Computational Seismology: Wave equations

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Goals of this lecture

- Presenting the basic **wave equations** used in this course
- Demonstrating the differences between **scalar** and **vectorial** wave equations
- Showing **first order** and **second order** wave equations
- Presenting **analytical solutions** for acoustic and elastic wave equations
- Introducing some **fundamental solutions** of elastic waves in homogeneous media

Scalar wave equation: analytical solutions

Scalar wave equation

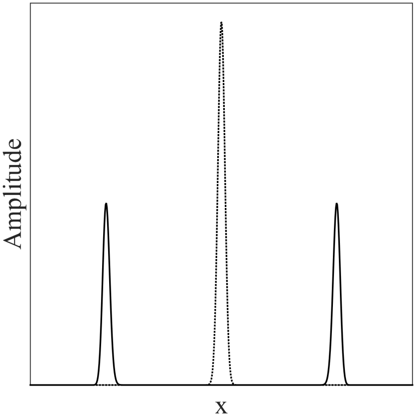
Acoustic wave equation

$$\partial_t^2 p \;=\; c^2 \Delta p + s$$

$p \rightarrow p(\mathbf{x}, t)$, pressure
$c \rightarrow c(\mathbf{x})$, velocity
$s \rightarrow s(\mathbf{x}, t)$, source term

Initial conditions

$$p(\mathbf{x}, t = 0) \;=\; p_0(\mathbf{x}, t)$$
$$\partial_t p(\mathbf{x}, t = 0) \;=\; 0$$



Snapshot of $p(\mathbf{x}, t)$ (solid line) after some time for initial condition $p_0(\mathbf{x}, t)$ (Gaussian, dashed line), 1D case.

Analytical solution: source-free case

Solution

$$p(x,t) \;=\; \frac{1}{2}p_0(ct-x) \;+\; \frac{1}{2}p_0(ct+x)$$

The solution shown in the movie has been obtained with a finite-difference approximation.

Acoustic wave equation: external source

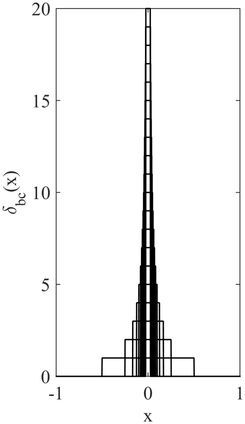
Green's Function G

$$\partial_t^2 G(\mathbf{x}, t; \mathbf{x}_0, t_0) - c^2 \Delta G(\mathbf{x}, t; \mathbf{x}_0, t_0) = \delta(\mathbf{x} - \mathbf{x}_0)\delta(t - t_0)$$

Delta function δ

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x) dx = 1 \, , \int_{-\infty}^{\infty} f(x)\delta(x) dx = f(0)$$



δ-generating function using boxcars.

Acoustic wave equation: analytical solutions

Green's functions for the inhomogeneous acoustic wave equation for all dimensions. $H(t)$ is the Heaviside function.

1D	2D	3D
$\frac{1}{2c}H(t-\frac{\lvert r\rvert}{c})$	$\frac{1}{2\pi c^2}\frac{H(t-\frac{\lvert r\rvert}{c})}{\sqrt{t^2-\frac{r^2}{c^2}}}$	$\frac{1}{4\pi c^2 r}\delta(t-r/c)$
$r=x$	$r=\sqrt{x^2+y^2}$	$r=\sqrt{x^2+y^2+z^2}$

Acoustic wave equation: analytical solutions