf{U} = \mathbf{U}^2, \quad \mathbf{A} \mathbf{A}^T = \mathbf{V}^T \mathbf{V} = \mathbf{V}^2. \quad (20)$$

The tensors  $\mathbf{U}$  and  $\mathbf{V}$  are known as the right and left stretch tensors. If  $\mathbf{R} = \mathbf{I}$ ,  $\mathbf{A} = \mathbf{U} = \mathbf{V}$  and the deformation is known as pure strain. If  $\mathbf{U} = \mathbf{V} = \mathbf{I}$ , the deformation corresponds to a rigid rotation. In, this case, from equations (17) and (18) we see that the lengths of the line elements and the angle between any two directions remain unchanged (as expected). Thus, the deformation generates zero strain.

Let us investigate the relation between the right stretch tensor and the stretch defined by equation (17). Introducing the first of equations (20) into equation (17) gives

$$\lambda(\mathbf{t}) = [\mathbf{t} \cdot (\mathbf{U}^2 \mathbf{t})]^{1/2} = |\mathbf{U}\mathbf{t}|. \quad (21)$$

Now let  $\mathbf{u}_i$  and  $\lambda_i$  indicate one of the eigenvectors of  $\mathbf{U}$  and the corresponding eigenvalue. Also let  $|\mathbf{u}_i| = 1$ . Furthermore, because  $\mathbf{U}$  is positive definite, its eigenvalues are all positive. Then, replacing  $\mathbf{t}$  in equation (21) by  $\mathbf{u}_i$  and using  $|\mathbf{U}\mathbf{u}_i| = \lambda_i$  gives  $\lambda(\mathbf{u}_i) = \lambda_i$ . For this reason, the  $\lambda_i$  ( $i = 1, 2, 3$ ) are known as the principal stretches.

To establish the properties of the left stretch tensor, we will consider

$$\mathbf{V}\mathbf{R}\mathbf{u}_i = \mathbf{R}\mathbf{U}\mathbf{u}_i = \lambda_i \mathbf{R}\mathbf{u}_i, \quad (22)$$

where equation (19) was used. This result shows that  $\mathbf{V}$  and  $\mathbf{U}$  have the same eigenvalues  $\lambda_i$  and that the eigenvectors of  $\mathbf{V}$ , equal to  $\mathbf{R}\mathbf{u}_i$ , are obtained by the rotation of those of  $\mathbf{U}$ .

Two additional important results are as follows. First,  $\mathbf{A}^T \mathbf{A}$  and  $\mathbf{U}$  have the same eigenvectors. This can be seen from

$$\mathbf{A}^T \mathbf{A} \mathbf{u}_i = \mathbf{U}^2 \mathbf{u}_i = \mathbf{U} \mathbf{U} \mathbf{u}_i = \lambda_i \mathbf{U} \mathbf{u}_i = \lambda_i^2 \mathbf{u}_i. \quad (23)$$

Second, the eigenvectors  $\mathbf{u}_i$ , which constitute an orthonormal basis, remain orthogonal after the deformation. To see that, use the second of equations (18) with  $\mathbf{t}_1 = \mathbf{u}_i$  and  $\mathbf{t}_2 = \mathbf{u}_j$ , the first of equations (20), and  $\lambda(\mathbf{u}_i) = \lambda_i$ . Then comparison with the first of equations (18) gives  $\gamma = 0$ . Therefore, the eigenvectors remain orthogonal, although they are rotated.

Now let us consider the sphere defined by  $|d\mathbf{x}|^2 = c^2$ , where  $c$  is a constant, in the deformed space centered at  $\mathbf{x}$ . In the undeformed state this corresponds to the reciprocal Lagrangian strain ellipsoid

$$d\mathbf{X} \cdot (\mathbf{A}^T \mathbf{A} d\mathbf{X}) = |d\mathbf{x}|^2 = c^2 \quad (24)$$

with the principal axes given by the  $\mathbf{u}_i$ . These axes are called the principal axes of strain (Love, 1944).

In view of the preceding results, the polar decomposition  $\mathbf{A} = \mathbf{R}\mathbf{U}$  can be interpreted as follows. First apply  $\mathbf{U}$  to the principal axes of strain, which results in a stretching of the axes according to  $\mathbf{U}\mathbf{u}_i = \lambda_i \mathbf{u}_i$ . Then, apply  $\mathbf{R}$ , which rotates the stretched axes, so that  $\mathbf{A}\mathbf{u}_i = \lambda_i \mathbf{R}\mathbf{u}_i$ . As shown previously, the three axes remain orthogonal after the deformation but not during its intermediate stages.

Finally, let us consider the displacement vector  $\mathbf{u}$ , defined as

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{x} - \mathbf{X}. \quad (25)$$

Solving for  $\mathbf{x}$  and applying the gradient operation to the resulting equation we get

$$\mathbf{A} = \mathbf{I} + \mathbf{D}, \quad (26)$$

where  $\mathbf{D}$  is the displacement gradient, which is a tensor with components  $D_{IJ} = \partial u_I / \partial X_J = u_{I,J}$ .

Now we will compare finite and infinitesimal deformations, the latter defined by the condition that  $\mathbf{U}$  and  $\mathbf{R}$  are close to the identity tensor. Thus we will write

$$\mathbf{U} = \mathbf{I} + \delta\mathbf{U}, \quad \mathbf{R} = \mathbf{I} + \delta\mathbf{R}, \quad (27)$$

with the components of  $\delta\mathbf{U}$  and  $\delta\mathbf{R}$  much smaller than one in absolute value. The tensor  $\delta\mathbf{R}$  is antisymmetric (see equation 10), while  $\delta\mathbf{U}$  must be symmetric because  $\mathbf{U}$  is also. Now introducing equations (27) into equation (19) and neglecting second-order terms gives

$$\mathbf{A} = (\mathbf{I} + \delta\mathbf{R})(\mathbf{I} + \delta\mathbf{U}) = \mathbf{I} + \delta\mathbf{U} + \delta\mathbf{R}. \quad (28)$$

Comparison of this result with equation (26) shows that

$$\mathbf{D} = \delta\mathbf{U} + \delta\mathbf{R}. \quad (29)$$

This decomposition of  $\mathbf{D}$  is unique (see the Appendix) and using it we can write

$$\delta\mathbf{U} = \frac{1}{2}(\mathbf{D} + \mathbf{D}^T), \quad \delta\mathbf{R} = \frac{1}{2}(\mathbf{D} - \mathbf{D}^T), \quad (30)$$

which are the results of the infinitesimal theory of elasticity, with  $\delta\mathbf{U}$  and  $\delta\mathbf{R}$  the strain and rotation tensors, respectively. Note the significant difference between the effects of finite and infinitesimal deformations on  $d\mathbf{X}$  (see equation 15). In the first case,  $\mathbf{U}$  and  $\mathbf{R}$  are applied sequentially,  $\mathbf{U}$  first and then  $\mathbf{R}$ ; the order of application cannot be reversed. In the second case, equation (28) shows that the strain and rotation effects are added together, which means that the order of application is immaterial. Clearly, the distinction between  $\mathbf{U}$  and  $\mathbf{V}$  disappears. In addition, in the infinitesimal case we can proceed further by introducing equation (28) into equation (15) and rearranging slightly, which gives

$$d\mathbf{x} = \delta\mathbf{U}d\mathbf{X} + (\mathbf{I} + \delta\mathbf{R})d\mathbf{X}. \quad (31)$$

This equation shows that the deformation of  $d\mathbf{x}$  has two contributions, one of which is the infinitesimal rotation due to the tensor  $\mathbf{I} + \delta\mathbf{R}$ . The corresponding rotation axis is defined by the vector associated with  $\delta\mathbf{R}$ , which is equal to  $(1/2)\nabla \times \mathbf{u}$ , while the rotation angle is equal to the absolute value of this vector (Pujol, 2003). Note that these results cannot be derived by consideration of  $\delta\mathbf{R}$  independently of the combination  $\mathbf{I} + \delta\mathbf{R}$ .

### Micropolar (Cosserat) Elasticity

In general, a system of forces acting on a rigid body is statically equivalent to a single force acting at an arbitrary point and a couple (e.g., Arya, 1990). This idea is used in continuum mechanics to introduce the stress vector, which is done by consideration of the forces across a surface element within a body. In this case, however, only the single force is considered (e.g., Love, 1944; Pujol, 2003). The contribution of the couple is neglected because the surface is allowed to go to zero, so that the arm of the couple, and thus its magnitude, go to zero (under the assumption that the forces involved in the couple remain bounded). The same argument is used to preclude the presence of body couples (Eringen, 1967, 1968). Although classical elasticity has been extraordinarily successful, alternative formulations have been presented. For example, Poisson in 1842 regarded the molecules in a crystal as small rigid bodies capable of rotation, an idea that was further elaborated by Voigt in 1877 (Love, 1944, p. 620). Rotations were treated systematically by the brothers E. and F. Cosserat (1909), who let each point have six degrees of freedom, three corresponding to position, as in the classical theory, and three corresponding to rotation. The latter was allowed by introducing a rigid trirectangular trihedral

(“trièdre trirectangle”), or orthogonal triad, at each point of the medium, and a consequence was the introduction of couple stresses. Bodies that allow them are known as polar. The work of the Cosserats turned out to be extremely important, although it went almost unnoticed for about 50 yr. For an overview and relevant references see, for example, Cowin (1970), Toupin (1964), and Truesdell and Noll (1965). The latter authors use the name directors to refer to the unit vectors of the triad (see also Ericksen and Truesdell, 1958). In the early 1960s several polar theories were introduced, but the one that seems to have gained the most popularity is the linear micropolar theory of Eringen (1966), which is a special case of the more general micromorphic theory developed by Eringen and Suhubi (1964). The micropolar theory corresponds to, and extends, the Cosserats’ theory.

The basics of the micropolar theory will be introduced here. This presentation follows closely a comprehensive review article by Eringen (1968), including the naming of his sections (here subsections). As in the previous section, the basic principles of continuum mechanics are assumed to be known; only the new features are given special attention. The reader is referred to the original article for the derivations of the results that are only quoted. Eringen (1999) introduced some modifications to the early formulation of the theory, some of which will be noted subsequently. The basic results, however, remain unchanged.

The micropolar theory of elasticity extends the classical continuum theory to bodies with microstructure. This includes, for example, crystalline solids, granular solids, and composite materials. The theory is already over 40 yr old, and there has been ample time to test it. As summarized by Eringen (1999), for bodies with periodic structures, such as crystals, and man-made structures, such as tall buildings, the theory has been successful. Unfortunately, for materials with arbitrary microstructure, such as earth materials, very little work has been done. As will be seen subsequently, the characterization of a micropolar medium requires six elastic coefficients whose experimental determination is very difficult (e.g., Lakes, 1995; Eringen, 1999). In fact, few experiments have been carried out (e.g., Gauthier, 1982; Lakes, 1982, 1986), and as far as the author is aware, earth materials have not been investigated. One of the main obstacles is the fact that micropolar effects can be observed only at short wavelengths and high frequencies and that they must have order of magnitudes similar to those of the characteristic length and time of the micropolar body (Eringen, 1999). Therefore, finding out whether micropolar rotations are relevant in earthquake seismology will require considerable effort, both experimental and theoretical. Regarding the latter, Ieşan (1981) and Teisseyre (1973) applied the micropolar and micromorphic theories to earthquake problems. The micropolar theory has also been applied to tectonic problems by Twiss and coworkers (e.g., Twiss *et al.*, 1991, 1993; Lewis *et al.*, 2007; Twiss, 2009).

### Deformation and Microdeformation

The elasticity theories that allow couple stresses assume the existence of an elastic continuum medium consisting of deformable points or point particles (Eringen and Suhubi, 1964; Toupin, 1964; Eringen, 1999). Clearly, this concept cannot be accommodated within classical continuum mechanics, and to handle it Eringen and Suhubi (1964) introduced macro-volume elements (corresponding to the deformable points) composed of microvolume elements.

Let us assume that a macrovolume  $\Delta V$  contains  $N$  microelements  $\Delta V^{(\alpha)}$ ,  $\alpha = 1, 2, \dots, N$ , each with density  $\rho^{(\alpha)}$ . Let the position of the  $\alpha$ th microelement be measured with respect to the center of mass  $P$  of the macrovolume  $\Delta V$ . Let  $\mathbf{X}$  indicate the position of  $P$ . Then, as Figure 2a shows, the position of  $\Delta V^{(\alpha)}$ , represented by the point  $Q$ , is given by

$$\mathbf{X}^{(\alpha)} = \mathbf{X} + \mathbf{\Xi}^{(\alpha)}. \quad (32)$$

After the deformation  $\Delta V$  becomes a new volume  $\Delta v$ , the center of mass moves to a point  $p$  with position vector  $\mathbf{x}$ , and  $\mathbf{X}^{(\alpha)}$  moves to  $\mathbf{x}^{(\alpha)}$ , given by

$$\mathbf{x}^{(\alpha)} = \mathbf{x}(\mathbf{X}, t) + \mathbf{\xi}^{(\alpha)}(\mathbf{X}, \mathbf{\Xi}^{(\alpha)}, t). \quad (33)$$

To proceed further it is necessary to relate  $\mathbf{\xi}^{(\alpha)}$  to  $\mathbf{\Xi}^{(\alpha)}$ . The following linear expression is physically justifiable for small  $\Delta V$

$$\mathbf{\xi}^{(\alpha)} = \chi_K(\mathbf{X}, t)\mathbf{\Xi}_K^{(\alpha)}, \quad \xi_k^{(\alpha)} = \chi_{kK}(\mathbf{X}, t)\Xi_K^{(\alpha)}, \quad (34)$$

where  $\chi_K$  ( $K = 1, 2, 3$ ) are three vector functions that represent the microdeformation and the sum convention over repeated indices was used. The right-hand equation is the

component form of the left-hand equation. Introducing this expression into equation (33) gives

$$\mathbf{x}^{(\alpha)} = \mathbf{x}(\mathbf{X}, t) + \chi_K(\mathbf{X}, t)\mathbf{\Xi}_K^{(\alpha)} \quad (35)$$

or, in component form,

$$x_k^{(\alpha)} = x_k(\mathbf{X}, t) + \chi_{kK}(\mathbf{X}, t)\Xi_K^{(\alpha)}, \quad k, K = 1, 2, 3. \quad (36)$$

Eringen (1999) has little use for the microvolumes, the superscript  $(\alpha)$  is no longer used, and gives cesium chloride (CsCl) in crystal form as an example of a deformable particle. In this case, a  $\text{Cs}^+$  ion is at the center of a cube surrounded by eight  $\text{Cl}^-$  ions placed on the corners of the cube (see also Toupin, 1964). The motion of the  $\text{Cs}^+$  and  $\text{Cl}^-$  ions are accounted for by  $\mathbf{x}(\mathbf{X}, t)$  and  $\chi_K(\mathbf{X}, t)$ , respectively

Let us introduce the inverse micromotion  $\mathfrak{X}_{Kk}$ , defined by

$$\chi_{kK}\mathfrak{X}_{Kl} = \delta_{kl}, \quad \chi_{kK}\mathfrak{X}_{Lk} = \delta_{KL}. \quad (37)$$

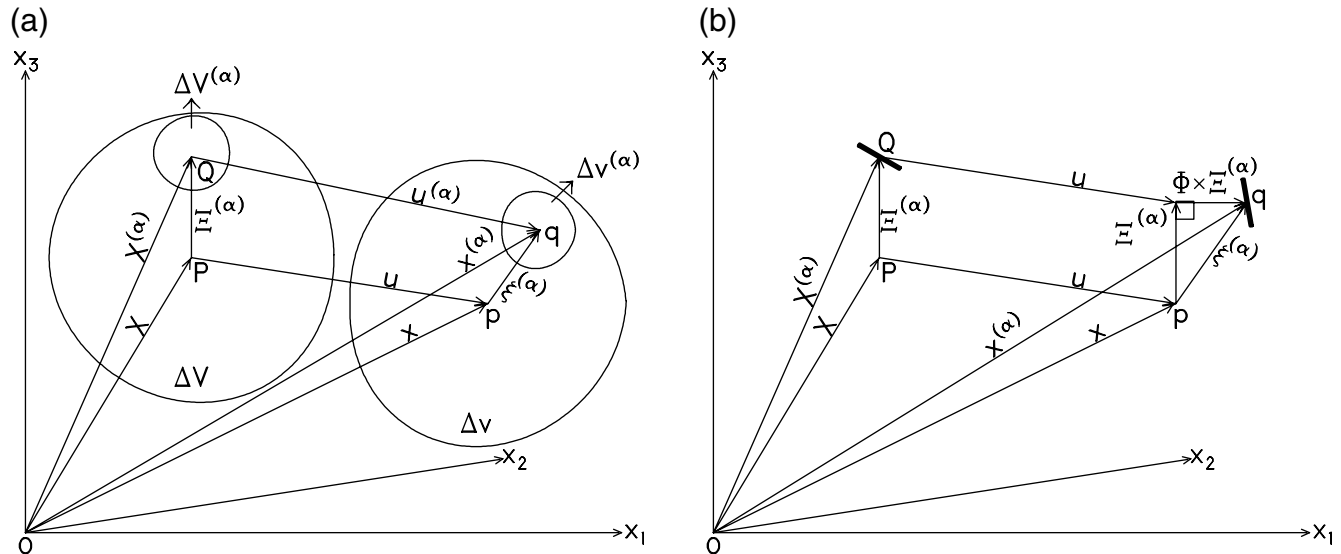
Multiplying  $\xi_k^{(\alpha)}$ , given by equation (34), by  $\mathfrak{X}_{Lk}$  and using equation (37) gives

$$\Xi_K^{(\alpha)} = \mathfrak{X}_{Kk}(\mathbf{x}, t)\xi_k^{(\alpha)}, \quad (38)$$

which in vector form becomes

$$\mathbf{\Xi}^{(\alpha)} = \mathfrak{X}_k(\mathbf{x}, t)\xi_k^{(\alpha)}. \quad (39)$$

The  $\chi_K$  and  $\mathfrak{X}_k$  constitute two independent sets and Eringen (1999) refers to them as directors, which at this point are deformable and constitute extra degrees of freedom with



**Figure 2.** (a) Schematic representation of a continuous medium with microstructure. The volume  $\Delta V$  represents a macroelement, which is composed of microelements with volumes  $\Delta V^{(\alpha)}$ ,  $\alpha = 1, 2, \dots$ . Capital and small letters represent the same variable before and after deformation. Point  $P$  corresponds to the center of mass of the macrovolume. (b) Similar to (a) for a micropolar medium, which can be represented by rigid elongated microelements (bold segments at  $Q$  and  $q$ ). After Eringen (1968).

respect to a classical continuum medium. Now it is possible to write the inverse motion corresponding to equation (36)

$$X_K^{(\alpha)} = X_K(\mathbf{x}, t) + \mathfrak{X}_{Kk}(\mathbf{x}, t)\xi_k^{(\alpha)}. \quad (40)$$

### Strain and Microstrain Tensors

The displacement vector  $\mathbf{u}^{(\alpha)}$  is defined as the difference between  $\mathbf{x}^{(\alpha)}$  and  $\mathbf{X}^{(\alpha)}$ :

$$\begin{aligned} \mathbf{u}^{(\alpha)} &= \mathbf{x} + \boldsymbol{\xi}^{(\alpha)} - (\mathbf{X} + \boldsymbol{\Xi}^{(\alpha)}) = \mathbf{u} + \boldsymbol{\xi}^{(\alpha)} - \boldsymbol{\Xi}^{(\alpha)}, \\ \mathbf{u} &\equiv \mathbf{x} - \mathbf{X}. \end{aligned} \quad (41)$$

The vector  $\mathbf{u}$  is the displacement vector of the classical theory, introduced in equation (25). This vector will have components  $U_K$  and  $u_k$  when referring to the material and spatial coordinates, respectively, which can be written as

$$U_K \equiv \mathbf{u} \cdot \mathbf{I}_K = x_k \delta_{kK} - X_K, \quad (42)$$

$$u_k \equiv \mathbf{u} \cdot \mathbf{i}_k = x_k - X_K \delta_{kK}, \quad (43)$$

where  $\mathbf{I}_K$  and  $\mathbf{i}_k$  indicate the unit vectors along the  $K$  and  $k$  axes in the material and spatial coordinates, respectively, and

$$\delta_{kK} \equiv \delta_{kK} \equiv \mathbf{i}_k \cdot \mathbf{I}_K. \quad (44)$$

Because the material and spatial coordinate systems have been chosen to be the same,  $\delta_{kK}$  is the standard Kronecker's delta. Therefore,  $x_k \delta_{kK} = x_K$  and  $X_K \delta_{kK} = X_k$ , and equations (42) and (43) can be rewritten as

$$U_k = x_k - X_k, \quad u_K = x_K - X_K. \quad (45)$$

We interchanged  $K$  and  $k$  because they are dummy indices. Next, differentiating  $U_k$  with respect to  $X_K$  and  $u_K$  with respect to  $x_k$  and rearranging gives

$$x_{k,K} = \delta_{kK} + U_{k,K} \equiv (\delta_{LK} + U_{L,K})\delta_{kL} \quad (46)$$

and

$$X_{K,k} = \delta_{Kk} - u_{K,k} \equiv (\delta_{lk} - u_{l,k})\delta_{Kl}. \quad (47)$$

The  $x_{k,K}$  and  $X_{K,k}$  are the components of the classical deformation gradients.

Now we will write  $\mathbf{u}^{(\alpha)}$  in material and spatial coordinates. Writing equation (41) in component form and using equations (34) and (38), we have

$$\begin{aligned} u_K^{(\alpha)} &= U_K + \chi_{KL}\Xi_L^{(\alpha)} - \Xi_K^{(\alpha)} \\ &= U_K + (\chi_{KL} - \delta_{KL})\Xi_L^{(\alpha)} \equiv U_K + \Phi_{KL}\Xi_L^{(\alpha)}, \end{aligned} \quad (48)$$

$$\begin{aligned} u_k^{(\alpha)} &= u_k + \xi_k^{(\alpha)} - \mathfrak{X}_{kl}\xi_l^{(\alpha)} \\ &= u_k + (\delta_{kl} - \mathfrak{X}_{kl})\xi_l^{(\alpha)} \equiv u_k + \phi_{kl}\xi_l^{(\alpha)}, \end{aligned} \quad (49)$$

where the tensors  $\Phi_{KL}$  and  $\phi_{kl}$  are defined by the identities. From these definitions we can also write

$$\chi_{kK} = \delta_{kK} + \Phi_{kK} = (\delta_{LK} + \Phi_{LK})\delta_{kL} \quad (50)$$

and

$$\mathfrak{X}_{Kk} = \delta_{Kk} - \phi_{Kk} = (\delta_{lk} - \phi_{lk})\delta_{Kl}. \quad (51)$$

The indices in these equations are different from those used in equations (48) and (49).

As in the classical theory, significant simplifications are possible when the equations are linearized, which requires that the absolute value of the tensor components be much smaller than one. Some of the consequences of this assumption are that the distinctions between material and spatial representations, between  $u_k$  and  $U_k$ , and between  $\phi_{kl}$  and  $\Phi_{KL}$  disappear. In the following the linear theory will be discussed, but the distinction between the two representations will be maintained for clarity.

In the linearized theory, deformation is represented by a strain tensor, similar to that of classical elasticity, and two microstrain tensors, which do not have classical analogues. These three tensors are derived from consideration of the squares of the differential line elements in the deformed body, and in the Lagrangian representation they will be indicated by  $E_{KL}$ ,  $\mathcal{E}_{KL}$ , and  $\Gamma_{KLM}$ , respectively, and defined by

$$E_{KL} = \frac{1}{2}(U_{K,L} + U_{L,K}), \quad (52)$$

$$\mathcal{E}_{KL} = \Phi_{KL} + U_{L,K}, \quad (53)$$

$$\Gamma_{KLM} = \Phi_{KL,M}. \quad (54)$$

In Eulerian coordinates the corresponding expressions are

$$e_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}), \quad (55)$$

$$\epsilon_{kl} = \phi_{kl} + u_{l,k}, \quad (56)$$

$$\gamma_{klm} = -\phi_{kl,m}. \quad (57)$$

Note the sign differences in equations (54) and (57).

### Micropolar Strains and Rotations

As equation (36) shows, the deformation described so far is very general and requires the knowledge of twelve functions, the three  $x_k$  and the nine  $\chi_{kK}$ . However, a significant simplification is achieved under the assumption that the medium is composed of rigid elongated microelements and that  $\Phi_{KL}$  and  $\phi_{kl}$  are antisymmetric, that is,

$$\Phi_{KL} = -\Phi_{LK}, \quad \phi_{kl} = -\phi_{lk} \quad (58)$$

(Eringen, 1966). With these properties and the linearity assumption, the  $\chi_{kk}$  and  $\mathfrak{X}_{kk}$  given by equations (50) and (51) become the components of tensors representing infinitesimal rotations (see equation 10). Let us investigate this type of deformation in more detail. The tensor  $\Phi_{KL}$  has associated with it a vector  $\Phi$  (the material microrotation vector) with components  $\Phi_K$  such that

$$\Phi_K = \frac{1}{2} \epsilon_{KLM} \Phi_{ML}, \quad \Phi_{KL} = -\epsilon_{KLM} \Phi_M \quad (59)$$

(see the Appendix). Using this expression for  $\Phi_{KL}$  in equation (50) gives

$$\chi_{kk} = \delta_{kk} - \epsilon_{kKM} \Phi_M. \quad (60)$$

Eringen (1999) obtained a similar equation (equation 1.3.15 in the reference) by first writing  $\xi$  as the result of a finite rotation (see equation 1) of angle  $\phi_m$  (similar to  $\Phi_M$ ) and then assuming that  $\phi_m$  is small and defined a micropolar continuum by the property that the directors are orthonormal, that is,

$$\chi_{kk} \chi_{IK} = \delta_{kl}, \quad \mathfrak{X}_{kk} \mathfrak{X}_{Lk} = \delta_{KL}. \quad (61)$$

This definition is consistent with equations (50) and (51) under the condition that the rotations are infinitesimal. Moreover, if the second of equations (61) is multiplied by  $\mathfrak{X}_{Ll}$  and equation (37) is used, we get  $\chi_{KL} = \mathfrak{X}_{Lk}$ , and equation (37) becomes

$$\chi_{kk} \chi_{IK} = \delta_{kl}, \quad \chi_{kk} \chi_{kL} = \delta_{KL} \quad (62)$$

so that the directors of a micropolar continuum are rigid.

To relate these results to those from classical linear elasticity recall that the classical rotation tensor  $R_{KL}$  is given by

$$R_{KL} = -R_{LK} = \frac{1}{2} (U_{K,L} - U_{L,K}) \quad (63)$$

(see equation 30). This tensor has an associated vector, indicated by  $\mathbf{R}$ , with components that satisfy

$$R_K = \frac{1}{2} \epsilon_{KLM} R_{ML} = \frac{1}{2} \epsilon_{KLM} U_{M,L}, \quad R_{KL} = -\epsilon_{KLM} R_M. \quad (64)$$

Then using

$$U_{K,L} \equiv E_{KL} + R_{KL} = E_{KL} - \epsilon_{KLM} R_M \quad (65)$$

and equation (59) with equations (53) and (54) gives

$$\mathcal{E}_{KL} = E_{KL} + \epsilon_{KLM} (R_M - \Phi_M), \quad (66)$$

$$\Gamma_{KLM} = -\epsilon_{KLN} \Phi_{N,M}. \quad (67)$$

In the Eulerian description the corresponding equations are

$$r_k = \frac{1}{2} \epsilon_{klm} r_{ml} = \frac{1}{2} \epsilon_{klm} u_{m,l}, \quad r_{kl} = -\epsilon_{klm} r_m, \quad (68)$$

$$u_{k,l} = e_{kl} + r_{kl} = e_{kl} - \epsilon_{klm} r_m, \quad (69)$$

$$e_{kl} = e_{kl} + \epsilon_{klm} (r_m - \phi_m), \quad (70)$$

$$\gamma_{klm} = \epsilon_{klm} \phi_{n,m}. \quad (71)$$

Note the sign differences in equations (67) and (71). Recall the capital and small letter convention to represent a variable in the Lagrangian and Eulerian descriptions, respectively. In particular, the  $\phi_m$  are the components of the spatial micro-rotation vector  $\phi$ . For future reference we also note the Eulerian expression corresponding to equation (60)

$$\mathfrak{X}_{Kl} = \delta_{Kl} + \epsilon_{Klm} \phi_m. \quad (72)$$

Now combine equations (34) and (60):

$$\xi_k^{(\alpha)} = (\delta_{kk} - \epsilon_{kKM} \Phi) \Xi_K^{(\alpha)} = \Xi_k^{(\alpha)} - (\Xi^{(\alpha)} \times \Phi)_k. \quad (73)$$

Therefore,

$$\xi^{(\alpha)} = \Xi^{(\alpha)} - \Xi^{(\alpha)} \times \Phi = \Xi^{(\alpha)} + \Phi \times \Xi^{(\alpha)}. \quad (74)$$

Comparison of this result with equation (8) shows that  $\xi^{(\alpha)}$  is obtained by a rotation of angle  $|\Phi|$  of  $\Xi^{(\alpha)}$  about a vector  $\Phi$ . Finally, introducing equation (74) into equation (33) and using equation (41) to write  $\mathbf{x}$  in terms of  $\mathbf{u}$ , we get the following expression for  $\mathbf{x}^{(\alpha)}$ :

$$\mathbf{x}^{(\alpha)} = \mathbf{X} + \mathbf{u} + \Xi^{(\alpha)} + \Phi \times \Xi^{(\alpha)}. \quad (75)$$

The geometric significance of this result is shown in Figure 2b.

#### Velocity, Acceleration, Microrotation, and Spin

Velocity, indicated by  $\mathbf{v}$ , is defined as the time rate of change of the position vector of a material point, that is,  $\mathbf{v} = \dot{\mathbf{x}}(\mathbf{X}, t)$ . Acceleration, indicated by  $\mathbf{a}$ , is the time rate of change of the velocity of a material point, that is,  $\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{x}}$ . These definitions apply to the motion of the center of mass  $\mathbf{X}$  of a macrovolume. Now the relative velocity and acceleration of a material point with coordinates  $\mathbf{X} + \Xi$  with respect to point  $\mathbf{X}$  will be introduced. The relative motion is given by equation (34), and the corresponding velocity and acceleration are given by

$$\dot{\xi} = \dot{\chi}_K(\mathbf{X}, t) \Xi_K, \quad \ddot{\xi} = \ddot{\chi}_K(\mathbf{X}, t) \Xi_K. \quad (76)$$

To simplify the notation the superscript  $(\alpha)$  will be dropped when not essential. These two expressions will be rewritten using equation (38). For the velocity we get

$$\dot{\xi} = \nu_k(\mathbf{x}, t)\xi_k, \quad \dot{\xi}_l = \nu_{lk}\xi_k, \quad (77)$$

where

$$\nu_k(\mathbf{x}, t) = \dot{\chi}_K[\mathbf{X}(\mathbf{x}, t), t]\mathcal{X}_{Kk}(\mathbf{x}, t), \quad \nu_{lk} = \dot{\chi}_{lK}\mathcal{X}_{Kk}. \quad (78)$$

The three vectors  $\nu_k$  are known as the gyration vectors and their components  $\nu_{lk}$  constitute the gyration tensor. For the acceleration we have

$$\ddot{\xi} = \dot{\nu}_k\xi_k + \nu_k\dot{\xi}_k = \dot{\nu}_k\xi_k + \nu_k\nu_{kl}\xi_l, \quad (79)$$

where equation (77) was used. Alternatively,

$$\ddot{\xi} = \alpha_k(\mathbf{x}, t)\xi_k, \quad \ddot{\xi}_k = \alpha_{kl}\xi_l, \quad (80)$$

where

$$\alpha_k(\mathbf{x}, t) = \dot{\nu}_k + \nu_m\nu_{mk}, \quad \alpha_{lk} = \dot{\nu}_{lk} + \nu_{lm}\nu_{mk}. \quad (81)$$

The  $\alpha_{lk}$  constitute the components of the spin tensor.

The previous expressions for  $\dot{\xi}$  and  $\ddot{\xi}$  are general. Because here we are dealing with micropolar media, equations (60) and (72) must be used. This gives

$$\nu_{kl} = -\epsilon_{klm}\dot{\Phi}_M + \epsilon_{kKM}\epsilon_{Klm}\dot{\Phi}_M\phi_m. \quad (82)$$

In the linearized theory the second term in this equation is neglected, and there is no distinction between the material and spatial descriptions. In this case we have

$$\nu_{kl} = -\epsilon_{klm}\dot{\phi}_m, \quad (83)$$

where the approximation  $\phi \approx \Phi$ , valid for the linear theory, was used. This result shows that  $\dot{\phi}_m$  is the vector associated with the antisymmetric tensor  $\nu_{kl}$ . Let us introduce the microgyration vector  $\nu$ , with components given by

$$\nu_k = \dot{\phi}_k. \quad (84)$$

Then

$$\nu_{kl} = -\epsilon_{klm}\nu_m, \quad \nu_k = \frac{1}{2}\epsilon_{klm}\nu_{ml}. \quad (85)$$

Introducing this  $\nu_{kl}$  in equation (77) allows us to get the following expression for  $\dot{\xi}$ :

$$\dot{\xi} = -\xi \times \nu. \quad (86)$$

Finally, the total velocity at a point with material coordinates  $\mathbf{X} + \Xi$  is given by

$$\mathbf{v}^{(\alpha)} = \dot{\mathbf{x}}^{(\alpha)} = \dot{\mathbf{x}} + \dot{\xi} = \mathbf{v} - \xi \times \nu, \quad (87)$$

where  $\mathbf{v}$  indicates the velocity of the centroid of the macrovolume, while the second term on the right gives the relative velocity about the centroid.

## Mechanical Balance Laws

Here we are interested in the conservation of mass and the balance of linear momentum and angular momentum (or moment of momentum). A basic assumption is that the properties of a macrovolume are obtained by averaging over its microvolumes, for which the classical balance laws are assumed to be valid. First we will consider the total mass. Let  $\rho_o^{(\alpha)}$ ,  $\Delta V_o^{(\alpha)}$ ,  $\rho^{(\alpha)}$ , and  $\Delta v^{(\alpha)}$  indicate the density and volume of a microelement before and after deformation, respectively. A basic principle is that the mass of each microelement remains constant during any deformation, that is,

$$\rho_o^{(\alpha)}\Delta V_o^{(\alpha)} = \rho^{(\alpha)}\Delta v^{(\alpha)} \quad (88)$$

(no summation over  $\alpha$ ). The total mass of the macrovolume before and after deformation is obtained by summing over all the corresponding microvolumes:

$$\rho_o\Delta V_o \equiv \sum_{\alpha}\rho_o^{(\alpha)}\Delta V_o^{(\alpha)} = \sum_{\alpha}\rho^{(\alpha)}\Delta v^{(\alpha)} \equiv \rho\Delta v. \quad (89)$$

This equation defines  $\rho_o(\mathbf{X})$  and  $\rho(\mathbf{x}, t)$  for the macrovolume  $\Delta V_o$  (equivalent to  $\Delta V$  in Fig. 2) before and after the deformation. The fact that the point  $\mathbf{X}$  is the center of mass of the macroelement means that

$$\sum_{\alpha}\rho_o^{(\alpha)}\Xi^{(\alpha)}\Delta V_o^{(\alpha)} = \mathbf{0}. \quad (90)$$

Introducing equations (39) and (88) into this equation gives

$$\mathcal{X}_k \sum_{\alpha}\rho^{(\alpha)}\xi_k^{(\alpha)}\Delta v^{(\alpha)} = \mathbf{0}. \quad (91)$$

Because  $\mathcal{X}_k$  is generally nonzero, the sum over  $\alpha$  must be equal to zero, which means that  $\mathbf{x}$  is the center of mass after the deformation.

Now compute the following second moments

$$\rho_o I_{KL} \Delta V_o \equiv \sum_{\alpha}\rho_o^{(\alpha)}\Xi_K^{(\alpha)}\Xi_L^{(\alpha)}\Delta V_o^{(\alpha)}. \quad (92)$$

Each term in the summation on the right-hand side can be interpreted as the product of two factors. One is a distance and the other is the product of a mass and a distance. The second factor is known as linear moment, while distance times linear moment is known as second moment (or quadratic moment, Synge, 1960). The second moments allow the introduction of inertia tensors, as follows. Introducing equations (38) and (88) into this equation gives

$$I_{KL} = i_{kl}\mathcal{X}_{Kk}\mathcal{X}_{Ll}, \quad (93)$$

where

$$\rho i_{kl}\Delta v = \sum_{\alpha}\rho^{(\alpha)}\xi_k^{(\alpha)}\xi_l^{(\alpha)}\Delta v^{(\alpha)}. \quad (94)$$

The quantities  $I_{KL}$  and  $i_{kl}$  are known as the material and spatial microinertia tensors. The following combinations occur frequently

$$J_{KL} = I_{MM}\delta_{KL} - I_{KL}, \quad j_{kl} = i_{mm}\delta_{kl} - i_{kl} \quad (95)$$

and are identical to the inertia tensors of rigid body dynamics.

The momentum of a macroelement is equal to the vector sum of the momenta of its microelements, namely

$$\begin{aligned} \Delta \mathbf{p} &= \sum_{\alpha} \rho^{(\alpha)} \mathbf{v}^{(\alpha)} \Delta v^{(\alpha)} = \sum_{\alpha} \rho^{(\alpha)} (\mathbf{v} + \dot{\boldsymbol{\xi}}) \Delta v^{(\alpha)} \\ &= \mathbf{v} \sum_{\alpha} \rho^{(\alpha)} \Delta v^{(\alpha)} + \boldsymbol{\nu} \times \sum_{\alpha} \rho^{(\alpha)} \boldsymbol{\xi} \Delta v^{(\alpha)} \\ &= \mathbf{v} \sum_{\alpha} \rho^{(\alpha)} \Delta v^{(\alpha)}, \end{aligned} \quad (96)$$

where equation (87) and the fact that  $\mathbf{x}$  is the center of mass were used. In the limit we can write  $d\mathbf{p} = \rho \mathbf{v} dv$ , and the total linear momentum of the body is given by

$$\mathbf{p} = \int_{\mathcal{V}} \rho \mathbf{v} dv, \quad (97)$$

where  $\mathcal{V}$  is the volume of the body after deformation.

The principle of linear momentum states that the time rate of  $\mathbf{p}$  is equal to the sum of all the applied forces, namely

$$\frac{d}{dt} \int_{\mathcal{V}} \rho \mathbf{v} dv = \int_{\mathcal{S}} \mathbf{t} d\mathbf{a} + \int_{\mathcal{V}} \rho \mathbf{f} dv, \quad (98)$$

where  $\mathbf{t}$  is the stress tensor,  $\mathbf{f}$  is the body force, and  $\mathcal{S}$  is the surface of  $\mathcal{V}$ . This equation is identical to that of classical continuum mechanics.

The angular momentum of a macroelement is equal to the vector sum of the angular momenta of its microelements, namely

$$\begin{aligned} \Delta \mathcal{M} &= \sum_{\alpha} \mathbf{x}^{(\alpha)} \times \rho^{(\alpha)} \mathbf{v}^{(\alpha)} \Delta v^{(\alpha)} \\ &= \sum_{\alpha} (\mathbf{x} + \boldsymbol{\xi}) \times \rho^{(\alpha)} (\mathbf{v} + \dot{\boldsymbol{\xi}}) \Delta v^{(\alpha)} \\ &= \mathbf{x} \times \mathbf{v} \sum_{\alpha} \rho^{(\alpha)} \Delta v^{(\alpha)} + \sum_{\alpha} \boldsymbol{\xi} \times \rho^{(\alpha)} \dot{\boldsymbol{\xi}} \Delta v^{(\alpha)}, \end{aligned} \quad (99)$$

where equations (33) and (87) were used and two terms vanish because  $\mathbf{x}$  is the center of mass. Then, using equations (86) and (89) in the limit we have

$$d\mathcal{M} = \mathbf{x} \times \rho \mathbf{v} dv + \rho \boldsymbol{\sigma} dv, \quad (100)$$

where

$$\rho \boldsymbol{\sigma} dv = \sum_{\alpha} \rho^{(\alpha)} \boldsymbol{\xi} \times (\boldsymbol{\nu} \times \boldsymbol{\xi}) \Delta v^{(\alpha)}. \quad (101)$$

The vector  $\boldsymbol{\sigma}$  is called the intrinsic spin, with components given by

$$\sigma_l = j_{kl} \nu_k. \quad (102)$$

This expression is obtained after expanding the vector products in equation (101) and then using equations (94) and (95). The total angular momentum of a micropolar body is given by

$$\mathcal{M} = \int_{\mathcal{V}} (\mathbf{x} \times \rho \mathbf{v} + \rho \boldsymbol{\sigma}) dv. \quad (103)$$

The only difference with the corresponding expression for the classical theory is the presence of the spin vector, introduced by the rotation of the microelements.

The principle of angular momentum states that the time rate of change of the angular momentum about a point is equal to the sum of all applied couples and the moment of all the forces about the same point. Thus,

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{V}} (\mathbf{x} \times \rho \mathbf{v} + \rho \boldsymbol{\sigma}) dv &= \int_{\mathcal{S}} (\mathbf{x} \times \mathbf{t} + \mathbf{m}) d\mathbf{a} \\ &+ \int_{\mathcal{V}} \rho (\mathbf{l} + \mathbf{x} \times \mathbf{f}) dv, \end{aligned} \quad (104)$$

where  $\mathbf{m}$  and  $\mathbf{l}$  are surface and body couples. As noted at the beginning of this section, in classical continuum mechanics the surface and body couples are neglected because the surface and volume elements are allowed to go to zero, which means that the arms of the couples, and thus the couples, vanish. In contrast, in micropolar elasticity the surface and volume elements remain finite, which means that the couples cannot be neglected.

### Stress and Couple Stress

Applying the well-known tetrahedron argument first with the principle of linear momentum and then with the principle of angular momentum (Eringen, 1962) gives

$$\mathbf{t}(\mathbf{n}) = n_k \mathbf{t}_k, \quad \mathbf{m}(\mathbf{n}) = n_k \mathbf{m}_k, \quad (105)$$

where  $\mathbf{n} = (n_1, n_2, n_3)$  is a vector normal to the surface on which  $\mathbf{t}$  acts and  $\mathbf{t}_k$  is the stress vector on the surface with normal vector  $\mathbf{i}_k$ . A similar definition applies to  $\mathbf{m}_k$ . Now writing  $\mathbf{t}_k$  and  $\mathbf{m}_k$  in component form

$$\mathbf{t}_k = t_{kl} \mathbf{i}_l, \quad \mathbf{m}_k = m_{kl} \mathbf{i}_l, \quad (106)$$

allows introducing the stress tensor  $t_{kl}$  and the couple stress tensor  $m_{kl}$ :

$$\mathbf{t}(\mathbf{n}) = t_{kl} n_k \mathbf{i}_l, \quad \mathbf{m}(\mathbf{n}) = m_{kl} n_k \mathbf{i}_l. \quad (107)$$

### Local Balance Laws

The result of these laws will be two equations of motion, obtained using the balance of linear momentum and the balance of angular momentum. Using Gauss' theorem with equations (98) and (104) we can convert the surface integrals to volume integrals. The total time derivatives can be

exchanged with the integrals and become partial derivatives with respect to time in the linear theory. As a result we get

$$\int_v [\mathbf{t}_{k,k} + \rho(\mathbf{f} - \dot{\mathbf{v}})] dv = \mathbf{0}, \quad (108)$$

$$\int_v [\mathbf{m}_{k,k} + \mathbf{i}_k \times \mathbf{t}_k + \rho(\mathbf{l} - \dot{\boldsymbol{\sigma}})] dv + \int_v \mathbf{x} \times [\mathbf{t}_{k,k} + \rho(\mathbf{f} - \dot{\mathbf{v}})] dv = \mathbf{0}, \quad (109)$$

where  $v$  is a small volume within the body. Because  $v$  is arbitrary, these two equations will be satisfied when the integrands are equal to zero. Then, using equations (105) and (107) we get

$$t_{lk,l} + \rho(f_k - \dot{v}_k) = 0, \quad (110)$$

$$m_{lk,l} + \epsilon_{kmn} t_{mn} + \rho(l_k - \dot{\sigma}_k) = 0. \quad (111)$$

These are the equations of motion for a micropolar medium and correspond to the local balance of momenta. Equation (110) is the equation of motion of classical elasticity. Equation (111) is an extension of the equation that is obtained when proving the classical result that the stress tensor is symmetric, that is,  $t_{lk} = t_{kl}$  (e.g., Atkin and Fox, 1980; Pujol, 2003). If  $\mathbf{m}_k$ ,  $\boldsymbol{\sigma}$ , and  $\mathbf{l}$  are all equal to zero, we recover the symmetry of the stress tensor. Therefore, in a micropolar medium the stress tensor is, in general, nonsymmetric (or asymmetric).

### Theory of Micropolar Elasticity

After having introduced strain and stress, it is necessary to establish the relation between them, which is done through constitutive equations. To establish them it is necessary to use thermodynamic arguments, which lead to the following relations between the stress and couple stress tensors and the free energy  $\Psi$  of the system:

$$t_{kl} = \rho \frac{\partial \Psi}{\partial \epsilon_{kl}}, \quad m_{kl} = \frac{\partial \Psi}{\partial \phi_{l,k}}. \quad (112)$$

In linear theory  $\Psi$  is expressed as a quadratic function of  $\epsilon_{k,l}$  and  $\phi_{k,l}$ :

$$\begin{aligned} \rho \Psi = & A_0 + A_{kl} \epsilon_{kl} + \frac{1}{2} A_{klmn} \epsilon_{kl} \epsilon_{mn} + B_{kl} \phi_{k,l} \\ & + \frac{1}{2} B_{klmn} \phi_{k,l} \phi_{m,n} + C_{klmn} \epsilon_{kl} \phi_{m,n}, \end{aligned} \quad (113)$$

where the subscripted  $A$  and  $B$  are functions of temperature only. Equations (112) and (113) are extensions of classical continuum mechanics results. Equation (113) must be invariant under orthogonal transformations (which include rotations and reflections, e.g., Eringen, 1962, 1967). However,

this condition is not met by the fourth and sixth terms, which include derivatives of the pseudovector  $\phi_k$ . In this case the transformation law under rotations and reflections is

$$\phi'_{k,l} = \pm a_{km} a_{ln} \phi_{m,n}, \quad (114)$$

where the  $a_{ij}$  are the components of the transformation matrix and the plus or minus signs depend on whether the determinant of the matrix is equal to 1 (pure rotation) or to  $-1$  (reflections are included). The tensor  $\epsilon_{kl}$  satisfies a relation similar to equation (114) with a positive sign only. As a consequence, the fourth and sixth terms of equation (113) change sign upon reflection while the other terms are always positive. Therefore, invariance requires  $B_{kl} = 0$  and  $C_{klmn} = 0$ .

Equation (113) also shows that the tensors in the third and fifth terms are symmetric with respect to the pairs of indices  $kl$  and  $mn$ , which means that  $A_{klmn} = A_{mnkl}$  and  $B_{klmn} = B_{mnkl}$ . Under these conditions equations (112) and (113) give

$$t_{kl} = A_{kl} + A_{klmn} \epsilon_{mn}, \quad (115)$$

$$m_{kl} = B_{klmn} \phi_{m,n}. \quad (116)$$

If the initial stress of the system is zero, as assumed here,  $A_{kl} = 0$ . If, in addition, the body is isotropic,

$$t_{kl} = \lambda \epsilon_{rr} \delta_{kl} + (\mu + \kappa) \epsilon_{kl} + \mu \epsilon_{lk}, \quad (117)$$

$$m_{kl} = \alpha \phi_{r,r} \delta_{kl} + \beta \phi_{k,l} + \gamma \phi_{l,k}, \quad (118)$$

where  $\mu + \kappa$  is a single coefficient written as a sum for convenience. In classical elasticity  $\kappa = 0$  because of the symmetry of the stress tensor. Using equation (70), equation (117) gives

$$t_{kl} = \lambda e_{rr} \delta_{kl} + (2\mu + \kappa) e_{kl} + \kappa \epsilon_{klm} (r_m - \phi_m). \quad (119)$$

If  $\kappa = \alpha = \beta = \gamma = 0$ , equation (118) vanishes and equation (119) becomes the Hooke's law of the classical theory of isotropic elasticity. Therefore, another difference between the linearized isotropic micropolar and classical elasticity theories is the presence in the former of four additional elastic moduli. Because Nowacki (1986) uses similar symbols but some of them have a different meaning, the equivalence between the two sets of symbols is given here. Using the subscripts  $E$  and  $N$  to identify the symbols used by Eringen and by Nowacki we have

$$\begin{aligned} \mu_E &= \mu_N - \alpha_N, & \kappa_E &= 2\alpha_N, & \lambda_E &= \lambda_N, \\ \gamma_E &= \gamma_N + \varepsilon_N, & \beta_E &= \gamma_N - \varepsilon_N, & \alpha_E &= \beta_N \end{aligned} \quad (120)$$

(Ostoja-Starzewski and Jasiuk, 1995).

## Field Equations and Propagation of Waves

The field equations are the equations of motion in terms of  $\mathbf{u}$  and  $\phi$ . Introducing the constitutive equations (118) and (119) into equations (110) and (111), using (69), (84), and (102), letting  $j_{kl} = j\delta_{kl}$  (isotropic solid), and assuming that  $j$  and  $\rho$  are constant gives

$$(\lambda + \mu)u_{l,lk} + (\mu + \kappa)u_{k,ll} + \kappa\epsilon_{klm}\phi_{m,l} + \rho(f_k - \ddot{u}_k) = 0, \quad (121)$$

$$(\alpha + \beta)\phi_{l,lk} + \gamma\phi_{k,ll} + \kappa\epsilon_{klm}u_{m,l} - 2\kappa\phi_k + \rho(l_k - j\ddot{\phi}_k) = 0. \quad (122)$$

In linear theory the approximations  $\ddot{u}_k \approx \partial^2 u_k / \partial t^2$  and  $\ddot{\phi}_k \approx \partial^2 \phi_k / \partial t^2$  are also introduced. In vector form equations (121) and (122) are

$$(\lambda + 2\mu + \kappa)\nabla\nabla \cdot \mathbf{u} - (\mu + \kappa)\nabla \times \nabla \times \mathbf{u} + \kappa\nabla \times \phi + \rho(\mathbf{f} - \ddot{\mathbf{u}}) = \mathbf{0}, \quad (123)$$

$$(\alpha + \beta + \gamma)\nabla\nabla \cdot \phi - \gamma\nabla \times \nabla \times \phi + \kappa\nabla \times \mathbf{u} - 2\kappa\phi + \rho(\mathbf{l} - j\ddot{\phi}) = \mathbf{0}. \quad (124)$$

Again, if  $\kappa = \alpha = \beta = \gamma = 0$ , we recover the equations of linear elasticity. In equations (123) and (124)  $\mathbf{u}$  and  $\phi$  are unknown,  $\mathbf{f}$  and  $\mathbf{l}$  must be prescribed, and the remaining parameters must be derived experimentally. These two equations are coupled, which means that solving them is considerably more complicated than in the classical case. An early and comprehensive analysis of the solutions corresponding to plane harmonic body waves in infinite media and in a half-space was provided by Parfitt and Eringen (1969) and is discussed in Eringen (1968). Assuming that  $\mathbf{f}$  and  $\mathbf{l}$  are both equal to zero, equations (123) and (124) can be rewritten as

$$(c_1^2 + c_3^2)\nabla\nabla \cdot \mathbf{u} - (c_2^2 + c_3^2)\nabla \times \nabla \times \mathbf{u} + c_3^2\nabla \times \phi = \ddot{\mathbf{u}}, \quad (125)$$

$$(c_4^2 + c_5^2)\nabla\nabla \cdot \phi - c_4^2\nabla \times \nabla \times \phi + \omega_0^2\nabla \times \mathbf{u} - 2\omega_0^2\phi = \ddot{\phi}, \quad (126)$$

where

$$c_1^2 = \frac{\lambda + 2\mu}{\rho}, \quad c_2^2 = \frac{\mu}{\rho}, \quad c_3^2 = \frac{\kappa}{\rho}, \quad c_4^2 = \frac{\gamma}{\rho j}, \quad c_5^2 = \frac{\alpha + \beta}{\rho j}, \quad \omega_0^2 = \frac{c_3^2}{j} = \frac{\kappa}{\rho j}. \quad (127)$$

The vectors  $\mathbf{u}$  and  $\phi$  can be decomposed into scalar and vector potentials as follows

$$\mathbf{u} = \nabla u + \nabla \times \mathbf{U}, \quad \nabla \cdot \mathbf{U} = 0, \quad (128)$$

$$\phi = \nabla \phi + \nabla \times \Phi, \quad \nabla \cdot \Phi = 0. \quad (129)$$

Introducing these expressions into equations (125) and (126) shows that they are satisfied if

$$(c_1^2 + c_3^2)\nabla^2 u = \ddot{u}, \quad (130)$$

$$(c_4^2 + c_5^2)\nabla^2 \phi - 2\omega_0^2\phi = \ddot{\phi}, \quad (131)$$

$$(c_2^2 + c_3^2)\nabla^2 \mathbf{U} + c_3^2\nabla \times \Phi = \ddot{\mathbf{U}}, \quad (132)$$

$$c_4^2\nabla^2 \Phi - 2\omega_0^2\Phi + \omega_0^2\nabla \times \mathbf{U} = \ddot{\Phi}. \quad (133)$$

The equations involving the scalar potentials ( $u$  and  $\phi$ ) are uncoupled while those involving the vector potentials ( $\mathbf{U}$  and  $\Phi$ ) are coupled. In Eringen (1999) some of the coefficients are defined differently: his  $c_1^2$ ,  $c_3^2$ , and  $c_2^2$  are given by the sums in parentheses in equations (130)–(132), respectively, while his  $\omega_0^2$  is equal to twice the  $\omega_0^2$  used here.

The main result of the Parfitt and Eringen (1969) analysis is the presence of four types of waves propagating with different velocities. As equations (130) and (131) show, two of the waves are longitudinal, with one of them similar to the classical dilatational waves and the other microrotational. The latter is dispersive and exists only above a cutoff frequency  $\omega_c = \sqrt{2}\omega_0$ . Below this frequency the waves become sinusoidal vibrations that decay with distance. The other two waves involve coupled transverse displacement and transverse microrotation. These waves are also dispersive, and the latter exists only for frequencies higher than  $\omega_c$ . The transverse displacement wave is similar to the shear wave of classical elasticity. Additional results can be found in, for example, Nowacki (1986) and Kulesh (2009) and references therein. The latter work suggests possible ways to test whether micropolar effects are present by analysis of surface waves. A special case corresponding to  $\nabla \cdot \phi = 0$  has been investigated by Grekova *et al.* (2009).

## Data and Resources

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## Appendix

### Basic Tensor Definitions and Properties

The sections on rotations and finite deformation are presented in coordinate-free form, that is, components are not used. Therefore, the resulting tensor equations are valid in any coordinate system and are simpler in appearance than when written in component form. The basic definitions and properties needed for an understanding of the coordinate-free equations are given next (see Ogden, 1997; Chadwick, 1999).

- (1) Let  $\mathbf{S}$  and  $\mathbf{T}$  indicate arbitrary second-order tensors and  $\mathbf{u}$  and  $\mathbf{v}$  indicate arbitrary vectors. The transpose  $\mathbf{S}^T$  of  $\mathbf{S}$ , the identity tensor  $\mathbf{I}$ , and the inner product  $\mathbf{ST}$  are defined by

$$\mathbf{v} \cdot (\mathbf{S}^T \mathbf{u}) = \mathbf{u} \cdot (\mathbf{S} \mathbf{v}), \quad (\text{A1})$$

$$\mathbf{I} \mathbf{u} = \mathbf{u}, \quad (\text{A2})$$

and

$$(\mathbf{ST})\mathbf{u} = \mathbf{S}(\mathbf{T}\mathbf{u}), \quad (\text{A3})$$

respectively. The dot stands for scalar product. In Cartesian component form  $\mathbf{ST}$  is written as  $S_{ik}T_{kj}$ , where the summation convention over repeated indices applies. In addition, the tensors  $\mathbf{S}$  and  $\mathbf{T}$  can be represented by  $3 \times 3$  matrices  $\mathbf{S}$  and  $\mathbf{T}$  with components  $S_{im}$  and  $T_{nj}$ , which shows that  $\mathbf{ST}$  is analogous to a matrix product and  $\mathbf{Su}$  to  $\mathbf{Su}$ .

- (2) A pair of juxtaposed vectors constitutes a dyad (e.g., Mase, 1970; Pujol, 2003). Given two vectors  $\mathbf{s}$  and  $\mathbf{t}$ , the dyad  $\mathbf{st}$  is defined by the following property:

$$(\mathbf{st})\mathbf{r} = (\mathbf{t} \cdot \mathbf{r})\mathbf{s}, \quad \text{any } \mathbf{r}. \quad (\text{A4})$$

In component form

$$(\mathbf{st})_{ij} = s_i t_j. \quad (\text{A5})$$

The tensor product of  $\mathbf{s}$  and  $\mathbf{t}$ , indicated by  $\mathbf{s} \otimes \mathbf{t}$  (e.g., Ogden, 1997; Chadwick, 1999), is equivalent to the dyad  $\mathbf{st}$ . A sum of dyads is known as a dyadic.

- (3) If vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  constitute an orthonormal basis,

$$\mathbf{aa} + \mathbf{bb} + \mathbf{cc} = \mathbf{I}. \quad (\text{A6})$$

To verify this relation consider an arbitrary vector  $\mathbf{v}$ , which can be written as  $\mathbf{v} = v_a \mathbf{a} + v_b \mathbf{b} + v_c \mathbf{c}$ . Then

$$(\mathbf{aa} + \mathbf{bb} + \mathbf{cc})(v_a \mathbf{a} + v_b \mathbf{b} + v_c \mathbf{c}) = \mathbf{v}. \quad (\text{A7})$$

Therefore, the sum of the three dyads is equal to  $\mathbf{I}$ . Here equation (A4) and the orthonormality of the basis vectors were used.

- (4) If  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$ , and  $\mathbf{d}$  are arbitrary vectors and  $\mathbf{S} = \mathbf{ab}$  and  $\mathbf{T} = \mathbf{cd}$ , then (A3) and (A4) give

$$(\mathbf{ab})(\mathbf{cd}) = (\mathbf{b} \cdot \mathbf{c})\mathbf{ad}. \quad (\text{A8})$$

- (5) The transpose of  $\mathbf{ab}$  is  $(\mathbf{ab})^T = \mathbf{ba}$ . This can be verified by substitution in equation (A1) and using equation (A4), which gives  $\mathbf{v} \cdot [(\mathbf{ba})\mathbf{u}] = (\mathbf{a} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{b}) = \mathbf{u} \cdot [(\mathbf{a} \cdot \mathbf{b})\mathbf{v}]$ .
- (6) The determinant of a tensor, indicated by  $\det$ , is equal to the determinant of the matrix of its components and does not depend on the choice of basis. Two properties of determinants to be used here are

$$\det(\mathbf{ST}) = \det \mathbf{S} \det \mathbf{T}, \quad \det \mathbf{S}^T = \det \mathbf{S}. \quad (\text{A9})$$

- (7) If  $\det \mathbf{T} \neq 0$ , the inverse tensor  $\mathbf{T}^{-1}$  is defined by

$$\mathbf{T}\mathbf{T}^{-1} = \mathbf{T}^{-1}\mathbf{T} = \mathbf{I}. \quad (\text{A10})$$

- (8) The eigenvectors and eigenvalues of a tensor  $\mathbf{T}$ , denoted by  $\mathbf{v}$  and  $\lambda$ , satisfy

$$\mathbf{T}\mathbf{v} = \lambda \mathbf{v} \quad (\text{A11})$$

and can be computed solving the following equation in

terms of Cartesian components:

$$\det(T_{ij} - \lambda \delta_{ij}) = 0. \quad (\text{A12})$$

This is an equation similar to the one that arises when solving the eigenvalue problem for matrices and as a consequence, the results derived for matrices carry verbatim to second-order tensors.

- (9) A tensor  $\mathbf{T}$  is said to be positive definite if  $\mathbf{v} \cdot (\mathbf{T}\mathbf{v}) > 0$  for all nonzero  $\mathbf{v}$  and positive semidefinite if  $\mathbf{v} \cdot (\mathbf{T}\mathbf{v}) \geq 0$ . The eigenvalues of positive definite and semidefinite tensors are positive and nonnegative, respectively. These results are similar to those obtained for matrices (e.g., Noble and Daniel, 1977; Pujol, 2007).

#### Proof of Equation (1)

Refer to Figure 1. The vector  $\mathbf{r}$  is rotated an angle  $\phi$  about  $\mathbf{a}$ . The segments  $\overline{BA}$  and  $\overline{BC}$  have the same lengths. The point  $D$  is the projection of point  $C$  on  $\overline{BA}$ . Let us introduce a set of orthonormal vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  with  $\mathbf{b}$  along  $\overline{BA}$ . The vector  $\mathbf{r}'$  will be written as the sum of three vectors

$$\mathbf{r}' = \mathbf{Q}\mathbf{r} = \overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BD} + \overrightarrow{DC}, \quad (\text{A13})$$

where

$$\overrightarrow{OB} = (\mathbf{r} \cdot \mathbf{a})\mathbf{a}, \quad \overrightarrow{BD} = d \cos \phi \mathbf{b}, \quad \overrightarrow{DC} = d \sin \phi \mathbf{c}, \quad (\text{A14})$$

and

$$d = |\overline{BA}| = |\overline{BC}|. \quad (\text{A15})$$

The vectors  $\mathbf{b}$  and  $\mathbf{c}$  are given by

$$\mathbf{b} = \frac{1}{d}\overrightarrow{BA} = \frac{1}{d}(\mathbf{r} - \overrightarrow{OB}) = \frac{1}{d}[\mathbf{r} - (\mathbf{r} \cdot \mathbf{a})\mathbf{a}], \quad (\text{A16})$$

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \frac{1}{d}\mathbf{a} \times [\mathbf{r} - (\mathbf{r} \cdot \mathbf{a})\mathbf{a}] = \frac{1}{d}\mathbf{a} \times \mathbf{r}. \quad (\text{A17})$$

Then

$$\begin{aligned} \mathbf{Q} &= (\mathbf{r} \cdot \mathbf{a})\mathbf{a} + \cos \phi [\mathbf{r} - (\mathbf{r} \cdot \mathbf{a})\mathbf{a}] + \sin \phi \mathbf{a} \times \mathbf{r} \\ &= \cos \phi \mathbf{r} + (1 - \cos \phi)(\mathbf{r} \cdot \mathbf{a})\mathbf{a} + \sin \phi \mathbf{a} \times \mathbf{r}. \end{aligned} \quad (\text{A18})$$

#### Proof of Equation (2)

Let  $\mathbf{r}$  in equation (A18) be equal to  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ . This gives

$$\begin{aligned} \mathbf{Q}\mathbf{a} &= \mathbf{a}, & \mathbf{Q}\mathbf{b} &= \cos \phi \mathbf{b} + \sin \phi \mathbf{c}, \\ \mathbf{Q}\mathbf{c} &= -\sin \phi \mathbf{b} + \cos \phi \mathbf{c}. \end{aligned} \quad (\text{A19})$$

These three equations show that the effect of  $\mathbf{Q}$  is a counterclockwise rotation of angle  $\phi$  of the basis vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$

about the  $\mathbf{a}$  axis. These equations also show that  $\mathbf{Q}$  can be written as follows:

$$\mathbf{Q} = \mathbf{a}\mathbf{a} + (\mathbf{b}\mathbf{b} + \mathbf{c}\mathbf{c}) \cos \phi - (\mathbf{b}\mathbf{c} - \mathbf{c}\mathbf{b}) \sin \phi \quad (\text{A20})$$

(Ogden, 1997; Chadwick, 1999). To verify this result apply  $\mathbf{Q}$  to  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  and use equation (A4) and the orthonormality of the three vectors.

#### Proof of Equation (3)

Let us apply  $\mathbf{Q}$  to  $\mathbf{r}$ . Letting

$$r_a = \mathbf{a} \cdot \mathbf{r}, \quad r_b = \mathbf{b} \cdot \mathbf{r}, \quad r_c = \mathbf{c} \cdot \mathbf{r}, \quad (\text{A21})$$

and using equation (A20) we get

$$\begin{aligned} \mathbf{Q}\mathbf{r} &= r_a \mathbf{a} + (r_b \mathbf{b} + r_c \mathbf{c}) \cos \phi - (r_c \mathbf{b} - r_b \mathbf{c}) \sin \phi \\ &= r_a \mathbf{a} + (r_b \cos \phi - r_c \sin \phi) \mathbf{b} + (r_b \sin \phi + r_c \cos \phi) \mathbf{c} \end{aligned} \quad (\text{A22})$$

(Chadwick, 1999). To interpret this equation, consider the vector  $\mathbf{r}^{bc} = (r_b, r_c)$  in the plane generated by the vectors  $\mathbf{b}$  and  $\mathbf{c}$ . Then the factors in parentheses correspond to the counterclockwise rotation of angle  $\phi$  of  $\mathbf{r}^{bc}$ .

#### Proof of Equations (5) and (6)

$\mathbf{Q}^T$  is given by

$$\mathbf{Q}^T = \mathbf{a}\mathbf{a} + (\mathbf{b}\mathbf{b} + \mathbf{c}\mathbf{c}) \cos \phi - (\mathbf{c}\mathbf{b} - \mathbf{b}\mathbf{c}) \sin \phi. \quad (\text{A23})$$

The product  $\mathbf{Q}^T \mathbf{Q}$  will include inner products such as

$$\begin{aligned} (\mathbf{a}\mathbf{a})(\mathbf{a}\mathbf{a}) &= (\mathbf{a} \cdot \mathbf{a})\mathbf{a}\mathbf{a} = \mathbf{a}\mathbf{a}, & (\mathbf{a}\mathbf{a})(\mathbf{b}\mathbf{b}) &= (\mathbf{a} \cdot \mathbf{b})\mathbf{a}\mathbf{b} = \mathbf{0}, \\ (\mathbf{b}\mathbf{c})(\mathbf{c}\mathbf{b}) &= (\mathbf{c} \cdot \mathbf{c})\mathbf{b}\mathbf{b} = \mathbf{b}\mathbf{b}, \end{aligned} \quad (\text{A24})$$

where equation (A8) was used. As a consequence

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{a}\mathbf{a} + \mathbf{b}\mathbf{b} + \mathbf{c}\mathbf{c} = \mathbf{I}, \quad (\text{A25})$$

where equation (A6) was used. Similarly,

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{I}. \quad (\text{A26})$$

This result and equation (A9) shows that

$$(\det \mathbf{Q})^2 = \det \mathbf{I} = 1. \quad (\text{A27})$$

Because  $\det \mathbf{Q} \neq 0$ ,  $\mathbf{Q}^{-1}$  exists. Next multiplication of equation (A25) on the right-hand side by  $\mathbf{Q}^{-1}$  gives

$$\mathbf{Q}^{-1} = \mathbf{Q}^T. \quad (\text{A28})$$

Now let us compute the scalar product of  $\mathbf{Q}\mathbf{r}_1$  and  $\mathbf{Q}\mathbf{r}_2$ , given

$$(\mathbf{Q}\mathbf{r}_1) \cdot (\mathbf{Q}\mathbf{r}_2) = \mathbf{r}_1 \cdot (\mathbf{Q}^T \mathbf{Q}\mathbf{r}_2) = \mathbf{r}_1 \cdot \mathbf{r}_2, \quad (\text{A29})$$

where equation (A1) was used with  $\mathbf{u} = \mathbf{Q}\mathbf{r}_2$ ,  $\mathbf{S} = \mathbf{Q}$ , and  $\mathbf{v} = \mathbf{r}_1$ .

Finally, the length of  $\mathbf{Q}\mathbf{r}$  can be computed using equation (A29) with  $\mathbf{r} = \mathbf{r}_1 = \mathbf{r}_2$ :

$$|\mathbf{Q}\mathbf{r}| = [(\mathbf{Q}\mathbf{r}) \cdot (\mathbf{Q}\mathbf{r})]^{1/2} = (\mathbf{r} \cdot \mathbf{r})^{1/2} = |\mathbf{r}|. \quad (\text{A30})$$

### Eigenvalues of $\mathbf{Q}$

Equation (A27) shows that  $\det \mathbf{Q} = \pm 1$ . If  $\det \mathbf{Q} = 1$  (or  $-1$ ),  $\mathbf{Q}$  is said to be a proper (or improper) orthogonal tensor. When dealing with matrices, an improper rotation matrix (its determinant is equal to  $-1$ ) includes reflections of coordinate axes, which in turn change the handedness of the coordinate system (e.g., from right- to left-handed). A general property of determinants is that it is invariant under a rotation of the basis vectors. Therefore, we can diagonalize  $\mathbf{Q}$ , with the diagonal elements equal to its eigenvalues  $\lambda_1$ ,  $\lambda_2$ , and  $\lambda_3$ , which gives

$$\det \mathbf{Q} = \lambda_1 \lambda_2 \lambda_3 = 1. \quad (\text{A31})$$

Another constraint on the eigenvalues of  $\mathbf{Q}$  can be derived as follows. Let  $\mathbf{v}_i$  and  $\lambda_i$  be an eigenvector and its corresponding eigenvalue. This means that

$$\mathbf{Q}\mathbf{v}_i = \lambda_i \mathbf{v}_i; \quad i = 1, 2, 3. \quad (\text{A32})$$

Now using equations (A30) and (A32) gives

$$|\mathbf{v}_i| = |\mathbf{Q}\mathbf{v}_i| = |\lambda_i \mathbf{v}_i|, \quad (\text{A33})$$

which in turn means that  $|\lambda_i| = 1$  (Noble and Daniel, 1977). Finally, the  $\lambda_i$  are the solutions of a cubic equation, which means that either the three of them are real, or one is real and the other two form a complex conjugate pair. In the first case we may have the values 1, 1, 1 or 1,  $-1$ ,  $-1$  (to satisfy  $\det \mathbf{Q} = 1$ ). In the second case we have 1,  $\exp(i\theta)$ ,  $\exp(-i\theta)$ . The two special subcases,  $\theta = 0$  and  $\theta = \pi$ , give the two possible sets of eigenvalues when they are all real. The eigenvectors corresponding to the complex eigenvalues have complex components. The angle  $\theta$  can be taken as the rotation angle  $\phi$  that appears in equation (1) (Jansen and Boon, 1967).

### Vector Associated with an Antisymmetric Tensor

A tensor  $\mathbf{T}$  is antisymmetric if  $\mathbf{T}^T = -\mathbf{T}$ . The corresponding expression in component form is  $T_{ij} = -T_{ji}$ , which is preserved under a rotation of coordinates. This implies that all the diagonal elements are identically equal to zero and that the tensor can be described by just three independent components. Therefore, the tensor can be associated with a vector  $\mathbf{t} = (t_1, t_2, t_3)$ , where each  $t_i$  is one of  $T_{12}$ ,  $T_{13}$ ,

or  $T_{23}$  (taken as the three independent components of  $T_{ij}$ ). These two entities are related as follows:

$$T_{ij} = \epsilon_{ijk} t_k, \quad (\text{A34})$$

$$t_i = \frac{1}{2} \epsilon_{ijk} T_{jk} \quad (\text{A35})$$

(e.g., Pujol, 2003). These two expressions are consistent with equations (59).

### Derivation of Equation (7) from Equation (2)

The dyadic  $\mathbf{bc} - \mathbf{cb}$  in equations (2) and (A20) is antisymmetric and in component form is written as  $b_i c_j - c_i b_j$ . The vector associated with this tensor (computed using equation A35) is  $\mathbf{b} \times \mathbf{c}$  (Pujol, 2003), which in turn is equal to  $\mathbf{a}$ . This result, in combination with equation (A34) gives the third term in equation (7). Now write the first two terms in equation (2) as

$$\begin{aligned} & \cos \phi (\mathbf{aa} + \mathbf{bb} + \mathbf{cc}) + (1 - \cos \phi) \mathbf{aa} \\ &= \cos \phi \mathbf{I} + (1 - \cos \phi) \mathbf{aa}, \end{aligned} \quad (\text{A36})$$

where equation (A6) was used. This result corresponds to the first two terms of equation (7) because the  $ij$  components of  $\mathbf{I}$  and  $\mathbf{aa}$  are  $\delta_{ij}$  and  $a_i a_j$ .

### Uniqueness of the Decomposition of a Tensor into Symmetric and Antisymmetric Parts

Assume that there are two decompositions, that is,

$$\mathbf{D} = \mathbf{S}_1 + \mathbf{A}_1 = \mathbf{S}_2 + \mathbf{A}_2, \quad (\text{A37})$$

where  $\mathbf{S}_1$  and  $\mathbf{S}_2$  are symmetric and  $\mathbf{A}_1$  and  $\mathbf{A}_2$  are antisymmetric. Then,

$$\mathbf{D}^T = \mathbf{S}_1 - \mathbf{A}_1 = \mathbf{S}_2 - \mathbf{A}_2. \quad (\text{A38})$$

From these two equations we get

$$\mathbf{S}_1 - \mathbf{S}_2 + (\mathbf{A}_1 - \mathbf{A}_2) = 0, \quad (\text{A39})$$

$$\mathbf{S}_1 - \mathbf{S}_2 - (\mathbf{A}_1 - \mathbf{A}_2) = 0. \quad (\text{A40})$$

From the sum of these two equations we get  $\mathbf{S}_1 = \mathbf{S}_2$  and from the difference  $\mathbf{A}_1 = \mathbf{A}_2$ . Therefore, the decomposition is unique.

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