Some basic maths for seismic data processing and inverse problems

(Refreshement only!)

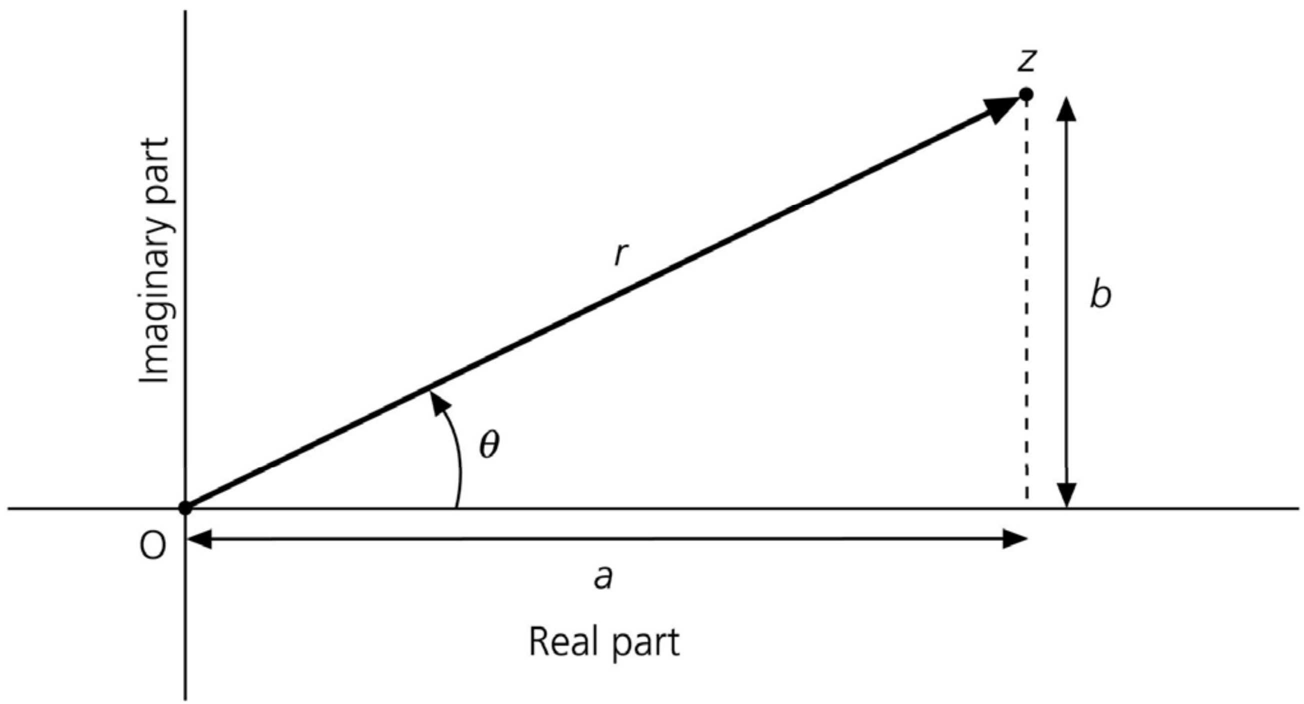
- Complex Numbers
- Vectors
  - Linear vector spaces
- Matrices
  - Determinants
  - Eigenvalue problems
  - Singular values
  - Matrix inversion

The idea is to illustrate these mathematical tools with examples from seismology

Complex numbers

$$z = a + ib = re^{i\phi} = r(\cos\phi + i\sin\phi)$$

Figure A.2-1: Representation of a complex number.



Complex numbers

conjugate, etc.

$$z^* = a - ib = r(\cos \phi - i \sin \phi)$$

$$= r \cos -\phi - ri \sin(-\phi) = r^{-i\phi}$$

$$\left| z^2 \right| = zz^* = (a + ib)(a - ib) = r^2$$

$$\cos \phi = (e^{i\phi} + e^{-i\phi}) / 2$$

$$\sin \phi = (e^{i\phi} - e^{-i\phi}) / 2i$$

Complex numbers

seismological application

Plane waves as superposition of harmonic signals using complex notation

$$\boxed{\begin{aligned} u_i(x_j, t) &= A_i \exp[ik(a_j x_j - ct)] \\ \mathbf{u}(\mathbf{x}, t) &= \mathbf{A} \exp[i\mathbf{kx} - \omega t] \end{aligned}}$$

Use this „Ansatz“ in the acoustic wave equation and interpret the consequences for wave propagation!

Vectors and Matrices

For discrete linear inverse problems we will need the concept of linear vector spaces. The generalization of the concept of size of a vector to matrices and function will be extremely useful for inverse problems.

Definition: Linear Vector Space. *A linear vector space over a field F of scalars is a set of elements V together with a function called addition from VxV into V and a function called scalar multiplication from FxV into V satisfying the following conditions for all $x,y,z \in V$ and all $\alpha,\beta \in F$*

1. $(x+y)+z = x+(y+z)$
2. $x+y = y+x$
3. There is an element 0 in V such that $x+0=x$ for all $x \in V$
4. For each $x \in V$ there is an element $-x \in V$ such that $x+(-x)=0$.
5. $\alpha(x+y)= \alpha x+ \alpha y$
6. $(\alpha + \beta)x= \alpha x+ \beta x$
7. $\alpha(\beta x)= \alpha\beta x$
8. $1x=x$

Matrix Algebra – Linear Systems

Linear system of algebraic equations

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\ldots\ldots\ldots\ldots$$

$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$$

... where the x_1, x_2, ... , x_n are the unknowns ...
in matrix form

$$\mathbf{Ax} = \mathbf{b}$$

Matrix Algebra – Linear Systems

$$\boxed{\mathbf{Ax} = \mathbf{b}}$$

where

$$\mathbf{A} = \left[\mathrm{a_{ij}}\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{11} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\mathbf{x} = \{x_i\} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$$

$$\mathbf{b} = \{b_i\} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix}$$

A is a *nxn* (square) matrix, and **x** and **b** are column vectors of dimension *n*

Matrix Algebra – Vectors

Row vectors

$$\mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

Column vectors

$$\mathbf{w} = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}$$

Matrix addition and subtraction

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \quad \text{with} \quad c_{ij} = a_{ij} + b_{ij}$$

$$\mathbf{D} = \mathbf{A} - \mathbf{B} \quad \text{with} \quad d_{ij} = a_{ij} - b_{ij}$$

Matrix multiplication

$$\mathbf{C} = \mathbf{AB} \quad \text{with} \quad c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

where **A** (size *lxm*) and **B** (size *mxn*) and *i=1,2,...,l* and *j=1,2,...,n*.
Note that in general **AB≠BA** but **(AB)C=A(BC)**

Matrix Algebra – Special

Transpose of a matrix

$$\mathbf{A} = [a_{ij}] \qquad \mathbf{A}^\mathbf{T} = [a_{ji}]$$

$$(\mathbf{AB})^\mathbf{T} = \mathbf{B}^\mathbf{T}\mathbf{A}^\mathbf{T}$$

Symmetric matrix

$$\mathbf{A} = \mathbf{A}^\mathbf{T}$$

$$a_{ij} = a_{ji}$$

Identity matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

with **AI=A, Ix=x**

Matrix Algebra – Orthogonal

Orthogonal matrices

a matrix is Q (nxn) is said to be orthogonal if

$$Q^T Q = I_n$$

... and each column is an orthonormal vector

$$q_i q_i = 1$$

... examples:

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

it is easy to show that :

$$Q^T Q = Q Q^T = I_n$$

if orthogonal matrices operate on vectors their size (the result of their inner product x.x) does not change -> Rotation

$$(Qx)^T (Qx) = x^T x$$

Matrix and Vector Norms

How can we compare the size of vectors, matrices (and functions!)? For scalars it is easy (absolute value). The generalization of this concept to vectors, matrices and functions is called a **norm**. Formally the norm is a function from the space of vectors into the space of scalars denoted by

$$\boxed{\|(.)\|}$$

with the following properties:

Definition: Norms.

1. $||v|| > 0$ for any $v \in 0$ and $||v|| = 0$ implies $v=0$
2. $||\alpha v||=|\alpha|\ ||v||$
3. $||u+v|| \leq ||v||+||u||$ *(Triangle inequality)*

We will only deal with the so-called l_p Norm.

The l_p-Norm

The l_p- Norm for a vector x is defined as ($p \geq 1$):

$$\boxed{\|x\|_{l_p} = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}}$$

Examples:

- for p=2 we have the ordinary euclidian norm: $\|x\|_{l_2} = \sqrt{x^T x}$

- for p= ∞ the definition is $\|x\|_{l_\infty} = \max_{1 \leq i \leq n} |x_i|$

- a norm for matrices is induced via $\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$

- for l_2 this means :
  $||A||_2$=maximum eigenvalue of $A^T A$

Matrix Algebra – Determinants

The determinant of a square matrix A is a scalar number denoted det (A) or |A|, for example

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

or

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

Matrix Algebra – Inversion

A square matrix is singular if det A=0. This usually indicates problems with the system (non-uniqueness, linear dependence, degeneracy ..)

Matrix Inversion

For a square and non-singular matrix ***A*** its inverse is defined such as

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

The cofactor matrix *C* of matrix *A* is given by

$$\mathbf{C}_{\mathbf{ij}} = (-1)^{i+j}\mathrm{M}_{\mathrm{ij}}$$

where M_{ij} is the determinant of the matrix obtained by eliminating the *i*-th row and the *j*-th column of ***A***.
The inverse of ***A*** is then given by

$$\mathbf{A}^{-1} = \frac{1}{\det A}\mathbf{C}^{T}$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Matrix Algebra – Solution techniques

... the solution to a linear system of equations is the given by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

The main task in solving a linear system of equations is finding the inverse of the coefficient matrix A.

Solution techniques are e.g.

Gauss elimination methods
Iterative methods

A square matrix is said to be positive definite if for any non-zero vector x

$$\mathbf{x}^{T} = \mathbf{A}\mathbf{x} > \mathbf{0}$$

... positive definite matrices are non-singular

Matrices –Systems of equations

Seismological applications

- Stress and strain tensors
- Tomographic forward and inverse problems
- Calculating interpolation or differential operators for finite-difference methods