Finite Elements – A practical introduction

- Introduction
  - Why Finite Elements
  - Domains of Applications
  - Applications in Geophysics
  - Brief history
  - Examples
- Review of Matrix Algebra

Finite elements ...

A general discretization procedure of continuum problems posed by mathematically defined statements

Finite Elements - the concept

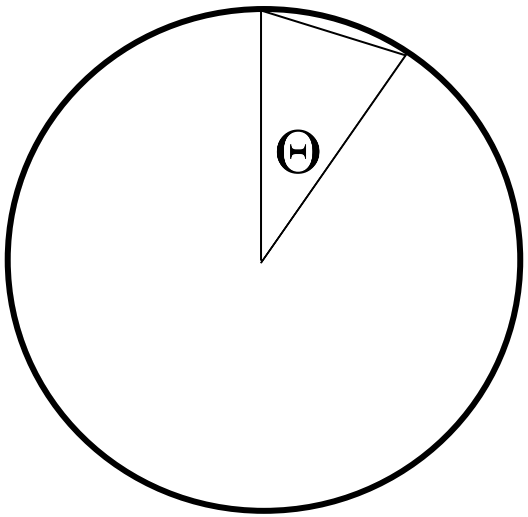
Basic principle: building a complicated object with simple blocks (e.g. LEGO) or divide a complicated object into manageable small pieces.

Example: approximation of an area of a circle

Area of one triangle: $$S_i = \frac{1}{2} R^2 \sin \theta_i$$

Area of the circle: $$S_N = \sum_{i=1}^{N} S_i = \frac{1}{2} R^2 N \sin(\frac{2\pi}{N})$$

$$\rightarrow \pi R^2 \quad as \quad N \rightarrow \infty$$



N total number of triangles

Finite Elements – the concept

How to proceed in FEM analysis:

- Divide stucture into pieces
- Describe behaviour of the physical quantities in each element
- Connect (assemble) the elements at the nodes to form an approximate system of equations for the whole structure
- Solve the system of equations involving unknown quantities at the nodes (e.g. displacements)
- Calculate desired quantities (e.g. strains and stresses) at selected elements

Finite Elements - Why?

FEM allows discretization of bodies with arbitrary shape. Originally designed for problems in static elasticity.

FEM is the most widely applied computer simulation method in engineering.

The required grid generation techniques are interfaced with graphical techniques (CAD).

Today numerous commercial FEM software is available (e.g. ANSYS, SMART)

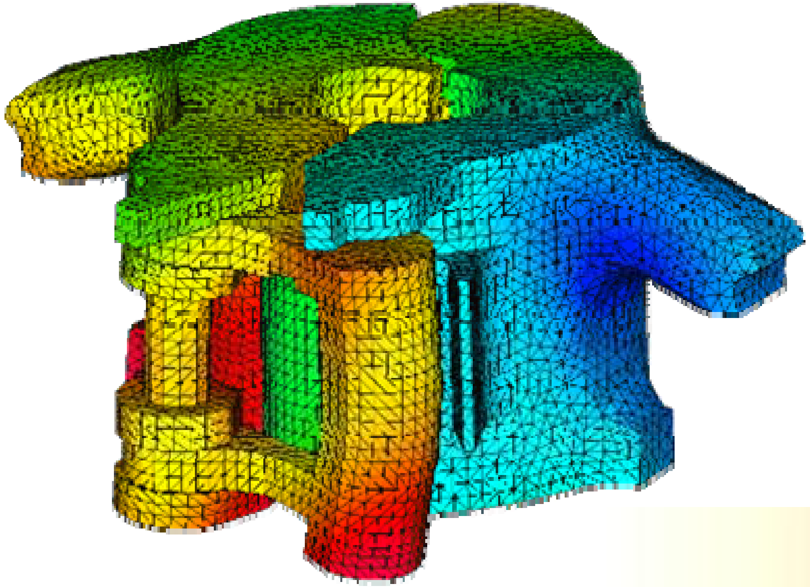
- Mechanical, Aerospace, Civil, Automobile Engineering
- Structure analysis (static, dynamic, linear, nonlinear)
- Thermal and fluid flows
- Electromagnetics
- Geomechanics
- Biomechanics
- ...

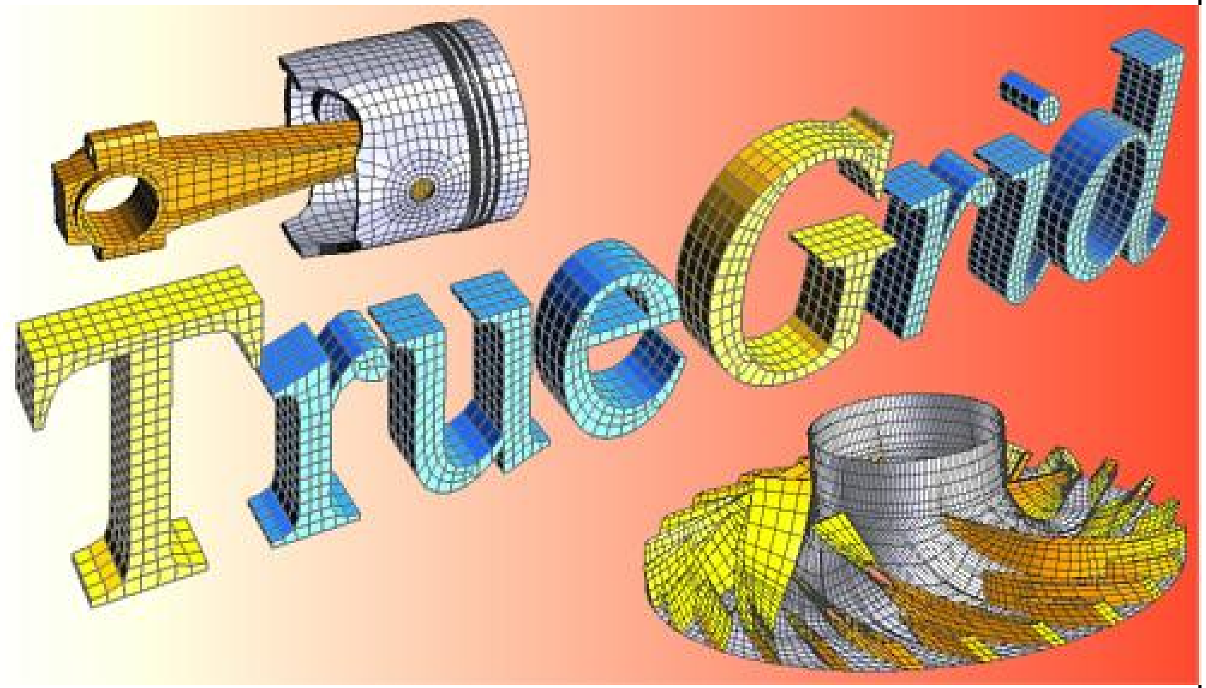
Finite Elements - Geophysics

- Crustal deformation
- Geophysical fluid mechanics
    - Geodynamics
    - Mantle Convection
- Electromagnetics
- Wave Propagation (FE and SE – spectral elements)
- Strong ground motion, earthquake engineering

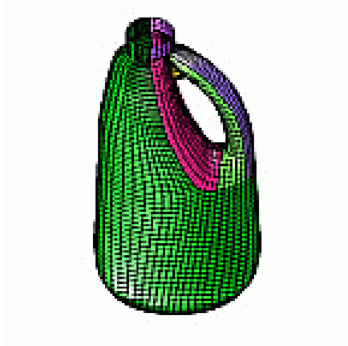
Finite Elements – History

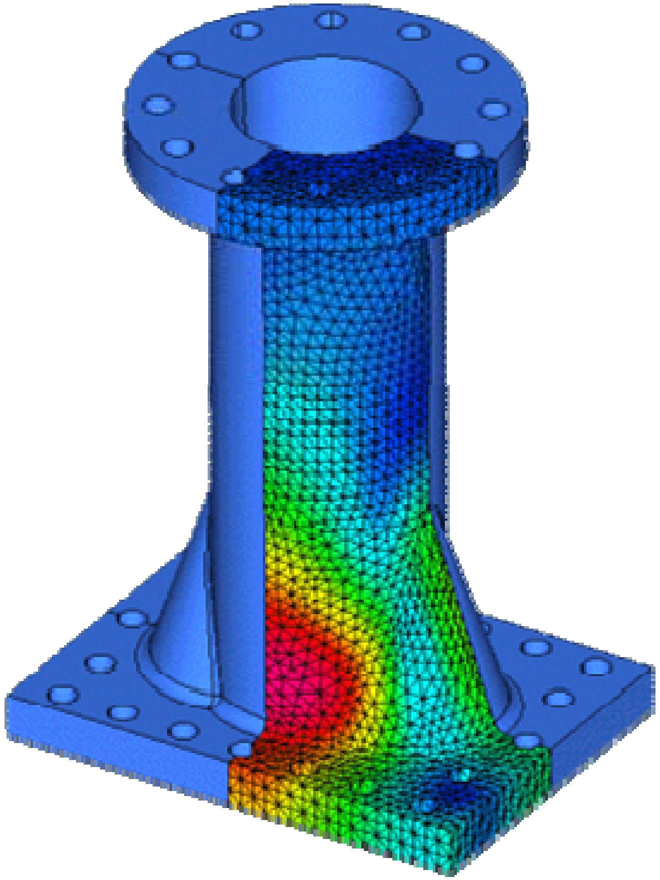
- 1943 – Courant (Variational Methods)
- 1956 – Turner, Clough, Martin, Topp (Stiffness)
- 1960 – Clough ("Finite Elements", plane problems)
- 1970s – applications on mainfraim computers
- 1980s – pre- and postprocessing on microcomputers
- 1990s - today applications to large structural systems

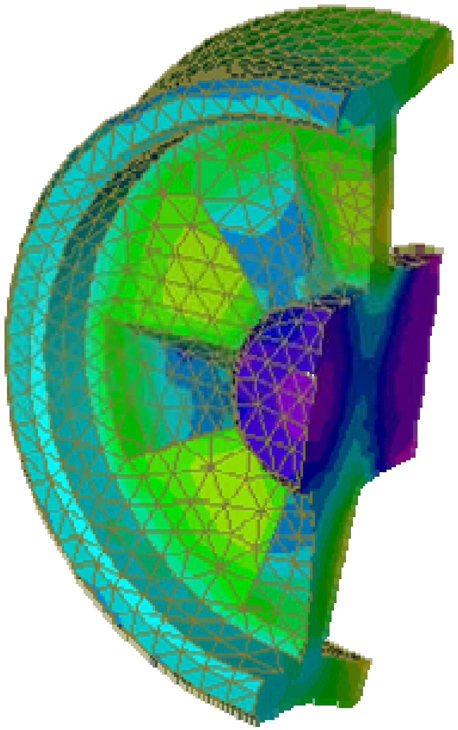


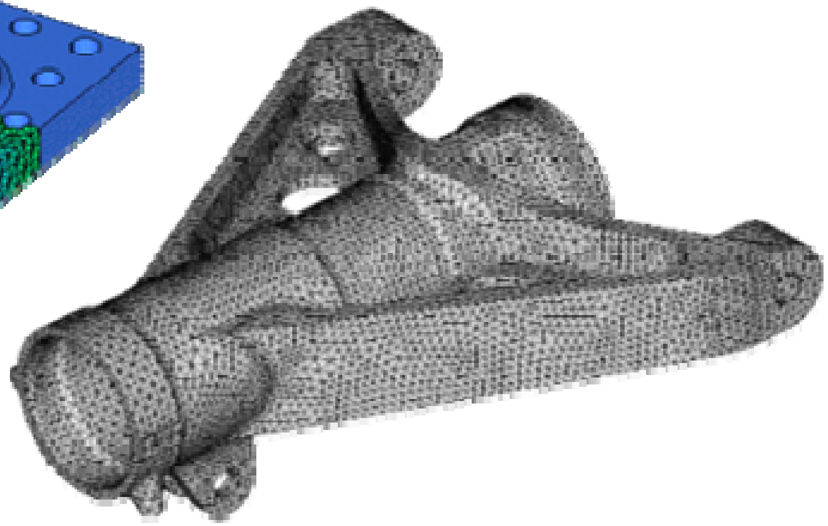

TrueGrid

Finite Elements - Examples

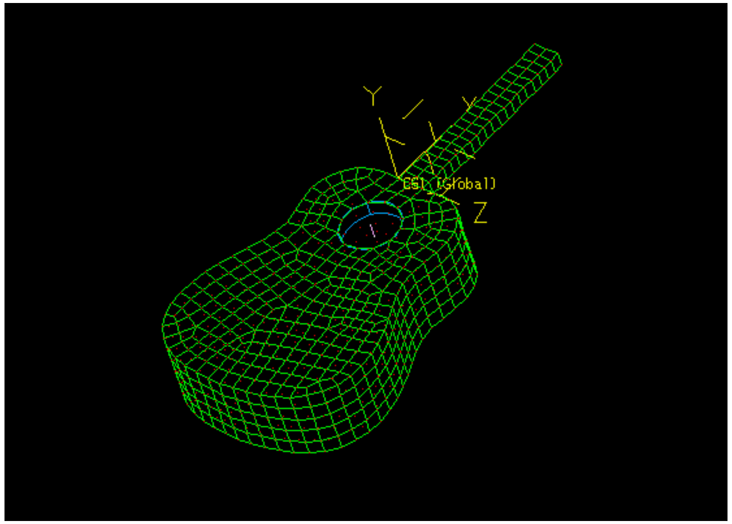


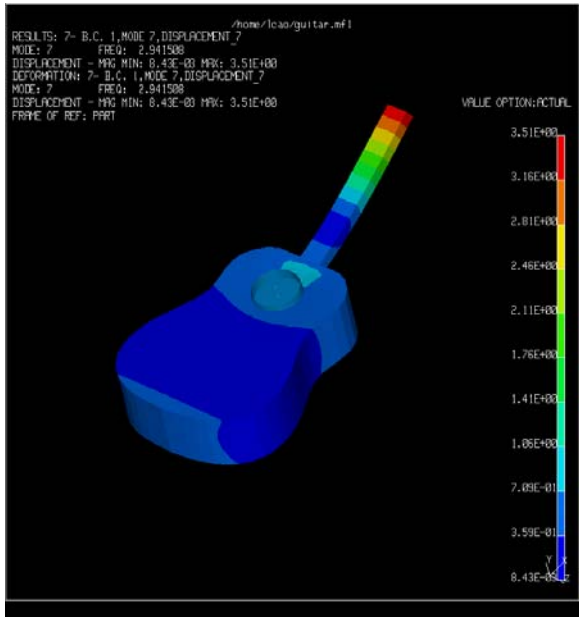


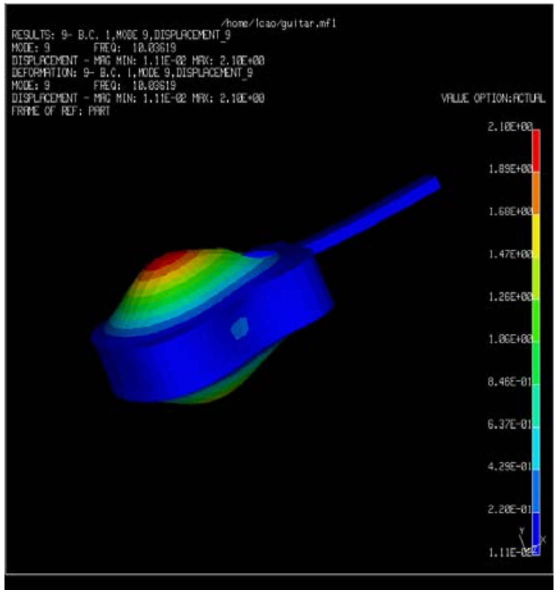


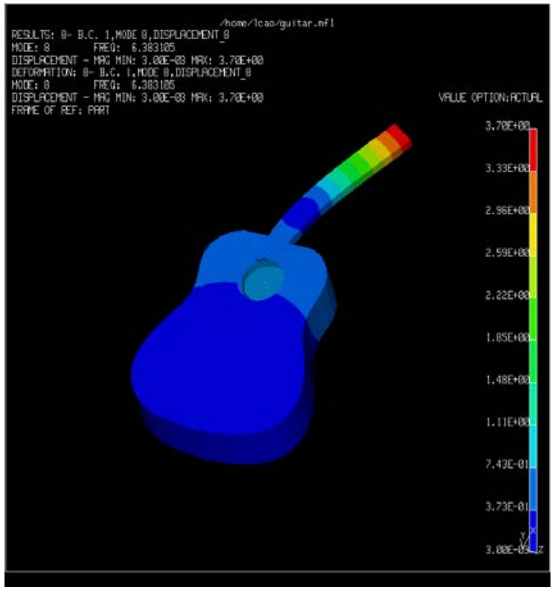


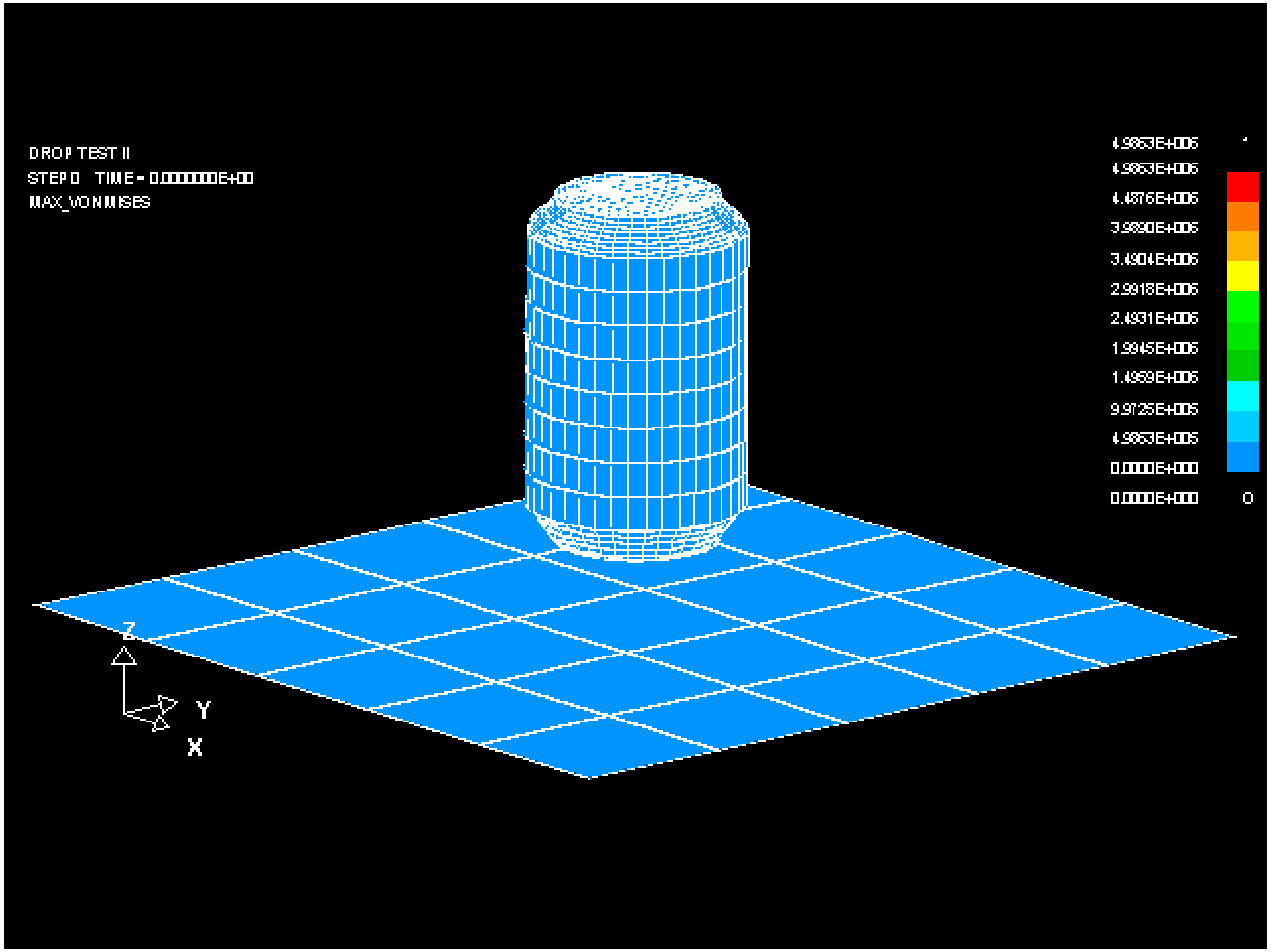
Finite Elements - Examples










DROP TEST II
STEP 0 TIME
MAX_VON MISES
4.9863E+006
4.9863E+006
4.4876E+006
3.9890E+006
3.4904E+006
2.9918E+006
2.4931E+006
1.9945E+006
1.4959E+006
9.9725E+005
4.9863E+005
0.0000E+000
0.0000E+000
0
Z
Y
X

Linear system of algebraic equations

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n} = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n} = b_2$$

$$\ldots\ldots\ldots\ldots$$

$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn} = b_n$$

... where the x_1, x_2, ... , x_n are the unknowns ...
in matrix form

$$\mathbf{Ax} = \mathbf{b}$$

Ax = b

where

$$\mathbf{A} = \left[a_{ij}\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{11} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\mathbf{x} = \left\{x_i\right\} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$$

$$\mathbf{b} = \left\{b_i\right\} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix}$$

A is a *nxn* (square) matrix, and **x** and **b** are column vectors of dimension *n*

Row vectors

$$\mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

Column vectors

$$\mathbf{w} = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}$$

Matrix addition and subtraction

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \quad \text{with} \quad c_{ij} = a_{ij} + b_{ij}$$

$$\mathbf{D} = \mathbf{A} - \mathbf{B} \quad \text{with} \quad d_{ij} = a_{ij} - b_{ij}$$

Matrix multiplication

$$\mathbf{C} = \mathbf{AB} \quad \text{with} \quad c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

where **A** (size *lxm*) and **B** (size *mxn*) and *i=1,2,...,l* and *j=1,2,...,n*.

Note that in general $\mathbf{AB} \neq \mathbf{BA}$ but **(AB)C=A(BC)**

Matrix Algebra - Special

Transpose of a matrix

$$\mathbf{A} = \left[a_{ij}\right] \qquad \mathbf{A}^{\mathsf{T}} = \left[a_{ji}\right]$$

$$(\mathbf{AB})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$$

Symmetric matrix

$$\mathbf{A} = \mathbf{A}^{\mathsf{T}}$$

$$a_{ij} = a_{ji}$$

Identity matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

with **AI=A, Ix=x**

The determinant of a square matrix A is a scalar number denoted det A or |A|, for example

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

or

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

A square matrix is singular if det A=0. This usually indicates problems with the system (non-uniqueness, linear dependence, degeneracy ..)

Matrix Inversion

For a square and non-singular matrix **A** its inverse is defined such as

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

The cofactor matrix *C* of matrix *A* is given by

$$\mathbf{C}_{ij} = (-1)^{i+j} \mathrm{M}_{ij}$$

where M_{ij} is the determinant of the matrix obtained by eliminating the *i*-th row and the *j*-th column of **A**.
The inverse of **A** is then given by

$$\mathbf{A}^{-1} = \frac{1}{\det A} \mathbf{C}^T$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Matrix Algebra – Solution techniques

... the solution to a linear system of equations is given by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

The main task in solving a linear system of equations is finding the inverse of the coefficient matrix A.

Solution techniques are e.g.

Gauss elimination methods
Iterative methods

A square matrix is said to be positive definite if for any non-zero vector x

$$\mathbf{x}^T = \mathbf{A}\mathbf{x} > 0$$

... positive definite matrices are non-singular

Matrices – Differentiation and Integration

Let

$$\mathbf{A}(t) = \lfloor a_{ij}(t) \rfloor$$

Then the differentiation of this matrix w.r.t. time is

$$\frac{d}{dt}\mathbf{A}(t) = \left[\frac{da_{ij}(t)}{dt} \right]$$

Likewise integration is defined by

$$\int \mathbf{A}(t)dt = \left\lfloor \int a_{ij}(t)dt \right\rfloor$$

Finite elements - elastostatics

The finite element method was originally derived for static problems in elasticity. It is informative to follow this historic route to introduce the basic concepts of elements, stiffness matrix, etc.

To introduce this concept we only need:

Hooke's law: $F=Ds$

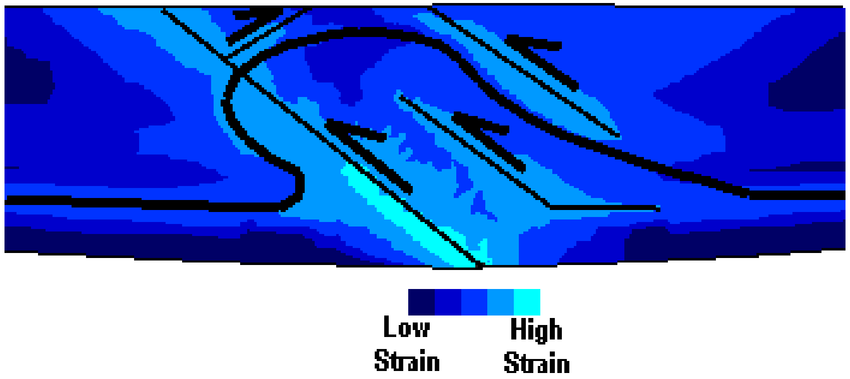
The principle of force balance $F_{tot}=F_1+F_2+F_3+F_4+$

The concept of work $W=\int F ds$

and strain energy $W=1/2\int \varepsilon_{ij}\, c_{ijkl}\, \varepsilon_{kl}$

… and a blackboard

Crustal Deformation



A piece of continental crust is forced to shorten as a velocity discontinuity is imposed along its base simulating subduction of the underlying mantle; in this example, the continental crust is assumed to be strongly rheologically layered leading to the formation of sub-horizontal decollements at mid-crustal levels and above the Moho. ...

From: J. Braun, Canberra

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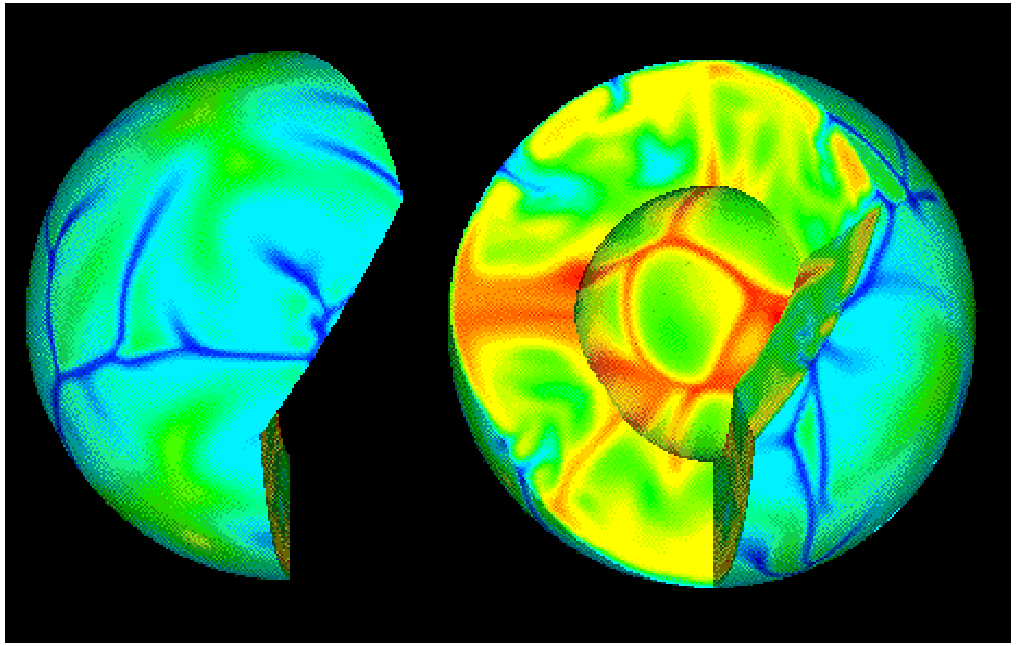
Mantle Convection



Thermal convection is modelled in 3-D a finite-element technique. The mesh has about 10 million grid points.
From Peter Bunge, Princeton.

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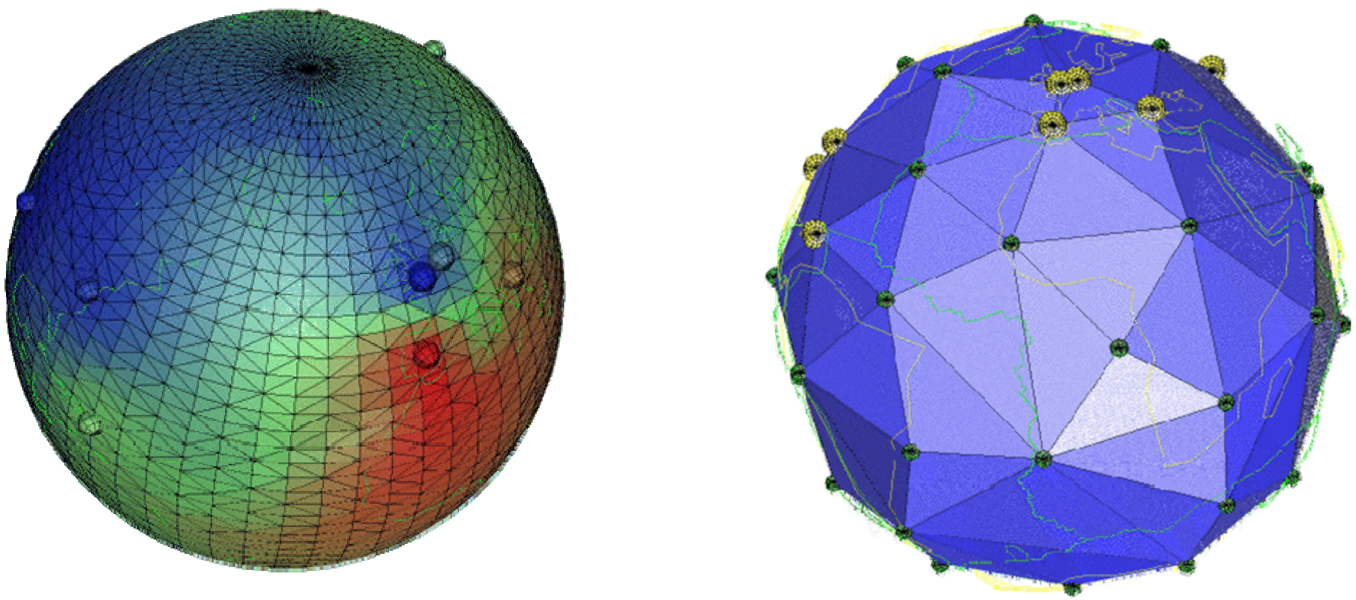

Gridding of a spherical surface used in FE modelling of the Earth's magnetic field. (From A. Schultz, Cambridge)

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