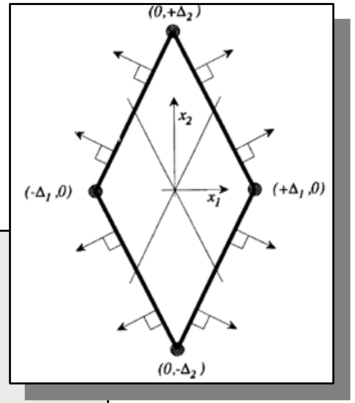
Finite volumes



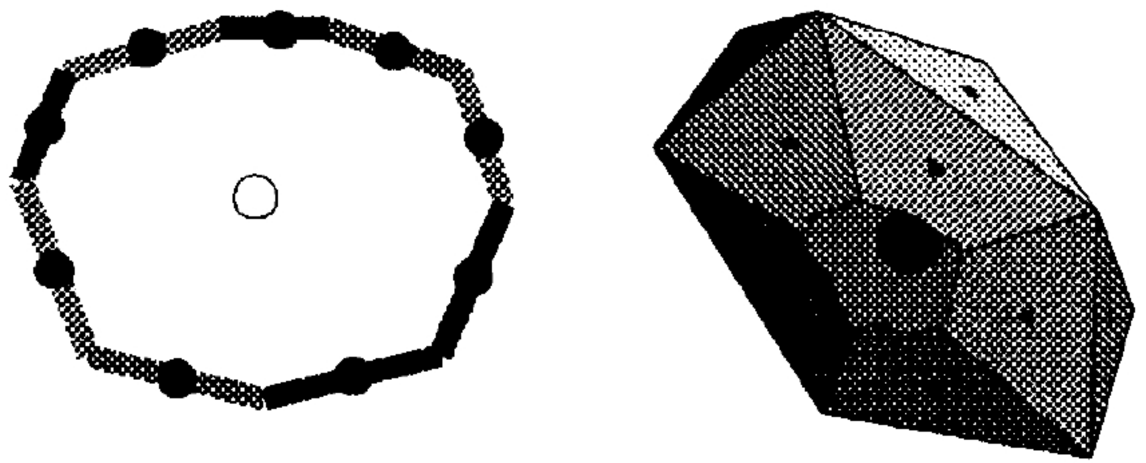
Finite volumes …

A numerical method based on a discrete version of Gauss´ theorem.

- The theoretical basis
- Derivation of weights for basic grid cells
- FV for hexagonal and irregular grids

… this lecture based on :
Dormy E. and Tarantola A., J. Geophys. Res., 100, 2123-2133, 1995.
Käser, M., Diplomarbeit LMU, 2000.

Finite volumes - basic theory

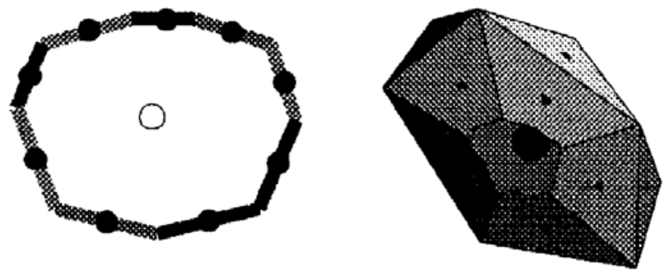


$$\partial_i V_{jk\ldots} \approx \frac{1}{\Delta S} \sum_{\alpha} \Delta L^{\alpha} \, n_i^{\alpha} \, V_{jk\ldots}^{\alpha}$$

$$\partial_i V_{jk\ldots} \approx \frac{1}{\Delta V} \sum_{\alpha} \Delta S^{\alpha} \, n_i^{\alpha} \, V_{jk\ldots}^{\alpha}$$

… as the figure suggests, the FV method is based on the idea of knowing a 3D field at the sides of a surface surrounding a **finite volume**. Is there a mathematical theorem relating the (vector) fields inside a volume with the values at its surface? …. Yes, it s **Gauss´ theorem** …

Finite volumes – Gauss´ theorem



$$\partial_i V_{jk\ldots} \approx \frac{1}{\Delta S}\sum_\alpha \Delta L^\alpha n_i^\alpha V_{jk\ldots}^\alpha \qquad \partial_i V_{jk\ldots} \approx \frac{1}{\Delta V}\sum_\alpha \Delta S^\alpha n_i^\alpha V_{jk\ldots}^\alpha$$

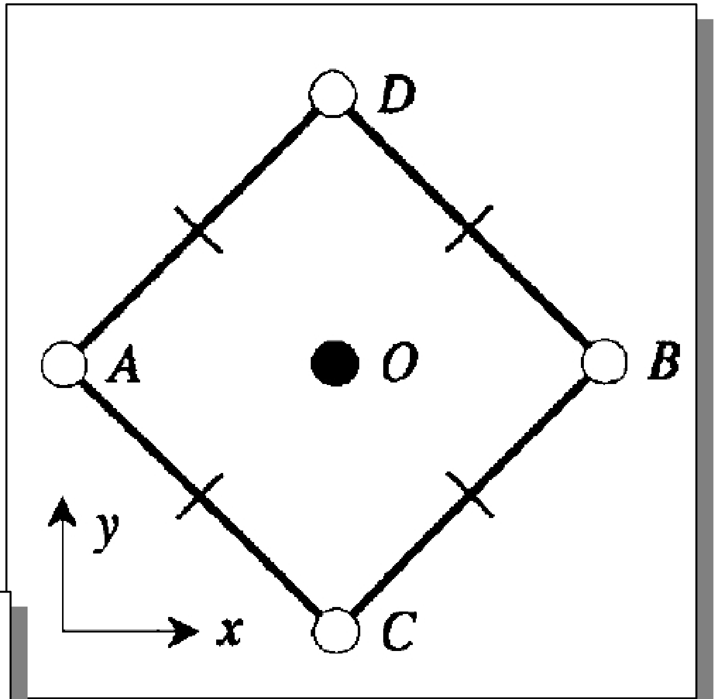
Gauss´ theorem:
(by the way one of the most important results on mathematical physics)

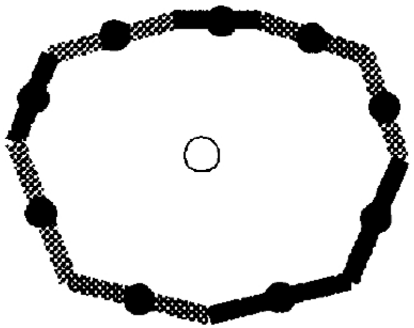
$$\int_V dV \partial_i w_i = \int_S dS n_i w_i$$

S	boundary surrounding V
V	volume inside S
w_i	vector field
n_i	unitary normal to the surface (pointing outwards)

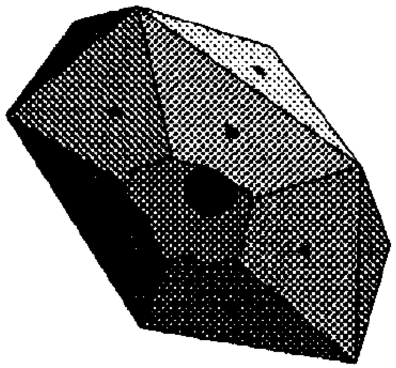
Question:

How can we approximate the gradient of a tensor field at a point P given the values a some points P_1, P_2, P_3, P_4, P_5, … around P ?





$$\partial_i V_{jk\ldots} \approx \frac{1}{\Delta S} \sum_{\alpha} \Delta L^{\alpha} n_i^{\alpha} V_{jk\ldots}^{\alpha}$$



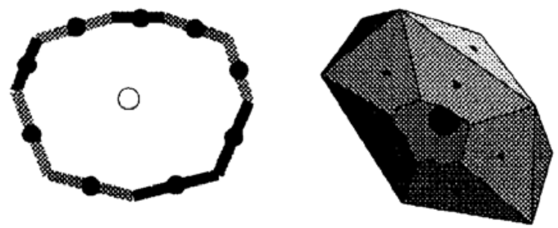
$$\partial_i V_{jk\ldots} \approx \frac{1}{\Delta V} \sum_{\alpha} \Delta S^{\alpha} n_i^{\alpha} V_{jk\ldots}^{\alpha}$$

$$\int_V dV \partial_i W_{jk} = \int_S dS n_i W_{jk}$$

$$\int_S dV \partial_i W_{jk} = \int_L dL n_i W_{jk}$$

Generalization

Gauss´ theorem:
Generalized to the gradient of for arbitrary tensor fields … (e.g. could also be a scalar field) …



$$\partial_i V_{jk\ldots} \approx \frac{1}{\Delta S} \sum_{\alpha} \Delta L^{\alpha} n_i^{\alpha} V_{jk\ldots}^{\alpha} \qquad \partial_i V_{jk\ldots} \approx \frac{1}{\Delta V} \sum_{\alpha} \Delta S^{\alpha} n_i^{\alpha} V_{jk\ldots}^{\alpha}$$

$$\int_V dV \partial_i W_{jk} = \int_S dS n_i W_{jk}$$

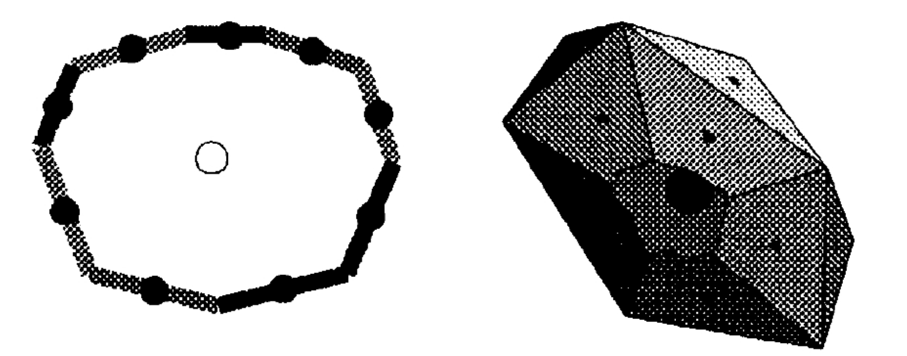
W_{jk}	arbitrary tensor field
V	volume inside S
S	surface around V
n_i	unitary normal to the surface (pointing outwards)

Finite volumes - 3D

Answer:

We simply need to turn Gauss´ theorem into a discrete version!

Assumption: smoothly varying W_{jk}



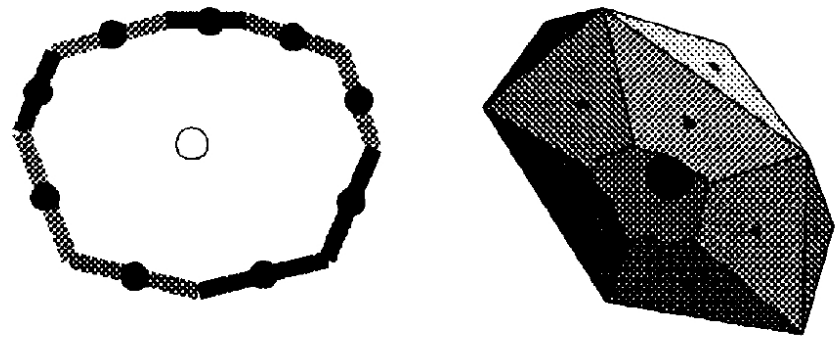
$$\partial_i V_{jk\dots} \approx \frac{1}{\Delta S}\sum_\alpha \Delta L^\alpha n_i^\alpha V_{jk\dots}^\alpha \qquad \partial_i V_{jk\dots} \approx \frac{1}{\Delta V}\sum_\alpha \Delta S^\alpha n_i^\alpha V_{jk\dots}^\alpha$$

$$\partial_i W_{jk} \approx \frac{1}{\Delta V}\sum_\alpha \Delta S_\alpha n_i^\alpha W_{jk}^\alpha$$

W_{jk}	arbitrary tensor field
ΔV	total volume
ΔS_α	surface segment
n_i	unitary normal to the surface
α	number of surface segments

Finite volumes - 2D

$$\partial_i W_{jk} \approx \frac{1}{\Delta S}\sum_{\alpha} \Delta L_{\alpha} n_i^{\alpha} W_{jk}^{\alpha}$$



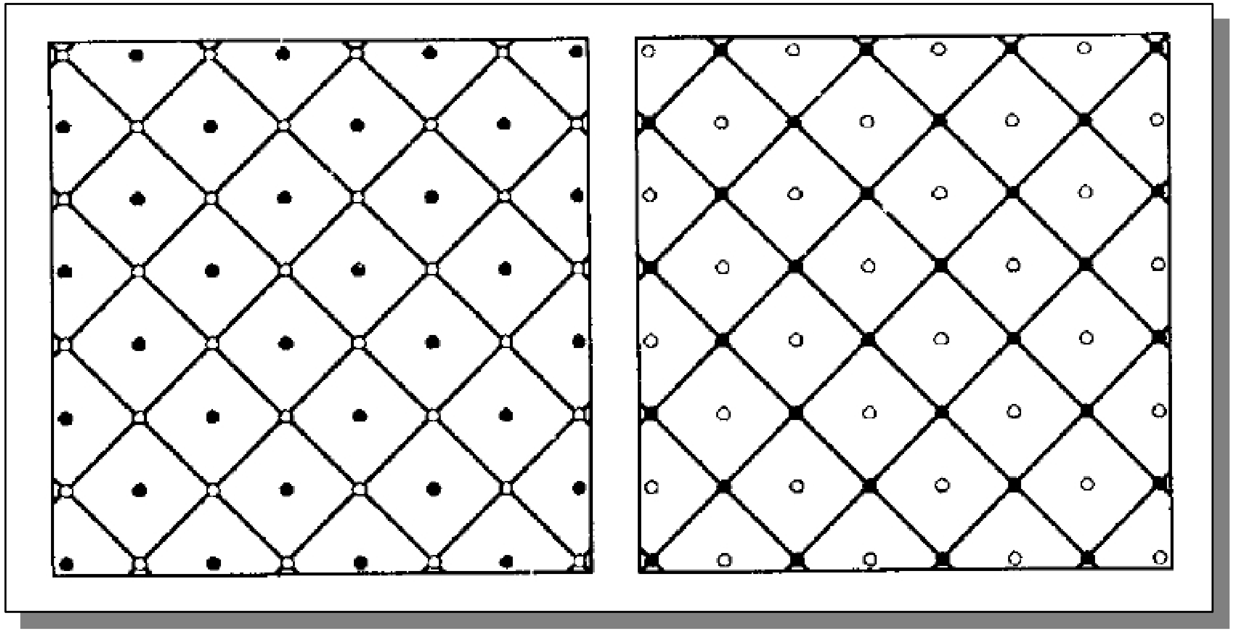
$$\partial_i V_{jk\ldots} \approx \frac{1}{\Delta S}\sum_{\alpha} \Delta L^{\alpha} n_i^{\alpha} V_{jk\ldots}^{\alpha} \qquad \partial_i V_{jk\ldots} \approx \frac{1}{\Delta V}\sum_{\alpha} \Delta S^{\alpha} n_i^{\alpha} V_{jk\ldots}^{\alpha}$$

W_{jk}	arbitrary tensor field
ΔS	total surface
ΔL_{α}	boundary segment
n_i	unitary normal to the surface
α	number of surface segments

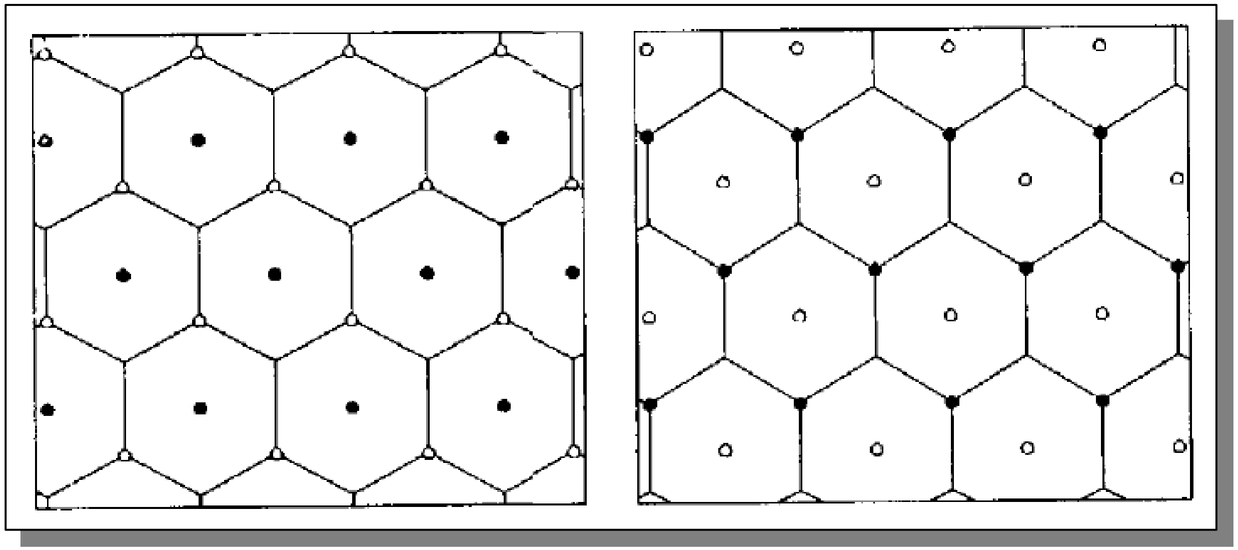
How can we use these ideas to solve p.d.e.´s ?

Finite volumes – space grids

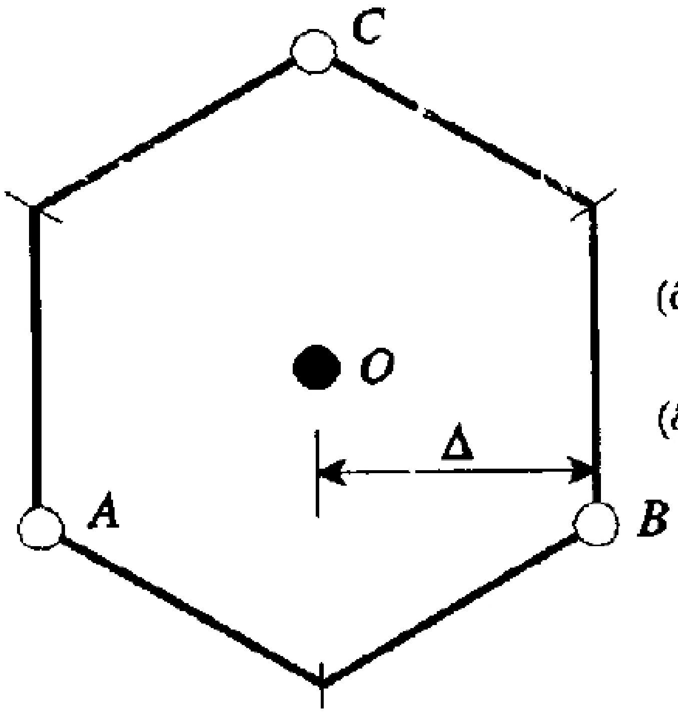
2D Euclidian space
- Lozenges
- staggered grid

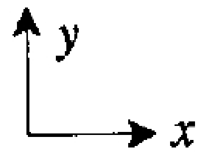


2D Euclidian space
- hexagons
- minimal grid



Finite volumes

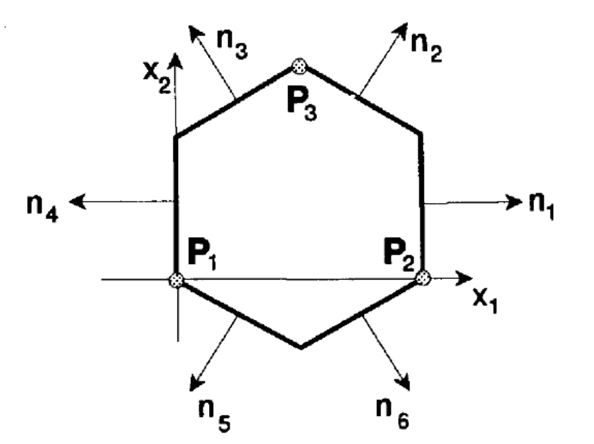




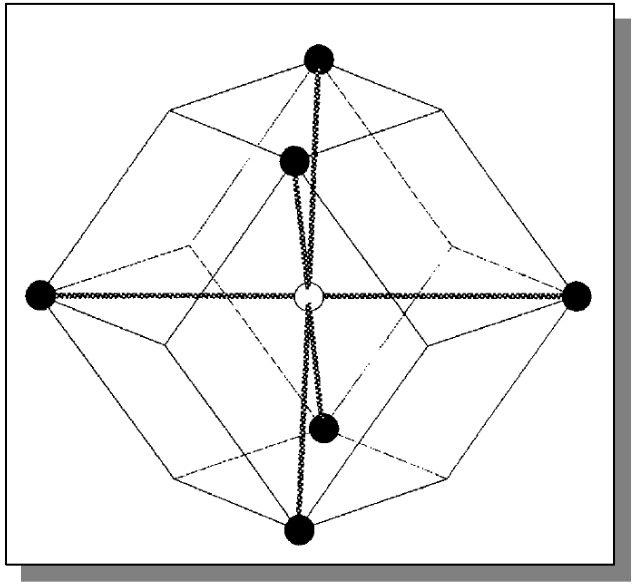
$$(\partial_x V_{ij\ldots})(\mathcal{O}) \approx \frac{V_{ij\ldots}(\mathcal{B}) - V_{ij\ldots}(\mathcal{A})}{2\Delta}$$

$$(\partial_y V_{ij\ldots})(\mathcal{O}) \approx \frac{V_{ij\ldots}(\mathcal{C}) - \frac{1}{2}(V_{ij\ldots}(\mathcal{B}) + V_{ij\ldots}(\mathcal{A}))}{\sqrt{3}\Delta}$$

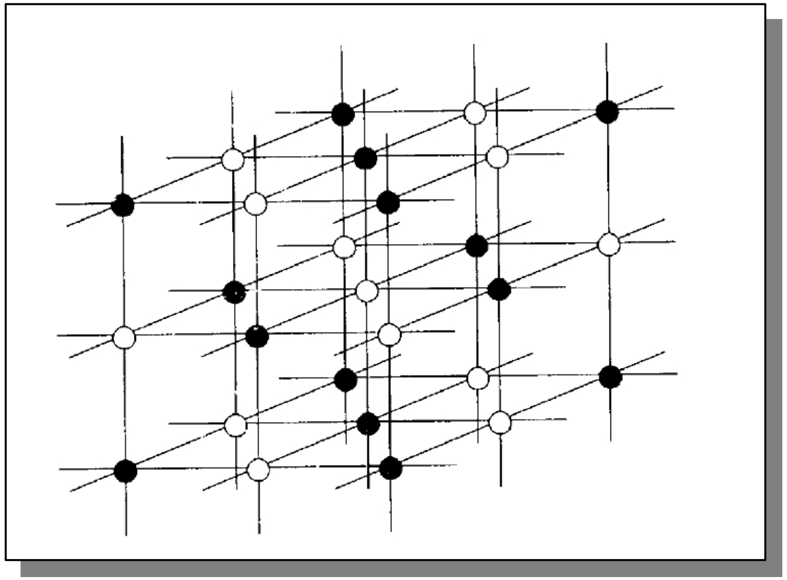
Minimal grid for finite volumes



Finite volumes - space grids



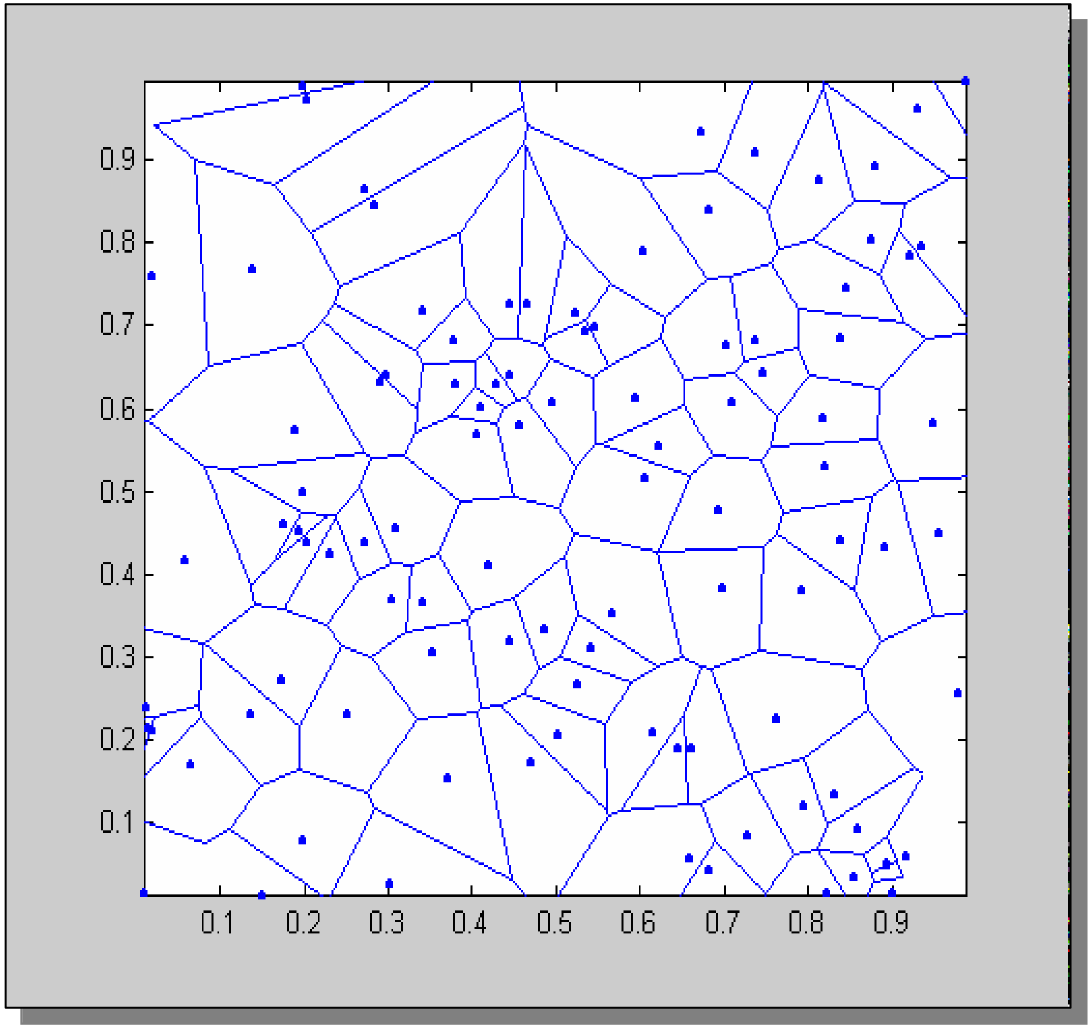
Voronoi cell for FD grid

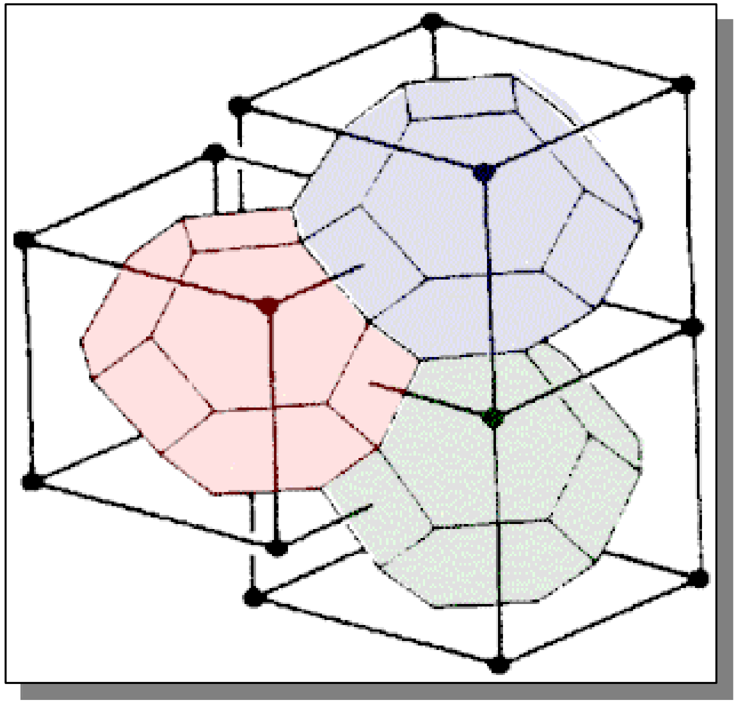


Classic FD grid in 3D

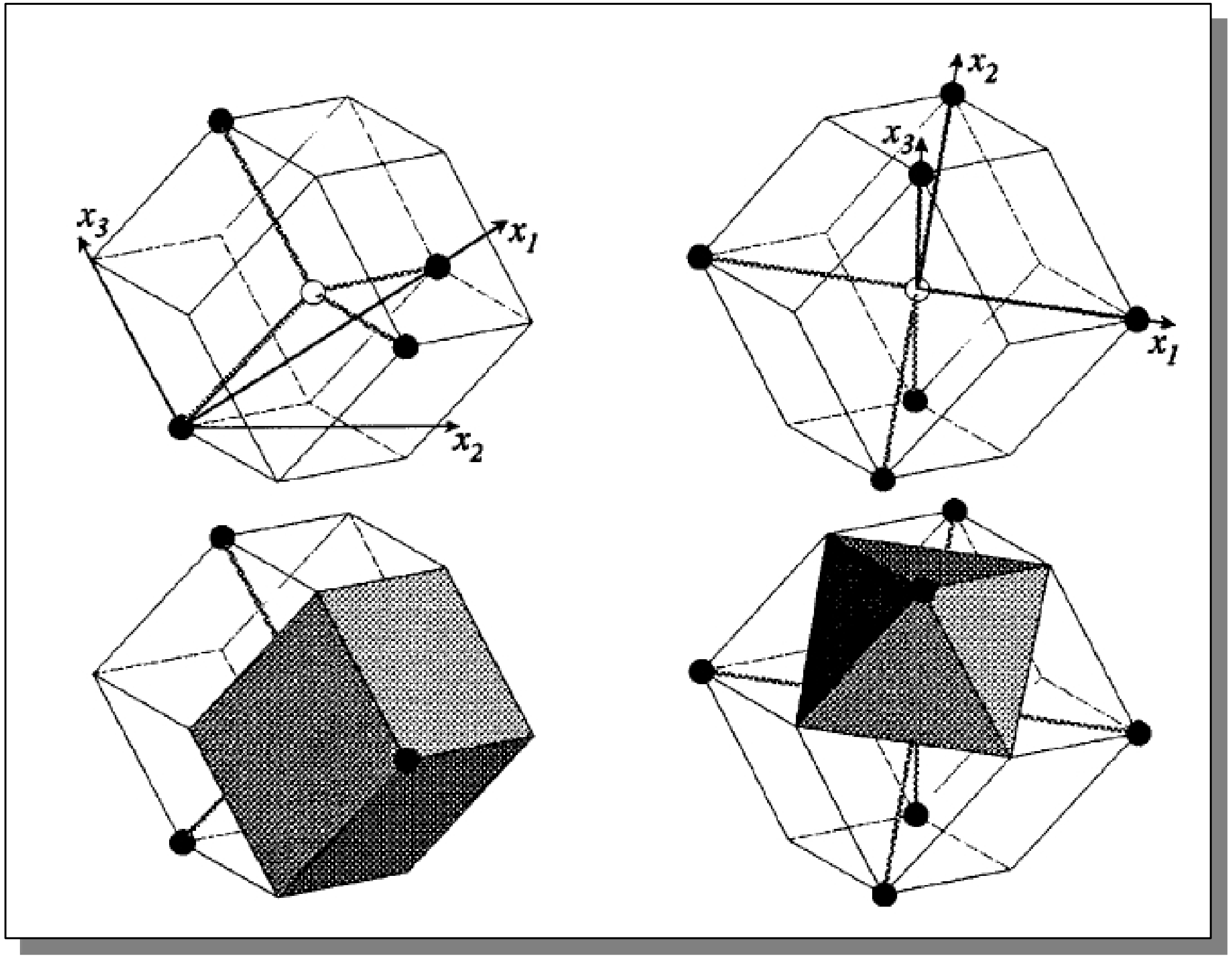
The Voronoi diagrams of an unstructured set of nodes divides the plane into a set of regions, one for each node, such that any point in a particular region is closer to that regions node than to any other.

Voronoi cells

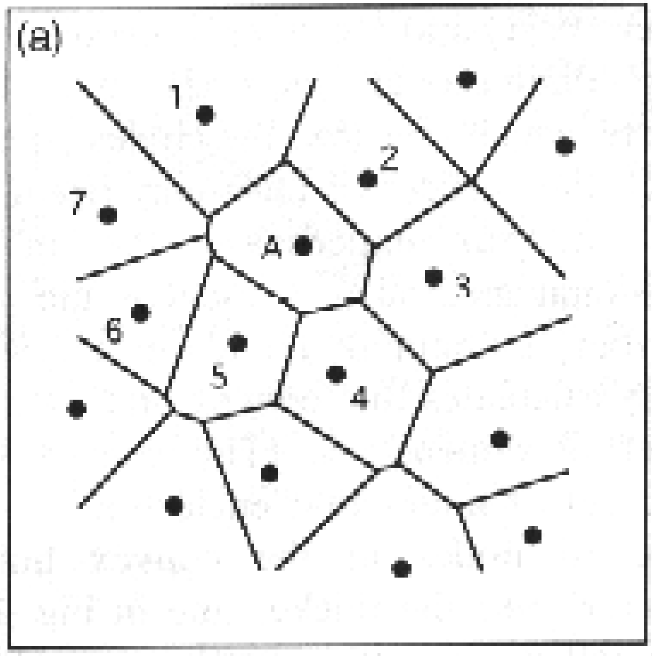


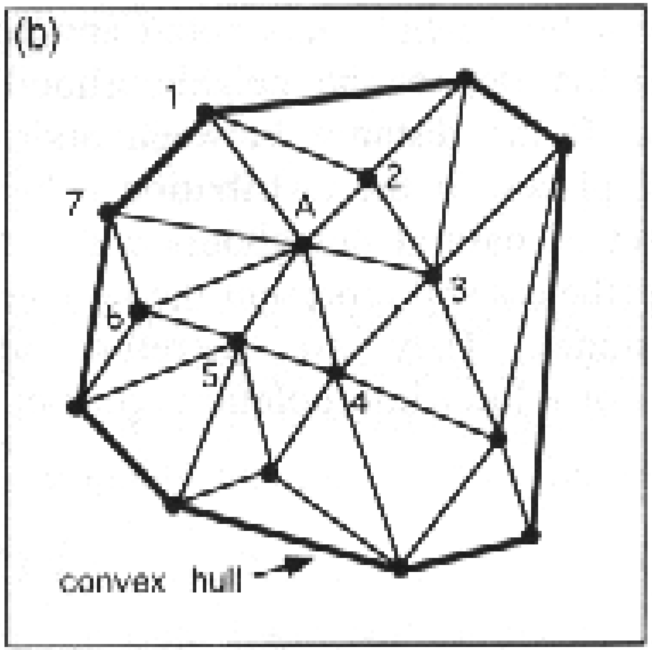


Finite volumes – volumes and surfaces



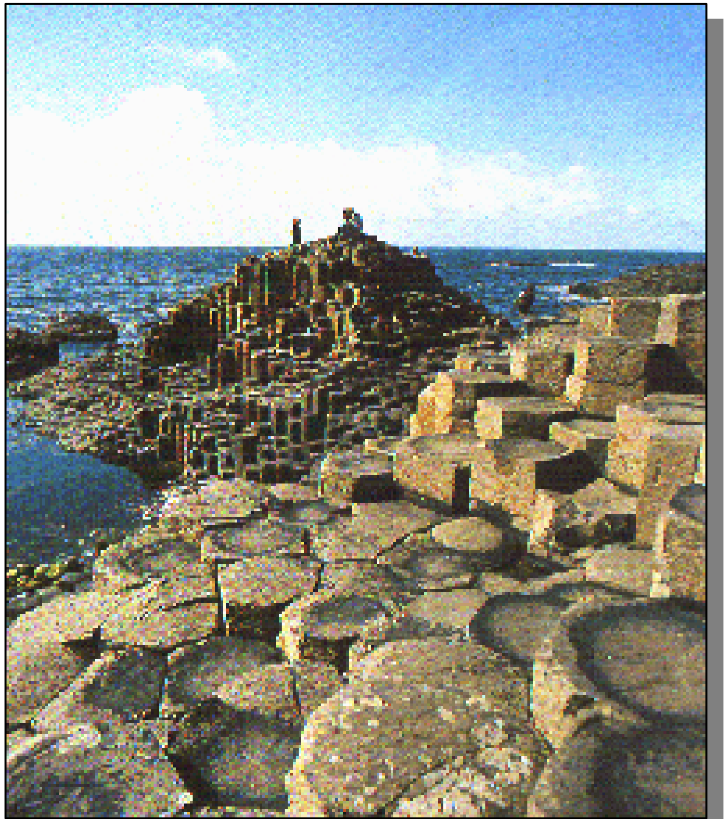
Voronoi and Delaunay



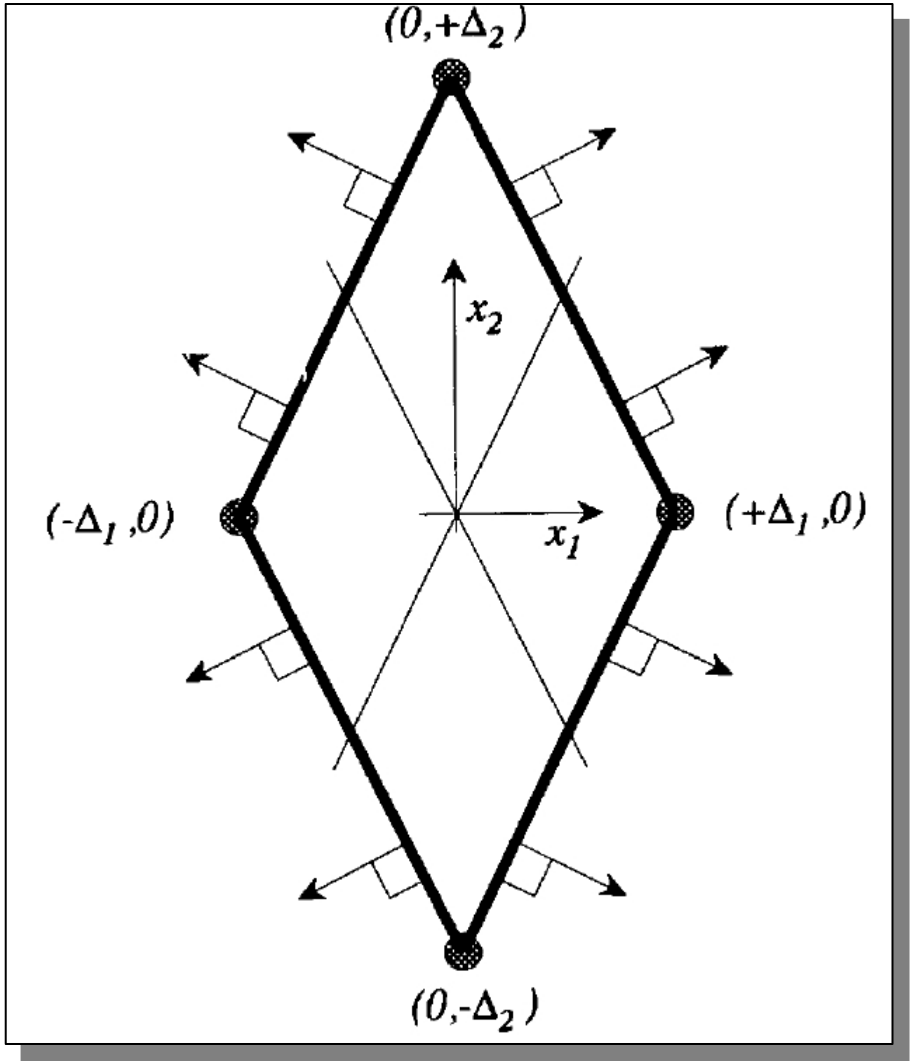


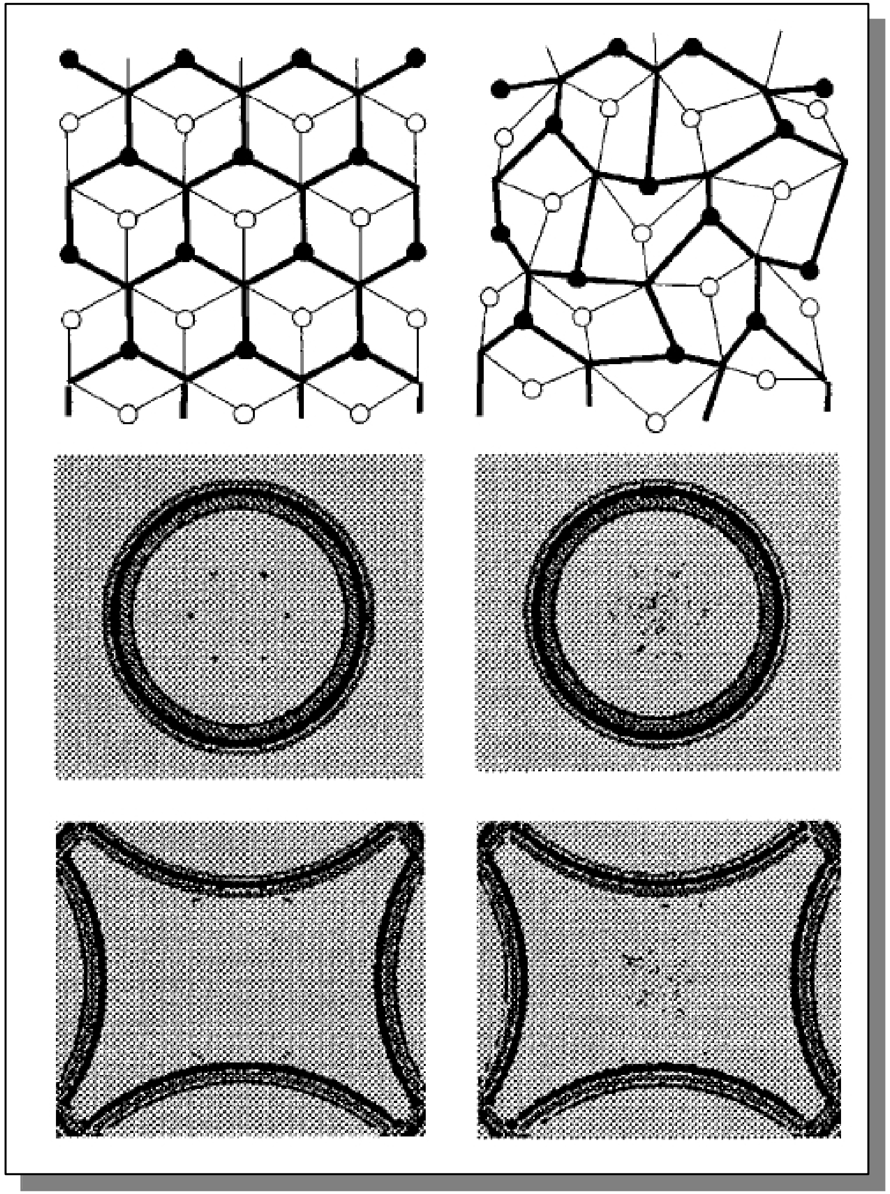
Delauney triangles are obtained by linking the vertices of neighbouring Voronoi cells

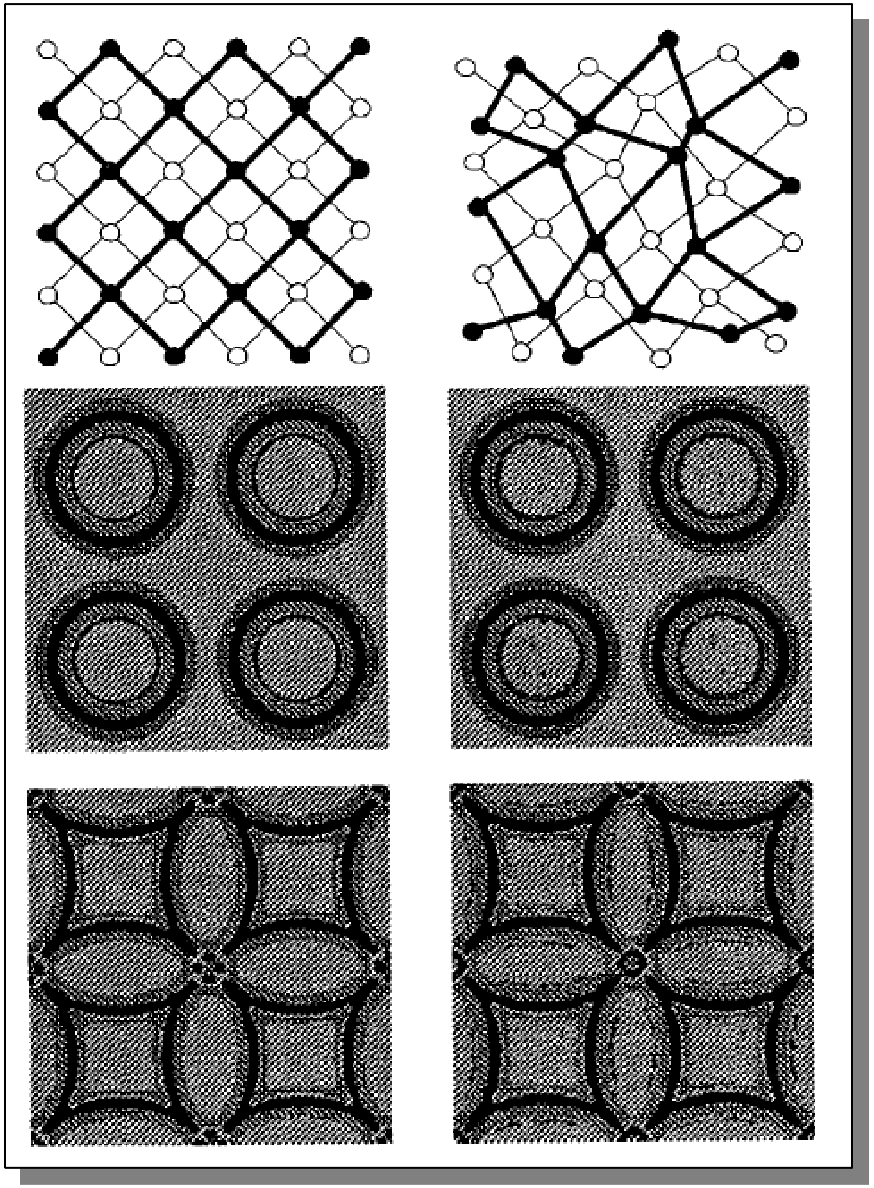
Voronoi Cells in Nature



Let us calculate the difference operators for a simples finite volume cell

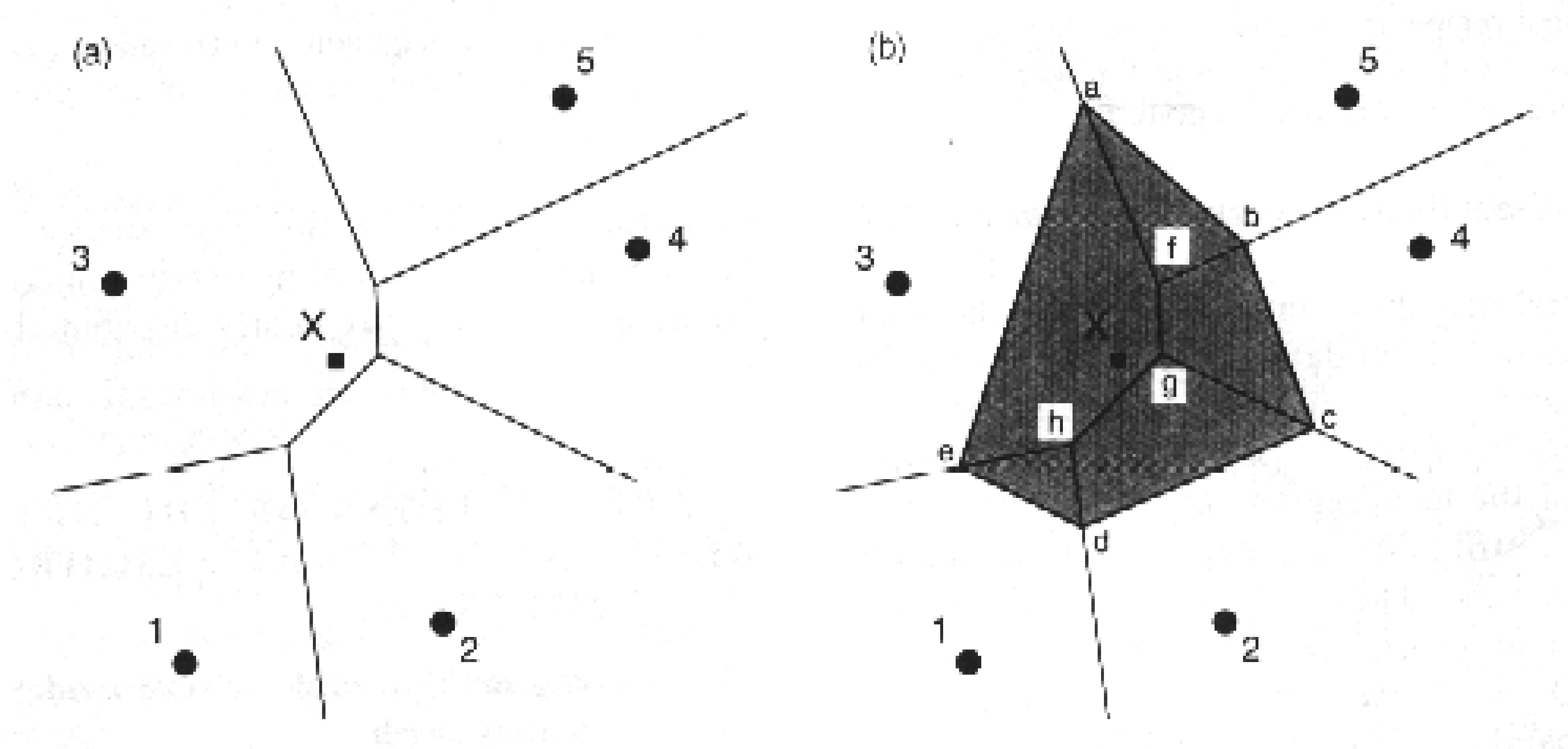




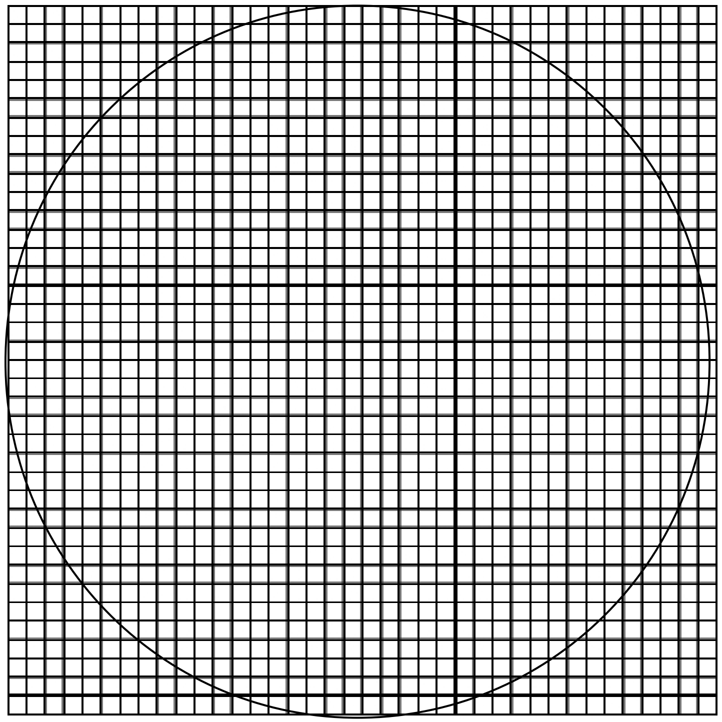


Natural Neighbours

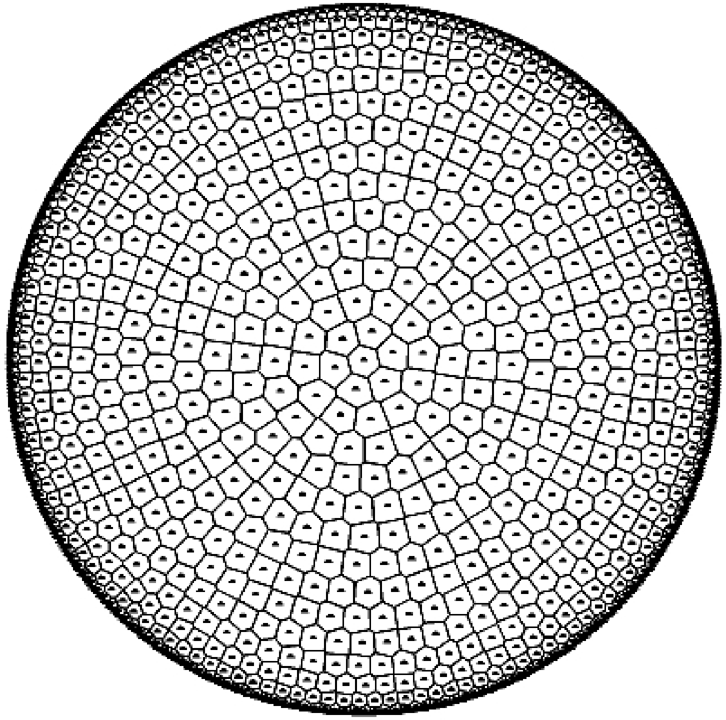
- Basis for local interpolation
- Linear interpolation using triangles
- Distance weighting
- Natural neighbour interpolation
- Differential weights
- Examples


(a)
5
3
4
X
1
2
(b)
a
5
b
f
4
3
X
g
h
c
e
d
1
2

Unstructured Grid Methods

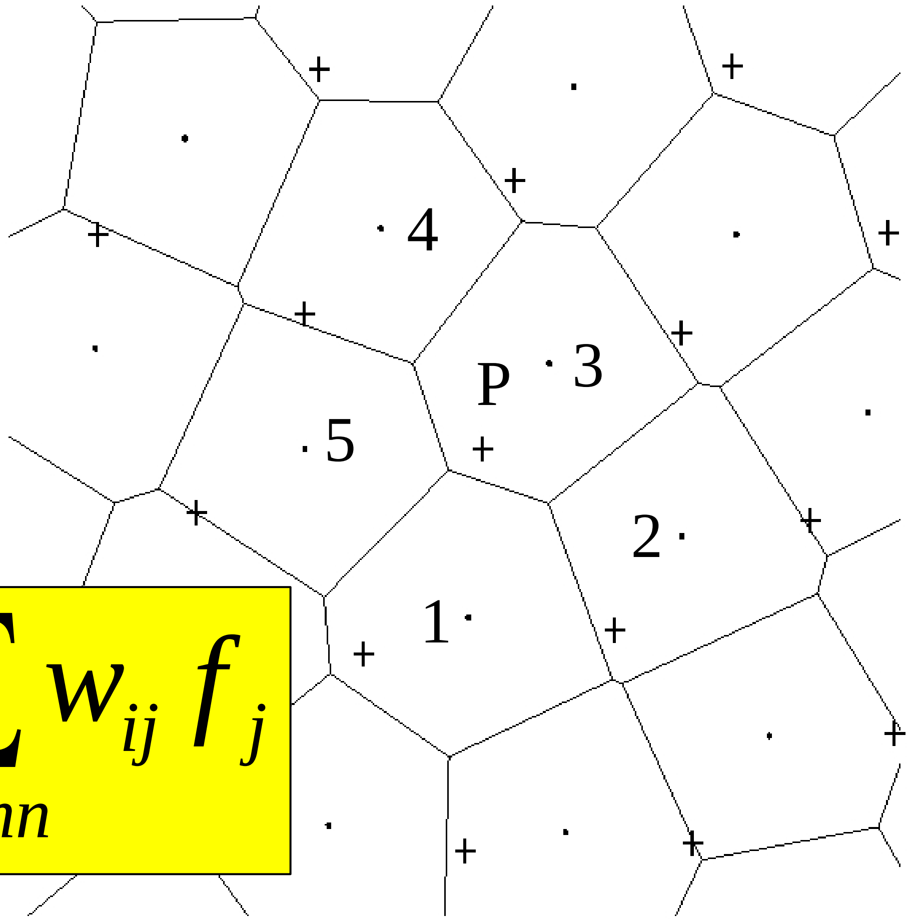


Regular Grid



Voronoi cells

Waves on unstructured grids



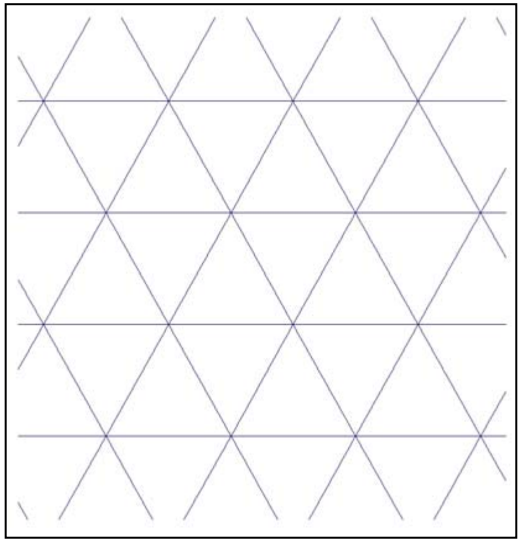
+ Primary grid (velocities)

. Secondary grid (stresses)

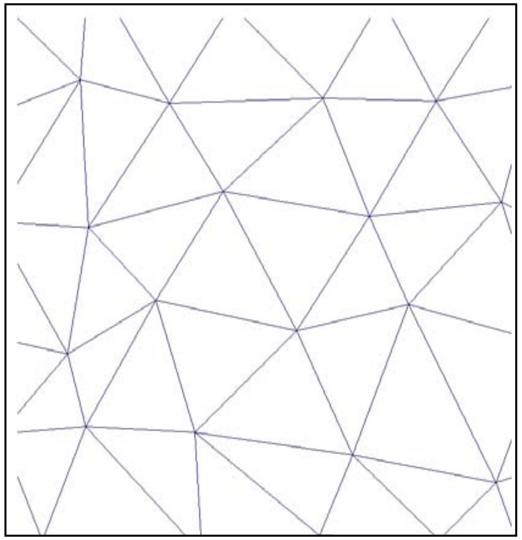
Point P with 5 natural neighbours

$$\partial f_P^i = \sum_{i=1,nn} w_{ij} f_j$$

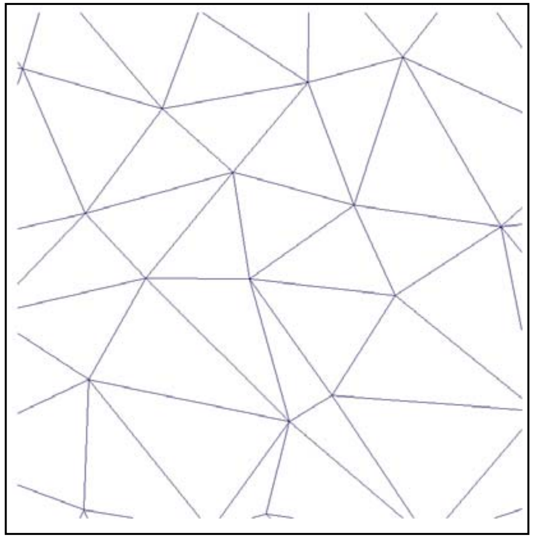
Triangular grid quality



q_{mean}=1.0



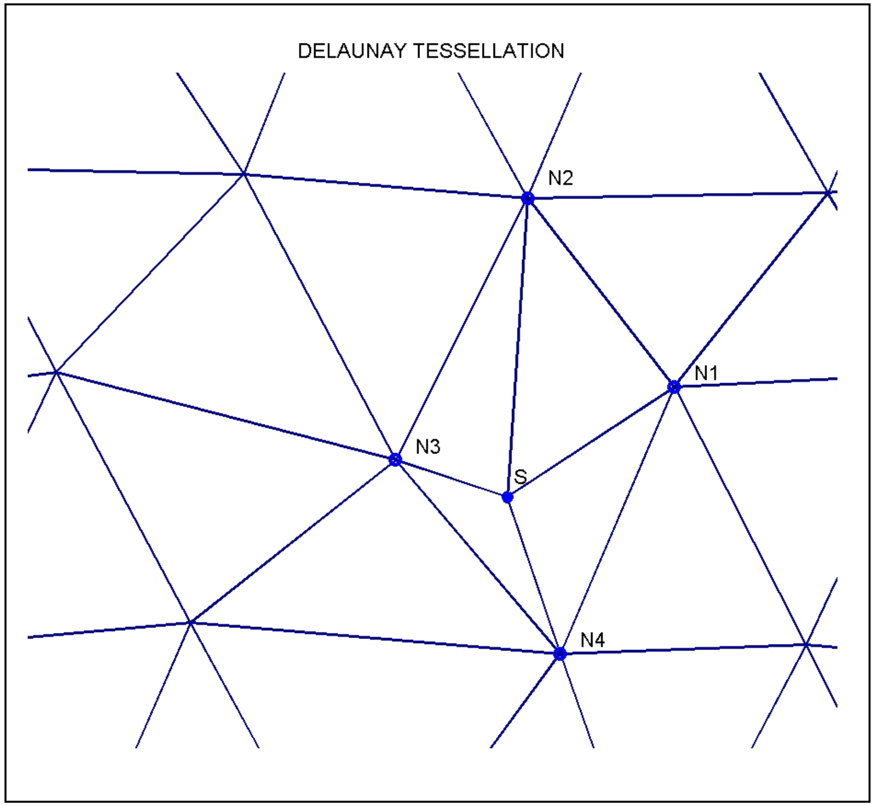
q_{mean}=0.9



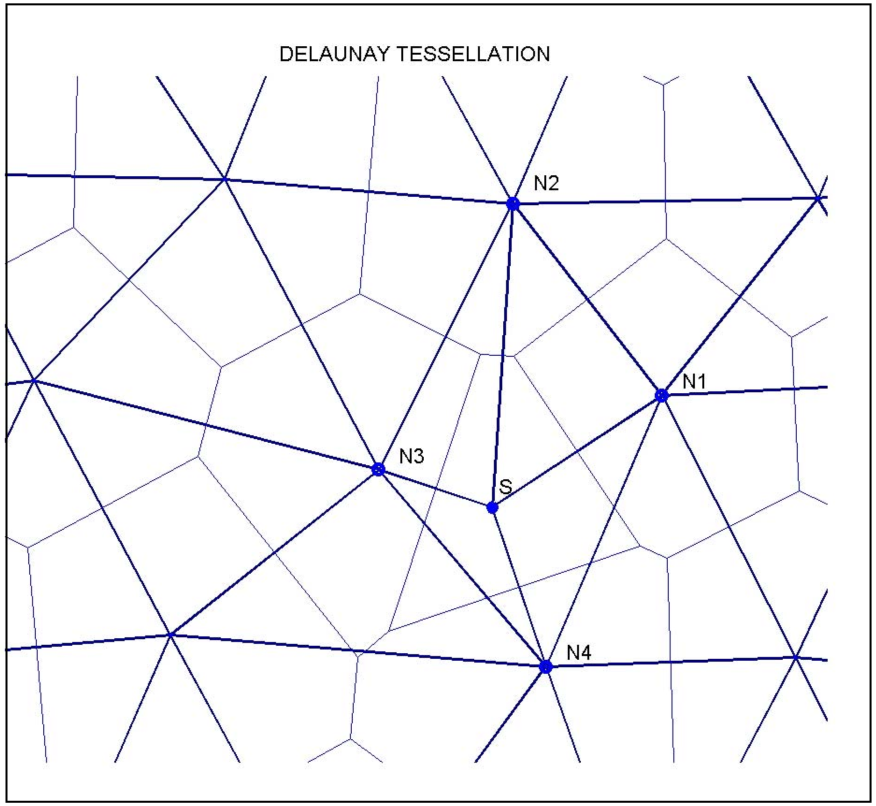
q_{mean}=0.8

$$q = \frac{4\sqrt{3}A}{a^2 + b^2 + c^2}$$

Triangle quality q



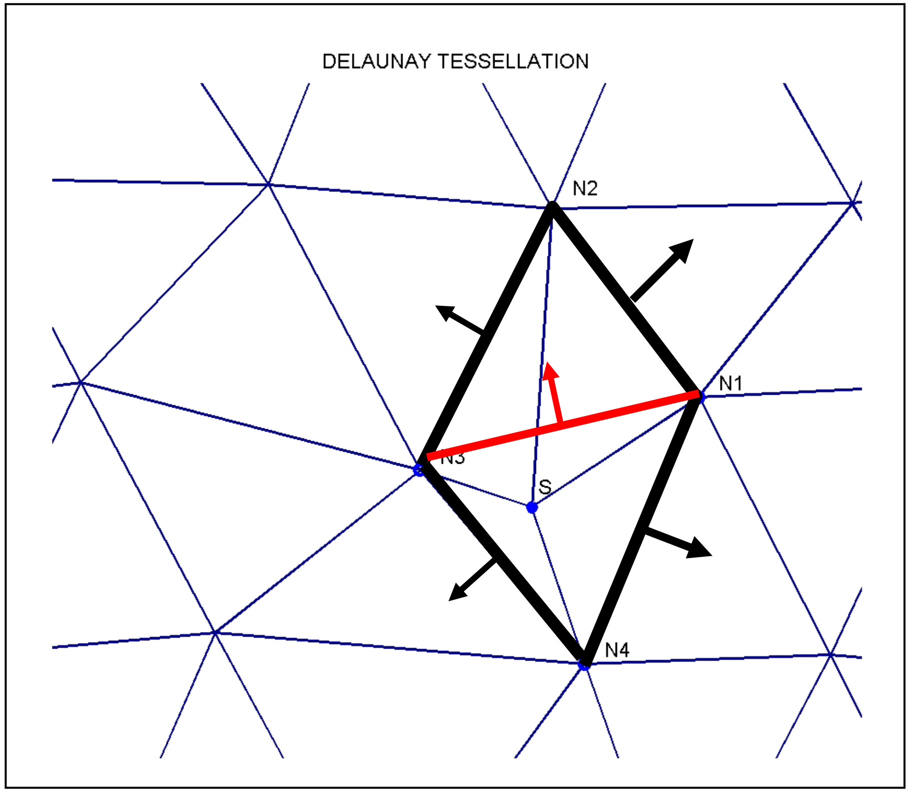
Triangulation



Voronoi Cells

Interpolation (and differential weights for natural neighbours are calculated using overlapping Voronoi cells).

Method 2: The Finite Volume Method



The Finite Volume method is based on a discretization of Gauss' Law

$$\partial_i f = \frac{1}{\Delta S} \sum_{j=1}^{NN} \Delta L_{ij} n_{ij} f_j$$

Note that the position of point S is irrelevant!

Surprising result! Using only three points is more accurate than using all natural neighbours!

Test Function

Test function f_p on primary grid points x_i:

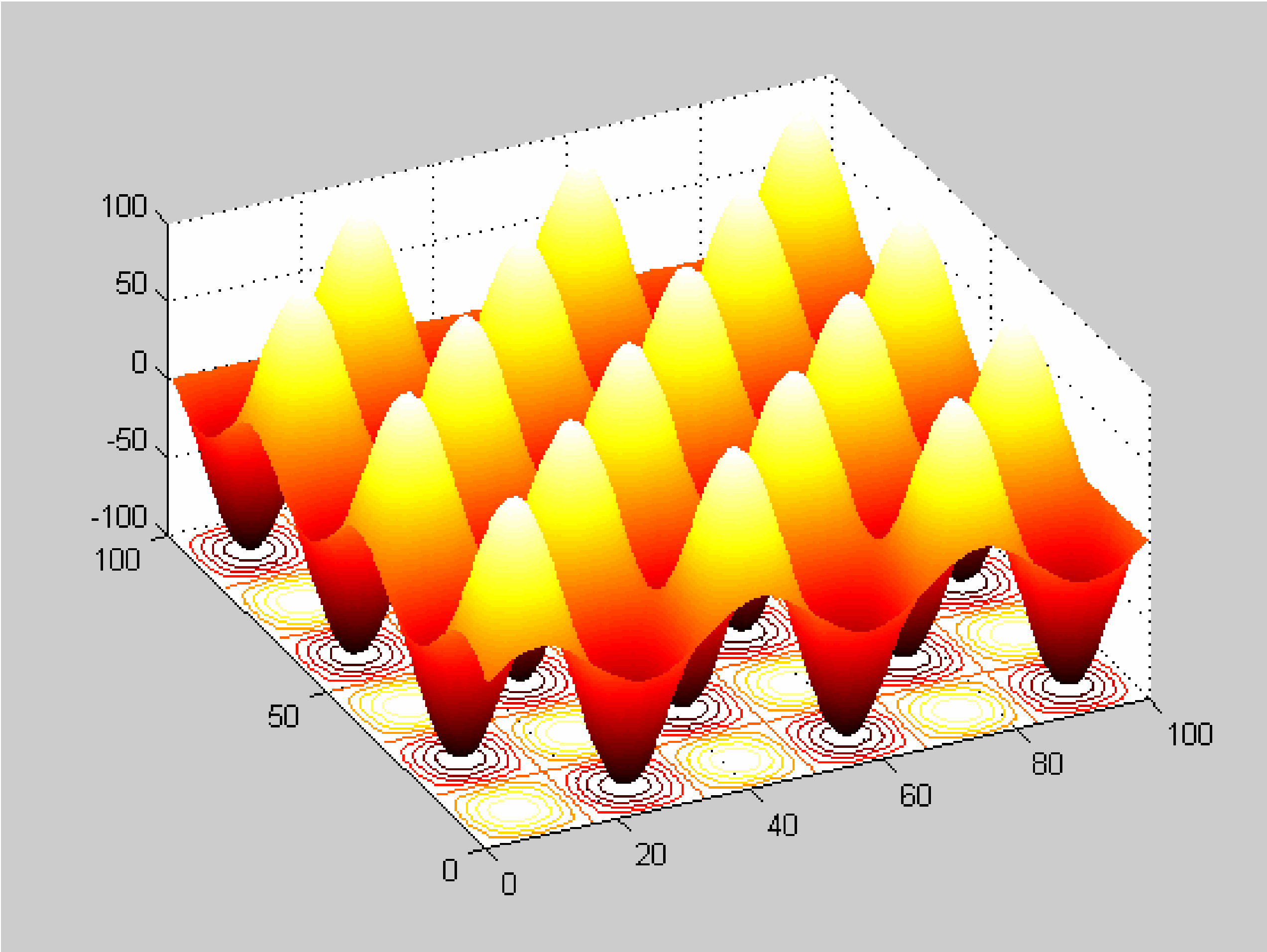
$$f_p(x_i) = \sin(\underline{k}\,\underline{x}_i - wt)$$

Analytical deriv ative $f^{(j)}$ on secondary grid points x_k:

$$f_s^{(j)}(x_k) = k_j \cos(\underline{k}\,\underline{x}_k - wt)$$

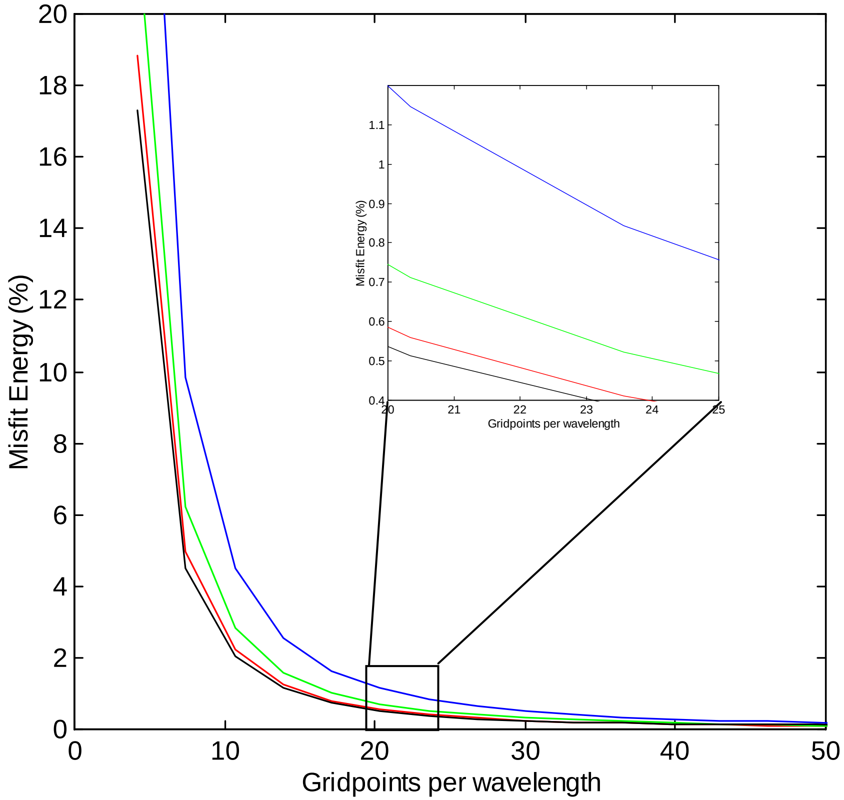
Error of numerical derivative on sec. grid

$$\varepsilon(\underline{k}, q_{mean}) = \frac{\sum_k (\tilde{f}^{(j)}(x_k) - f(x_k))^2}{\sum_k f^2(x_k)}$$


100
50
0
-50
-100
100
50
0
0
20
40
60
80
100

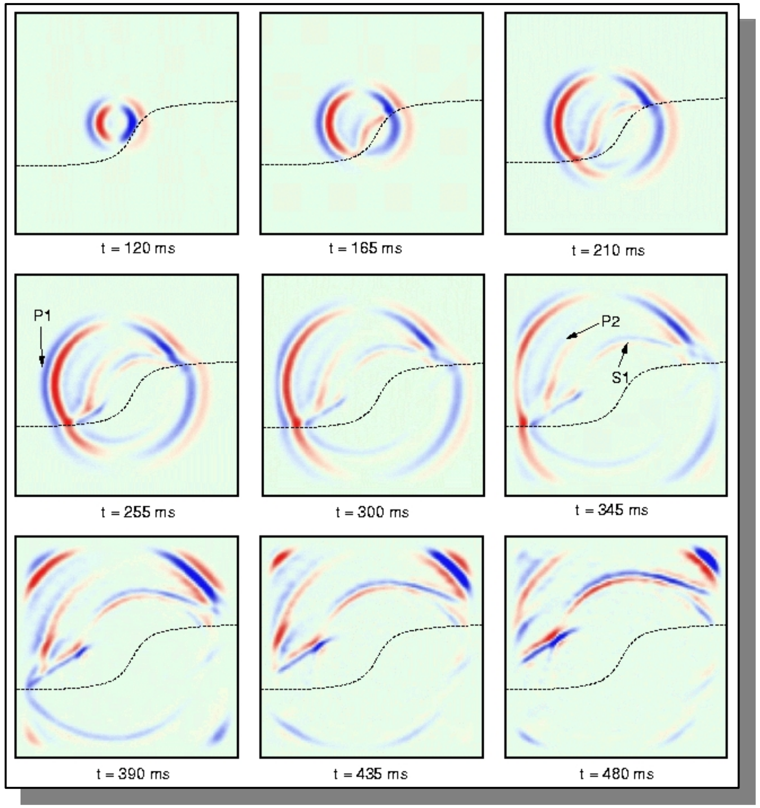
Error of space derivative

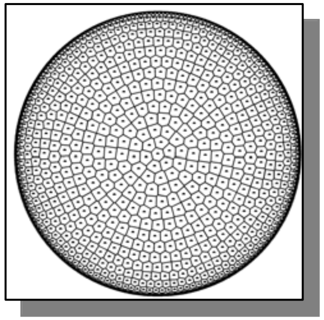
Irregular Grid - q^{mean} =0.8

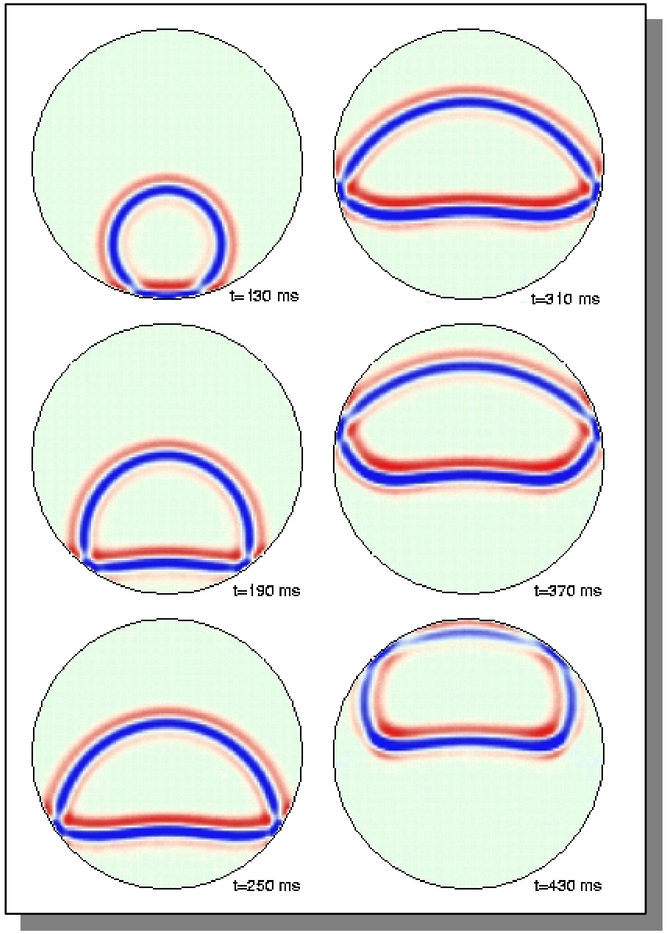


black - Magnier
green - NN
blue - FV(NN)
red - FV (3 points)

Waves with natural neighbours







Käser, Igel, Sambridge, Braun, 2001
Käser and Igel, 2001

Finite volumes: summary

The **finite volume method** is an elegant approach to solving partial differential equations on unstructured grids.

The finite volume method is based on a discretization of **Gauss´ theorem**.

The FV method is frequently applied to **flow problems**. High-order approaches have been recently developed.

The FV method requires the calculation of volumes and surfaces for each cell. This requires the calculation of **Voronoi cells** and **triangulation**.