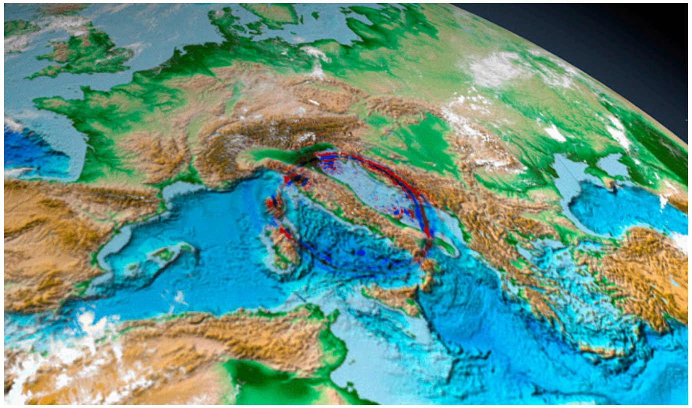
Seismic waves: A primer



- What are the governing equations for elastic wave propagation?
- What are the most fundamental results in simple media?
- How do we describe and input seismic sources (superposition principle)?
- What are consequences of the reciprocity principle?
- What rheologies do we need (stress-strain relation)?
- 3-D heterogeneities and scattering
- Green's functions, numerical solvers as linear systems

Goal: You know what to expect when running a wave simulation code!

Wave Equations

The elastic wave equation (strong form)

$$\rho \partial_t^2 u_i = \partial_j (\sigma_{ij} + M_{ij}) + f_i$$

$$\sigma_{ij} = c_{ijkl} \varepsilon_{kl}$$

$$\varepsilon_{kl} = \frac{1}{2} \left(\partial_k u_l + \partial_l u_k \right)$$

This is the displacement – stress formulation

The elastic wave equation – the cast

$$\rho \partial_t^2 u_i = \partial_j (\sigma_{ij} + M_{ij}) + f_i$$

$$\sigma_{ij} = c_{ijkl} \varepsilon_{kl}$$

$$\varepsilon_{kl} = \frac{1}{2} \left(\partial_k u_l + \partial_l u_k \right)$$

$\rho \rightarrow \rho(\mathbf{x})$	Mass density
$u_i \rightarrow u_i(\mathbf{x}, t)$	Displacement vector
$\sigma_{ij} \rightarrow \sigma_{ij}(\mathbf{x}, t)$	Stress tensor (3x3)
$M_{ij} \rightarrow M_{ij}(\mathbf{x}, t)$	Moment tensor (3x3)
$f_i \rightarrow f_i(\mathbf{x}, t)$	Volumetric force
$c_{ijkl} \rightarrow c_{ijkl}(\mathbf{x})$	Tensor of elastic constants (3x3x3x3)
$\varepsilon_{kl} \rightarrow \varepsilon_{kl}(\mathbf{x}, t)$	Strain tensor (3x3)

The elastic wave equation

$$\rho \partial_t v_i = \partial_j (\sigma_{ij} + M_{ij}) + f_i$$

$$\dot{\sigma}_{ij} = c_{ijkl} \dot{\varepsilon}_{kl}$$

$$\dot{\varepsilon}_{kl} = \frac{1}{2} \left(\partial_k v_l + \partial_l v_k \right)$$

$$v_i = \dot{u}_i = \partial_t u_i$$

This is the velocity – stress formulation.
This is a *coupled* formulation.

1D elastic wave equation

$$\rho \ddot{u} = \partial_x \mu \partial_x u + f$$

This is a scalar wave equation descriptive of transverse motions of a string

3D acoustic wave equation

$$\ddot{p} = c^2 \Delta p + s$$

This is the constant density acoustic wave equation (sound in a liquid or gas)

$$c \rightarrow c(x)$$

P-velocity

$$p \rightarrow p(x,t)$$

Pressure

$$s \rightarrow s(x,t)$$

Sources

.

$$\Delta \rightarrow \begin{pmatrix} \partial_x^2 \\ \partial_y^2 \\ \partial_z^2 \end{pmatrix}$$

Laplace Operator

This is equation is still tremendously important in exploration seismics!

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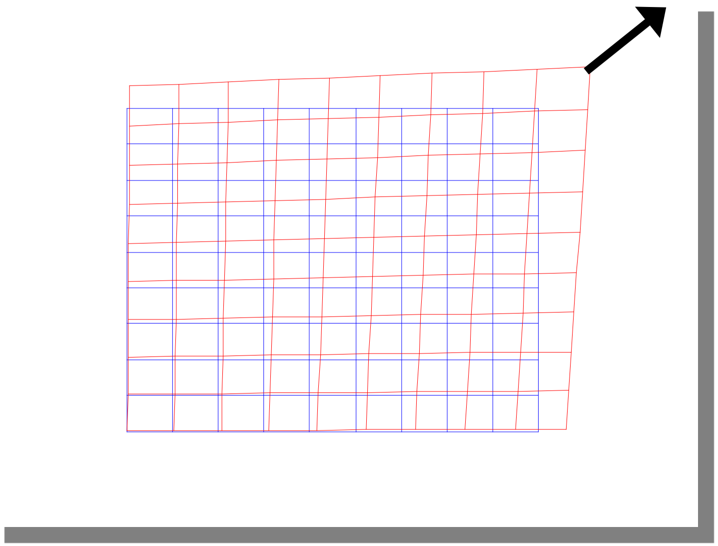
Rheologies

Stress and strain

To first order the Earth's crust deforms like an elastic body when the deformation (strain) is small.

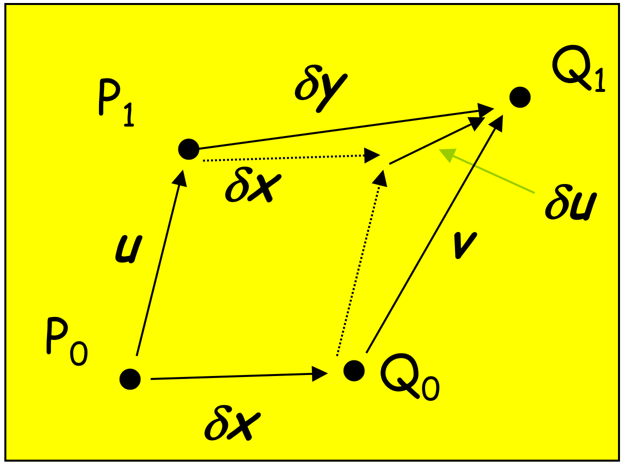
In other words, if the force that causes the deformation is stopped the rock will go back to its original form.



The change in shape (i.e., the deformation) is called strain, the forces that cause this strain are called stresses.

Linear Elasticity – symmetric part

The partial derivatives of the vector components



$$\frac{\partial u_i}{\partial x_k}$$

represent a second-rank tensor which can be resolved into a symmetric and anti-symmetric part:

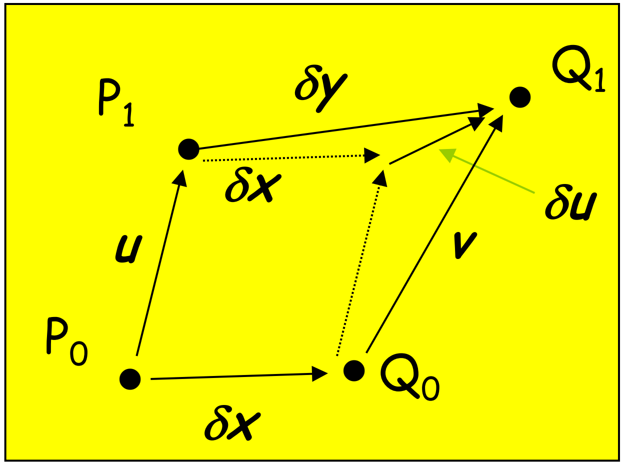
$$\delta u_i = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)\delta x_k - \frac{1}{2}\left(\frac{\partial u_k}{\partial x_i} - \frac{\partial u_i}{\partial x_k}\right)\delta x_k$$

- symmetric
- deformation

- antisymmetric
- pure rotation

The symmetric part is called the deformation tensor

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$



and describes the relation between deformation and displacement in linear elasticity. In 2-D this tensor looks like

$$\varepsilon_{ij} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{\partial u_y}{\partial y} \end{bmatrix}$$

Can strain be directly measured?

Stress tensor

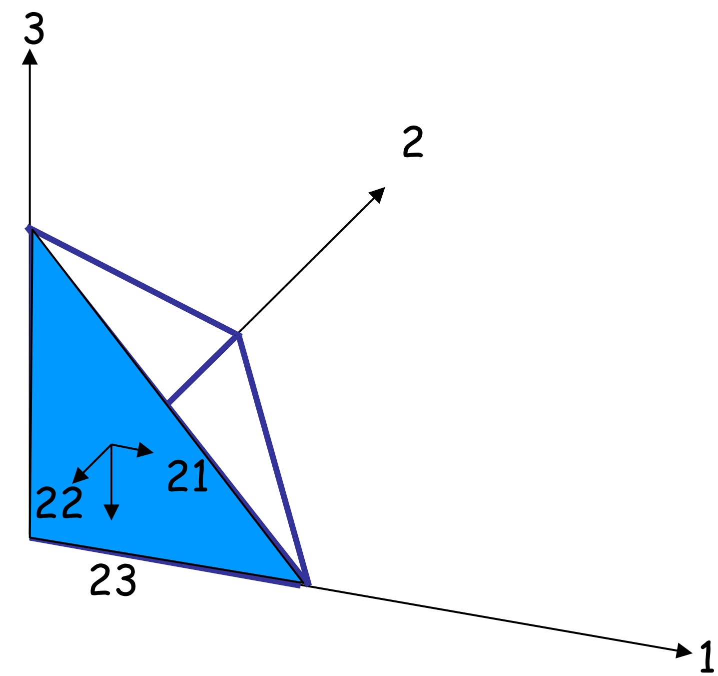
... in components we can write this as

$$t_i = \sigma_{ij} n_j$$

where σ_{ij} ist the stress tensor and n_j is a surface normal.
The stress tensor describes the forces acting on planes within a body. Due to the symmetry condition

$$\sigma_{ij} = \sigma_{ji}$$

there are only six independent elements.



σ_{ij}

- i: The vector normal to the corresponding surface
- j: The direction of the force vector acting on that surface

Stress - Glossary

Stress units	bars (10^6dyn/cm^2), 1N=10^5 dyn (cm g/s^2) 10^6Pa=1MPa=10bars 1 Pa=1 N/m^2 At sea level p=1bar At depth 3km p=1kbar
maximum compressive stress	the direction perpendicular to the minimum compressive stress, near the surface mostly in horizontal direction, linked to tectonic processes.
principle stress axes	the direction of the eigenvectors of the stress tensor

Can stress be directly measured?

Other rheologies (not further explored in this course)

Viscoelasticity
- the loss of energy due to internal friction
- possibly frequency-dependent
- different for P and S waves (why?)
- described by Q
- Not easy to implement numerically for time-domain methods

Porosity
- Effects of pore space (empty, filled, partially filled) on stress-strain
- Frequency-dependent effects
- Additional wave types (slow P wave)
- Highly relevant for reservoir wave propagation

Plasticity
- permanent deformation due to changes in the material as a function of deformation or stress
- resulting from (micro-) damage to the rock mass
- often caused by damage on a crystallographic scale
- important close to the earthquake source
- not well constrained by observations

Stress-strain relation

The relation between stress and strain in general is described by the tensor of elastic constants c_{ijkl}

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}$$

Generalised Hooke's Law

From the symmetry of the stress and strain tensor and a thermodynamic condition if follows that the maximum number if independent constants of c_{ijkl} is 21. In an isotropic body, where the properties do not depend on direction the relation reduces to

$$\sigma_{ij} = \lambda\Theta\delta_{ij} + 2\mu\varepsilon_{ij}$$

Hooke's Law

where l and m are the Lame parameters, q is the dilatation and d_{ij} is the Kronecker delta.

$$\Theta\delta_{ij} = \varepsilon_{kk}\delta_{ij} = \left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right)\delta_{ij}$$

Seismic Waves

Consequences of the equations of motion

$$\rho \ddot{u}_i = f_i + \partial_j \sigma_{ij}$$

What are the solutions to this equation? At first we look at infinite homogeneous isotropic media, then:

$$\sigma_{ij} = \lambda \theta \delta_{ij} + 2\mu \varepsilon_{ij}$$

$$\sigma_{ij} = \lambda \partial_k u_k \delta_{ij} + \mu(\partial_i u_j + \partial_j u_i)$$

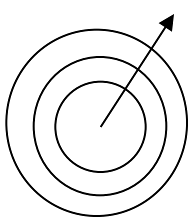
$$\rho \partial_t^2 u_i = f_i + \partial_j \left(\lambda \partial_k u_k \delta_{ij} + \mu(\partial_i u_j + \partial_j u_i) \right)$$

$$\rho \partial_t^2 u_i = f_i + \lambda \partial_i \partial_k u_k + \mu \partial_i \partial_j u_j + \mu \partial_j^2 u_i$$

Spherical Waves

$$\ddot{p} = c^2 \Delta p$$

Let us assume that h is a function of the distance from the source



$$\Delta p = \partial_r^2 p + \frac{2}{r}\partial_r p = \frac{1}{c^2}\ddot{p}$$

where we used the definition of the Laplace operator in spherical coordinates
let us define

$$p = \frac{\overline{p}}{r}$$

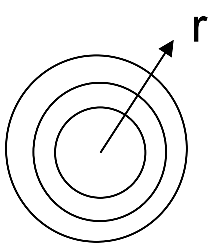
to obtain

with the known solution $$\overline{p} = f(r - \alpha t)$$

Geometrical spreading

so a disturbance propagating away with spherical wavefronts decays like

$$p = \frac{1}{r} f(r - \alpha t) \qquad p \approx \frac{1}{r}$$



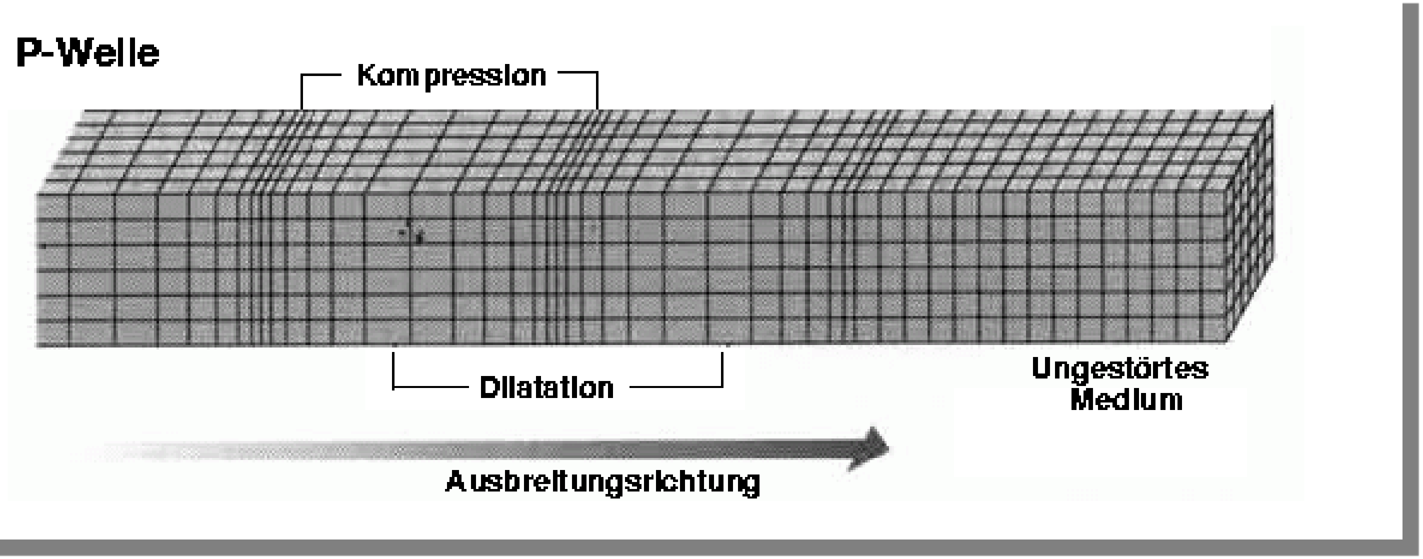
... this is the geometrical spreading for spherical waves, the amplitude decays proportional to 1/r.

If we had looked at cylindrical waves the result would have been that the waves decay as (e.g. surface waves)

$$p \approx \frac{1}{\sqrt{r}}$$

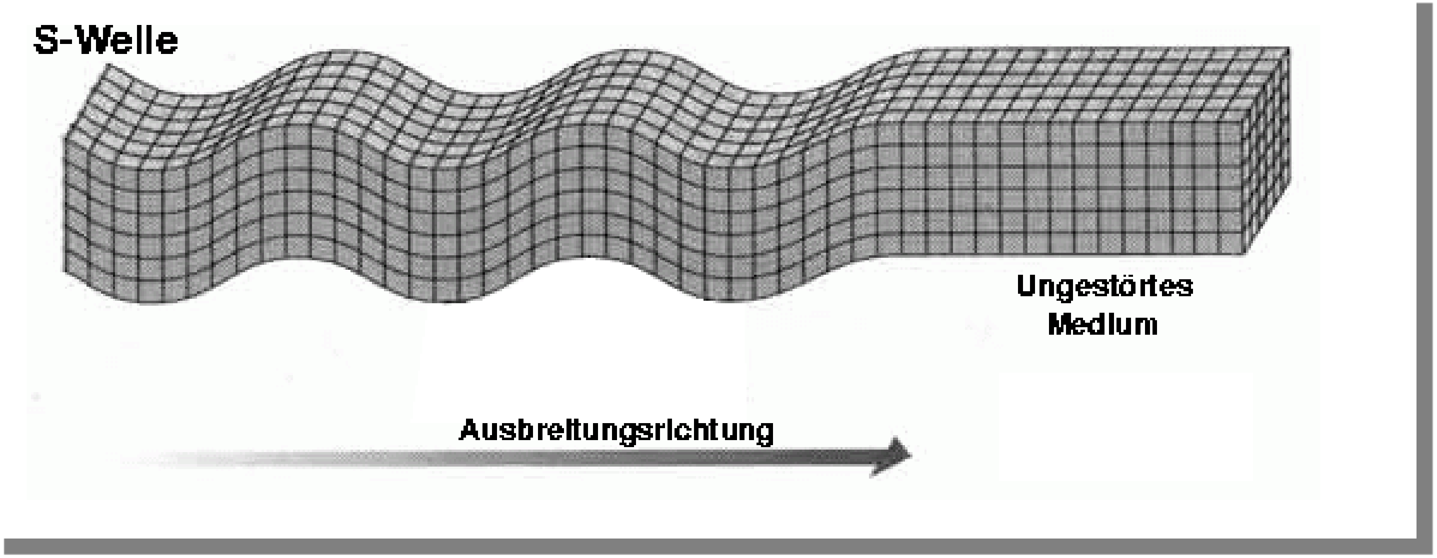
Seismic wave types
P - waves

P – primary waves – compressional waves – longitudinal waves



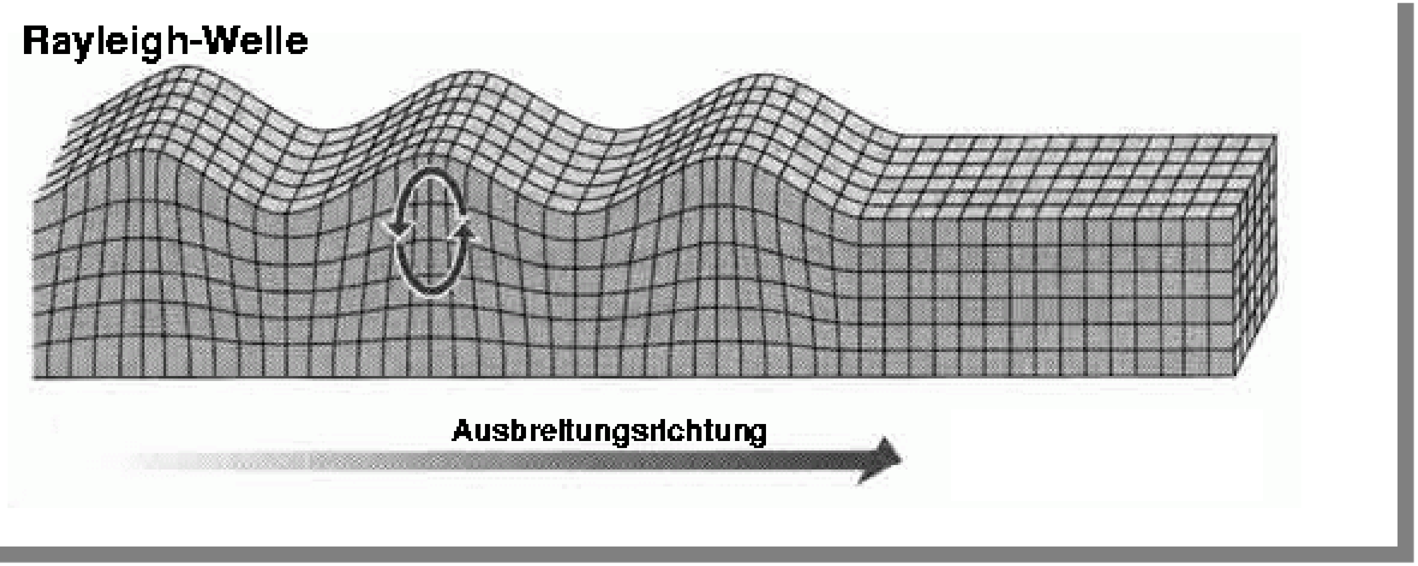
Seismic wave types
S - waves

S – waves – secondary waves – shear waves – transverse waves



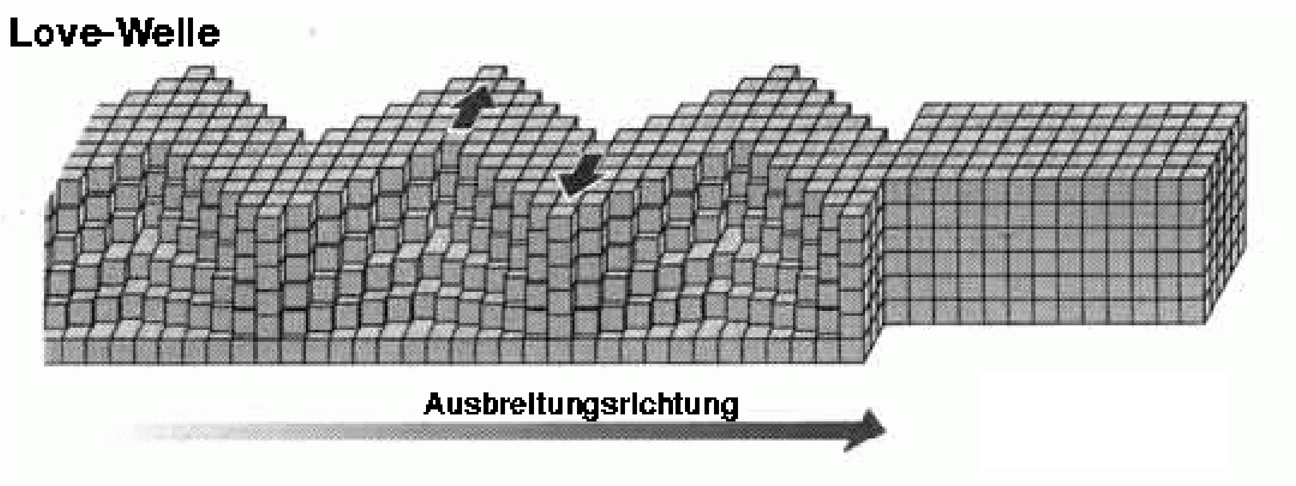
Rayleigh waves

Rayleigh waves – polarized in the plane through source and receiver – superposition of P and SV waves



Seismic wave types
Love waves

Love waves – transversely polarized – superposition of SH waves in layered media



Seismic wave velocities strongly depend on

- rock type (sediment, igneous, metamorphic, volcanic)
- porosity
- pressure and temperature
- pore space content (gas, liquid)

$$v = \sqrt{\frac{ElasticModuli}{Density}}$$

P-waves

$$v_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

approximately

$$v_p = \sqrt{3} v_s$$

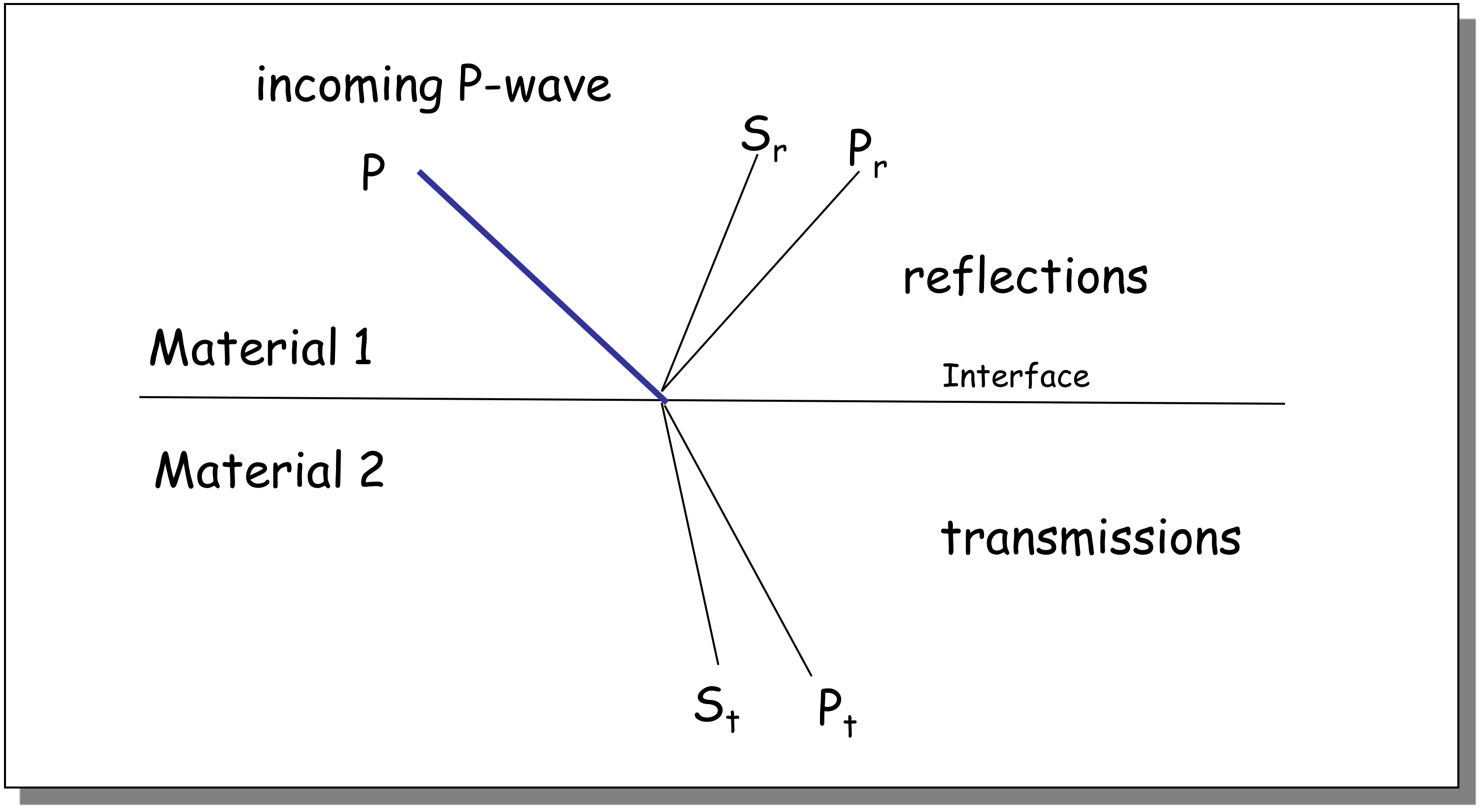
S-waves

$$v_s = \sqrt{\frac{\mu}{\rho}}$$

Reflection, Transmission

Reflection and transmission at boundaries
oblique incidence - conversion

P waves can be converted to S waves and vice versa. This creates a quite complex behavior of wave amplitudes and wave forms at interfaces. This behavior can be used to constrain the properties of the material interface.

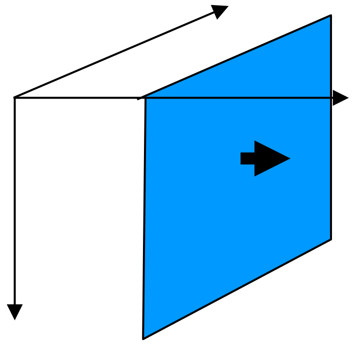


Boundary conditions: internal interfaces

Boundary conditions: free surface

Rayleigh wave displacement

Displacement in the x-z plane for a plane harmonic surface wave propagating along direction x



$$u_x = C(e^{-0.8475kz} - 0.5773e^{-0.3933kz})\sin k(ct - x)$$

$$u_z = C(-0.8475e^{-0.8475kz} + 1.4679e^{-0.3933kz})\cos k(ct - x)$$

This development was first made by Lord Rayleigh in 1885. It demonstrates that YES there are solutions to the wave equation propagating along a **free surface!**

Some remarkable facts can be drawn from this particular form:

Lamb's Problem

-the two components are out of phase by p

- for small values of z a particle describes an ellipse and the motion is retrograde

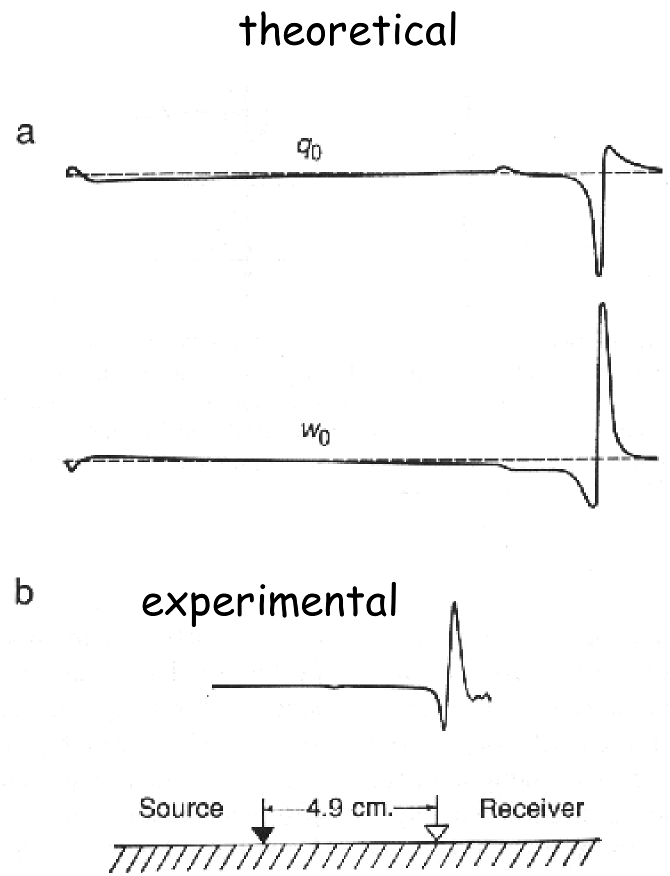
- at some depth z the motion is linear in z

- below that depth the motion is again elliptical but prograde

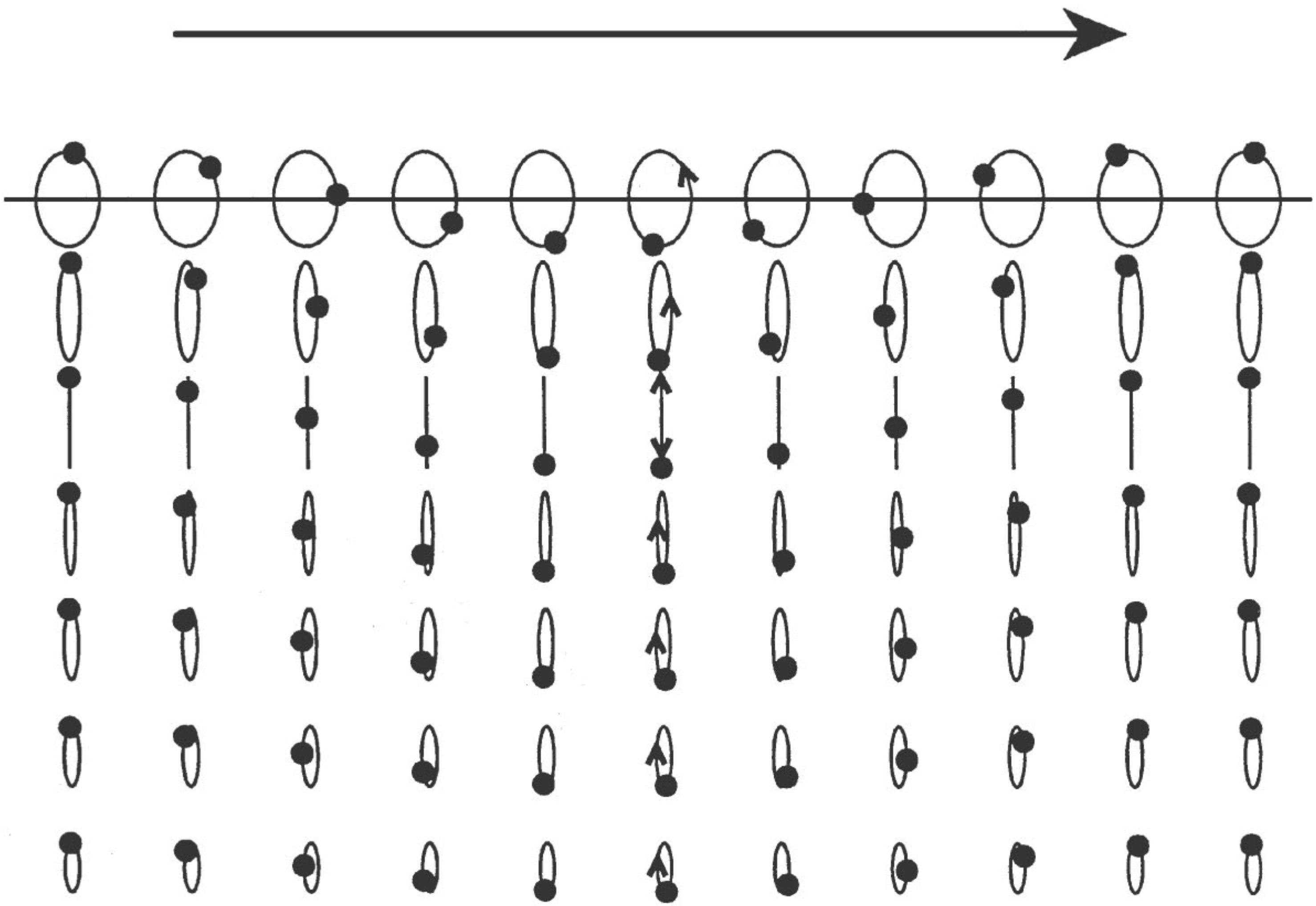
- the phase velocity is independent of k: **there is no dispersion** for a homogeneous half space

- the problem of a vertical point force at the surface of a half space is called **Lamb's problem** (after Horace Lamb, 1904).

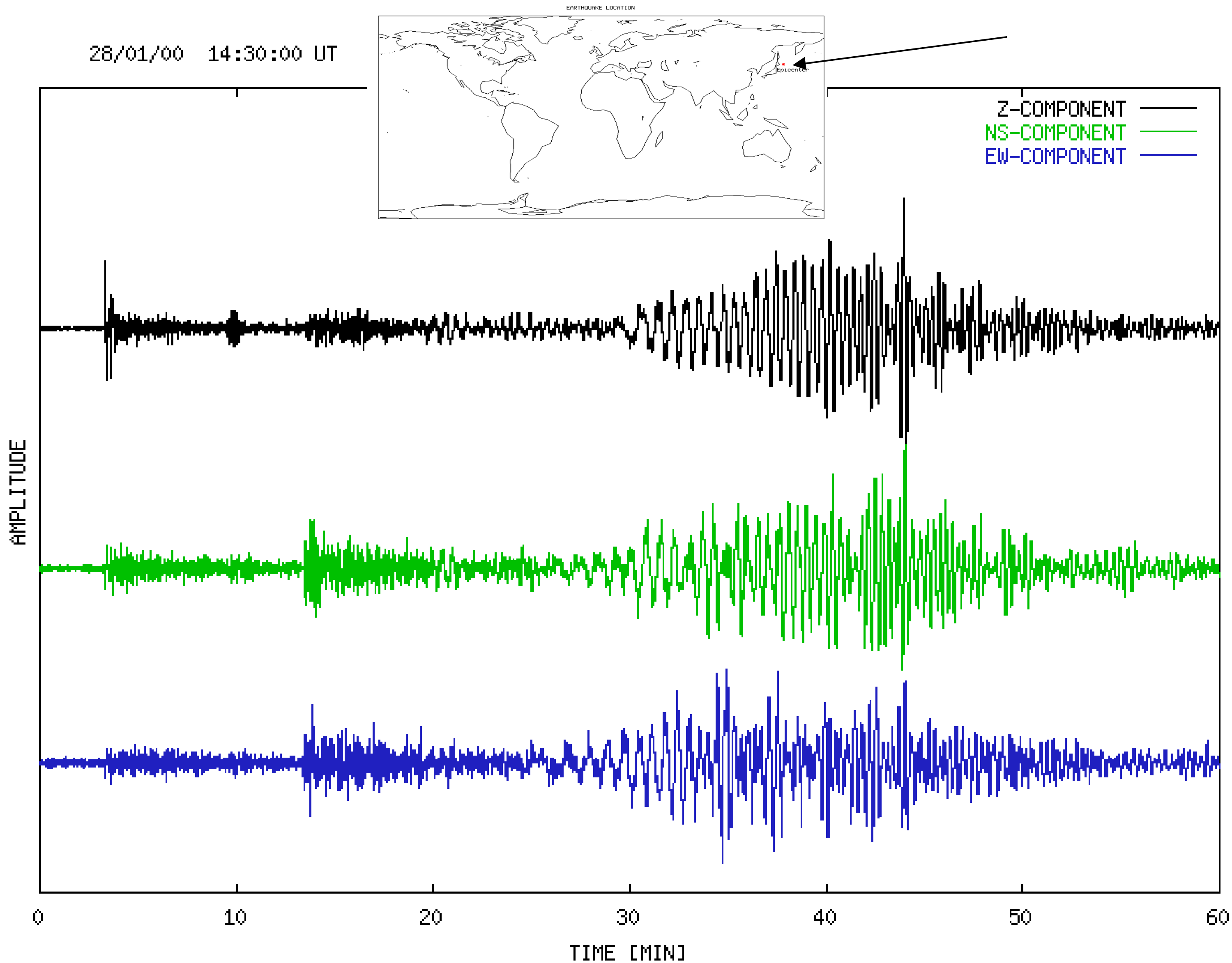
- Right Figure: radial and vertical motion for a source at the surface



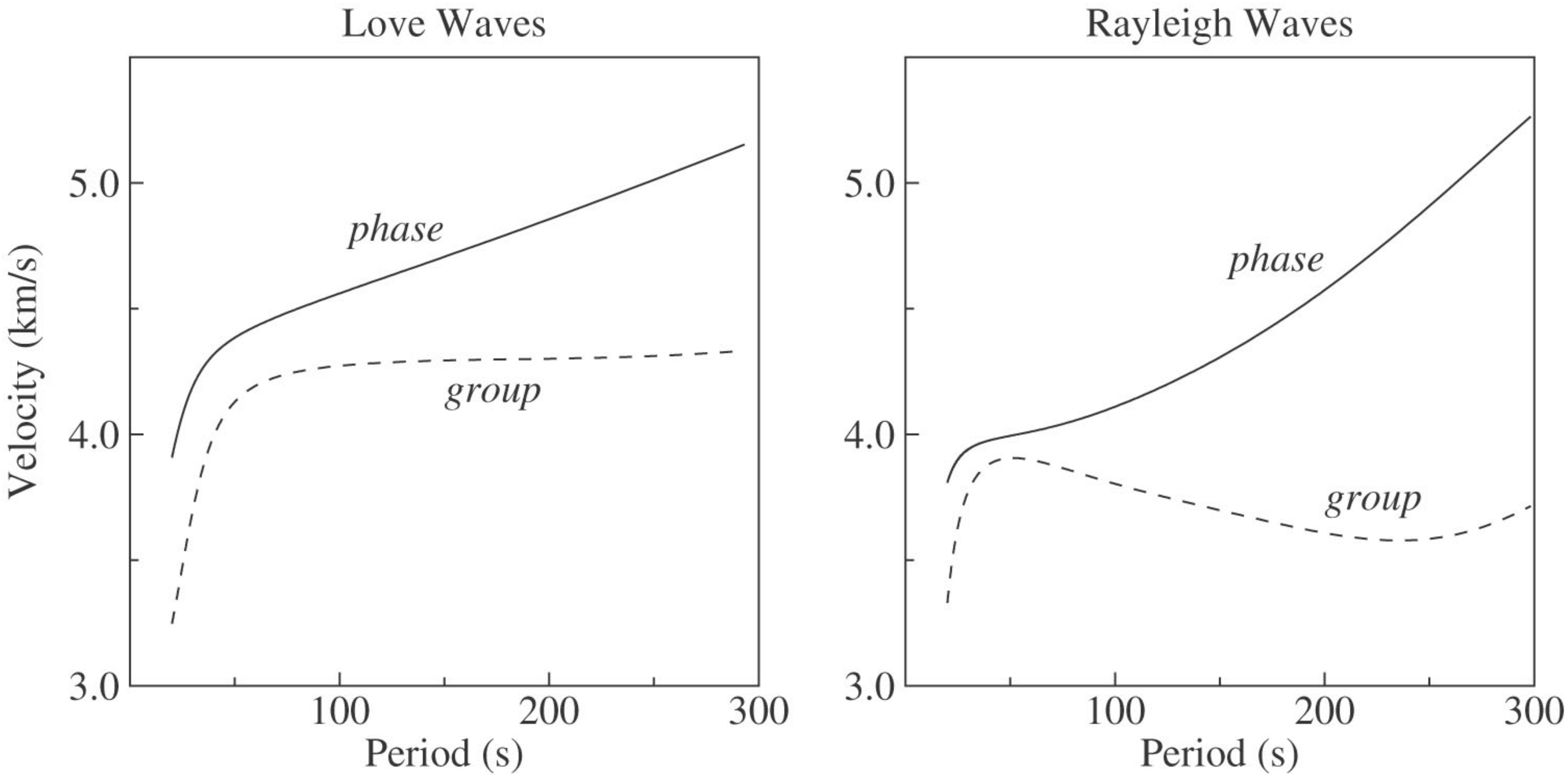
Particle Motion Rayleigh waves



Data Example



Surface wave dispersion



Surface waves summary

- Elastic surface waves (Love and Rayleigh) in nature generally show dispersive behavior (later we will see that there is also dispersive behaviour due to numerical effects!)

- Surface waves are a consequence of the free-surface boundary condition. We thus might expect that – when using numerical approximations there might be differences concerning the accurate implementation of this boundary condition.

- The accurate simulation of surface waves plays a dominant role in global and regional (continental scale) seismology and is usually not so important in exploration geophysics.

Seismic sources

Radiation from a point double-couple source

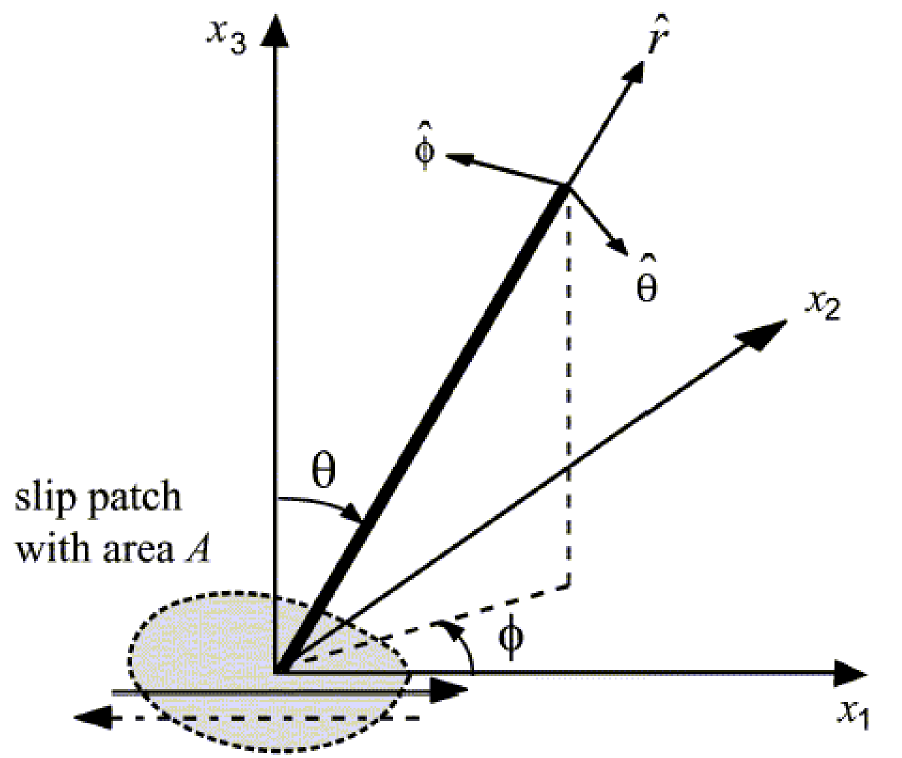


FIGURE 5 Cartesian and polar coordinate systems for analysis of radiation by a slip patch with area A and average slip $\langle \Delta u(t) \rangle$.

Geometry we use to express the seismic wavefield radiated by point double-couple source with area A and slip Du

Here the fault plane is the x_1x_2-plane and the slip is in x_1-direction. Which stress components are affected?

Radiation from a point source

$$
\begin{aligned}
\boldsymbol{u}(\boldsymbol{x}, t) = & \frac{1}{4\pi\rho} \boldsymbol{A}^N \frac{1}{r^4} \int_{r/v_P}^{r/v_S} \tau M_0(t-\tau)\, d\tau \\
& + \frac{1}{4\pi\rho v_P^2} \boldsymbol{A}^{IP} \frac{1}{r^2} M_0(t - r/v_P) \\
& + \frac{1}{4\pi\rho v_S^2} \boldsymbol{A}^{IS} \frac{1}{r^2} M_0(t - r/v_S) \\
& + \frac{1}{4\pi\rho v_P^3} \boldsymbol{A}^{FP} \frac{1}{r} \dot{M}_0(t - r/v_P) \\
& + \frac{1}{4\pi\rho v_S^3} \boldsymbol{A}^{FS} \frac{1}{r} \dot{M}_0(t - r/v_S).
\end{aligned}
$$

… one of the most important results of seismology!
… Let's have a closer look …

u	ground displacement as a function of space and time
r	density
r	distance from source
V_s	shear velocity
V_p	P-velocity
N	near field
IP/S	intermediate field
FP/S	far field
M_0	seismic moment

$$
\begin{aligned}
\boldsymbol{A}^N &= 9 \sin 2\theta \cos\phi \hat{\boldsymbol{r}} - 6(\cos 2\theta \cos\phi \hat{\boldsymbol{\theta}} - \cos\theta \sin\phi \hat{\boldsymbol{\phi}}), \\
\boldsymbol{A}^{IP} &= 4 \sin 2\theta \cos\phi \hat{\boldsymbol{r}} - 2(\cos 2\theta \cos\phi \hat{\boldsymbol{\theta}} - \cos\theta \sin\phi \hat{\boldsymbol{\phi}}), \\
\boldsymbol{A}^{IS} &= -3 \sin 2\theta \cos\phi \hat{\boldsymbol{r}} + 3(\cos 2\theta \cos\phi \hat{\boldsymbol{\theta}} - \cos\theta \sin\phi \hat{\boldsymbol{\phi}}), \\
\boldsymbol{A}^{FP} &= \sin 2\theta \cos\phi \hat{\boldsymbol{r}}, \\
\boldsymbol{A}^{FS} &= \cos 2\theta \cos\phi \hat{\boldsymbol{\theta}} - \cos\theta \sin\phi \hat{\boldsymbol{\phi}},
\end{aligned}
$$

Radiation from a point source

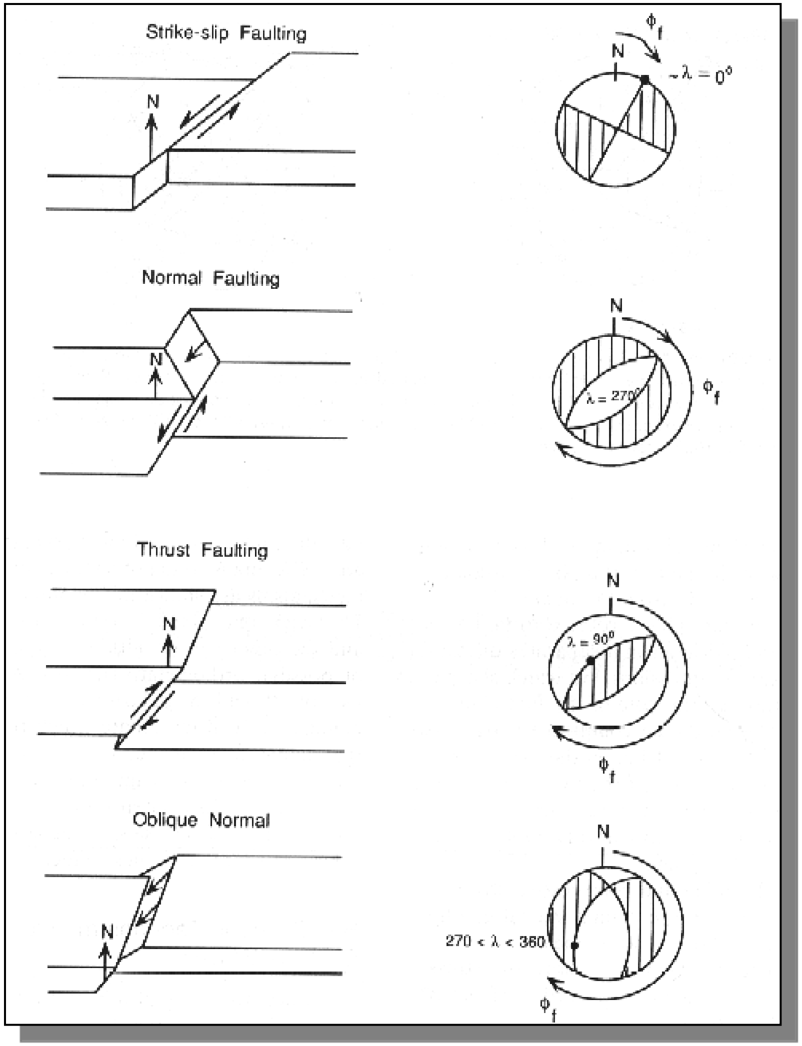
$$
\begin{aligned}
\boldsymbol{u}(\boldsymbol{x}, t) = & \frac{1}{4\pi\rho} \mathbf{A}^{N} \frac{1}{r^4} \int_{r/v_P}^{r/v_S} \tau M_0(t-\tau)\, d\tau \\
& + \frac{1}{4\pi\rho v_P^2} \mathbf{A}^{IP} \frac{1}{r^2} M_0(t - r/v_P) \\
& + \frac{1}{4\pi\rho v_S^2} \mathbf{A}^{IS} \frac{1}{r^2} M_0(t - r/v_S) \\
& + \frac{1}{4\pi\rho v_P^3} \mathbf{A}^{FP} \frac{1}{r} \dot{M}_0(t - r/v_P) \\
& + \frac{1}{4\pi\rho v_S^3} \mathbf{A}^{FS} \frac{1}{r} \dot{M}_0(t - r/v_S).
\end{aligned}
$$

Near field term contains the static deformation

Intermediate terms

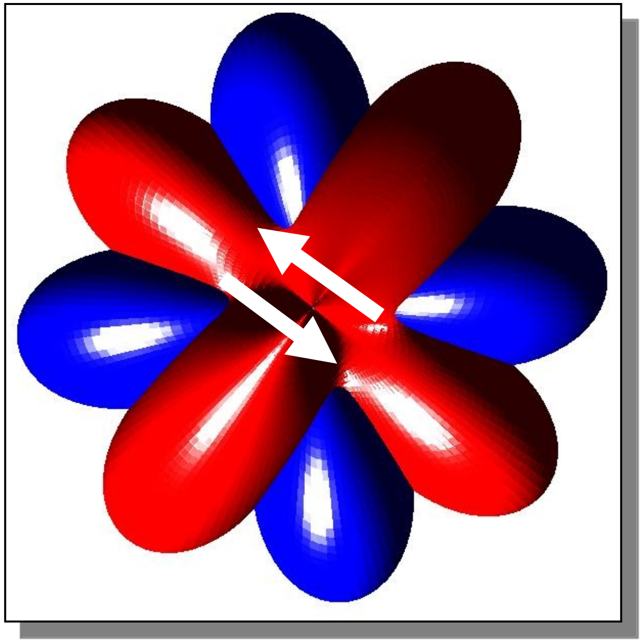
Far field terms: the main ingredient for source inversion, ray theory, etc.

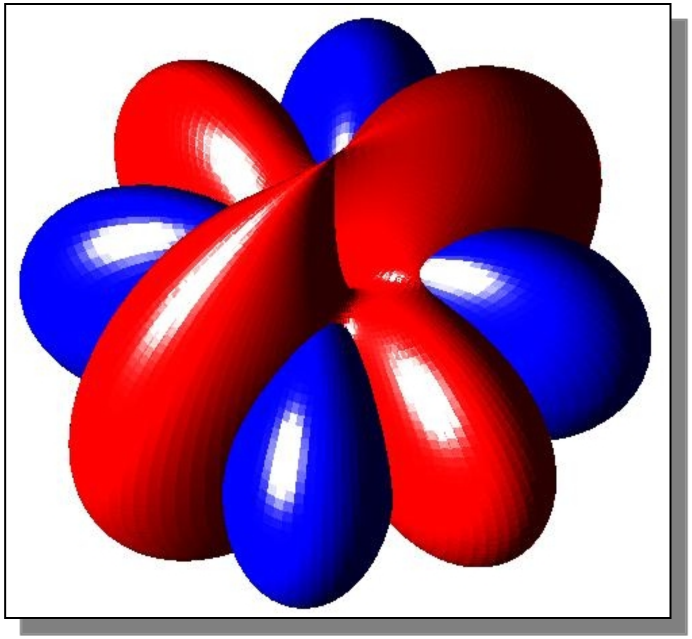
Source mechanisms



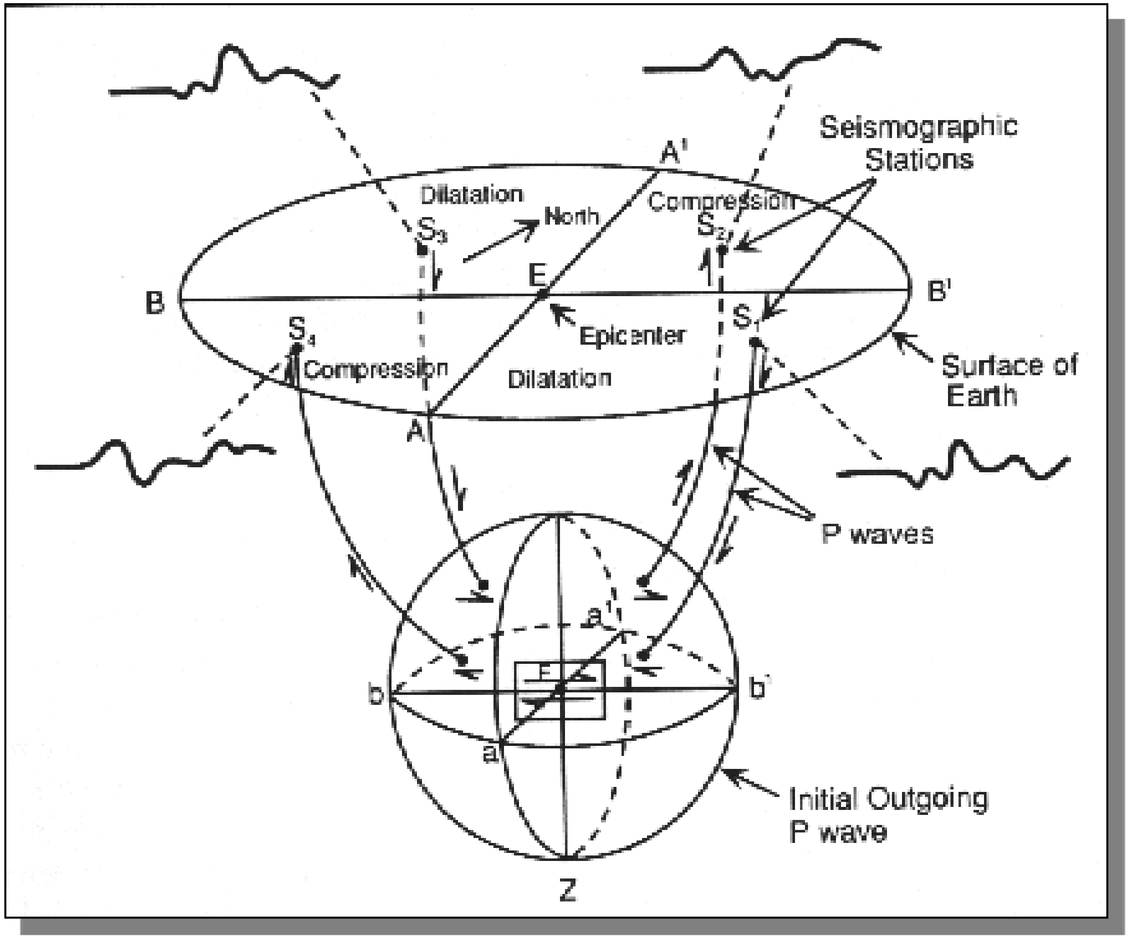
Basic fault types and their appearance in the focal mechanisms. Dark regions indicate compressional P-wave motion.

Far field P – blue
Far field S - red





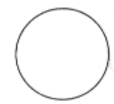
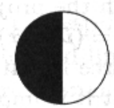
Radiation from shear dislocation



First motion of P waves at seismometers in various directions.

The polarities of the observed motion is used to determine the point source characteristics.

Beachballs and moment tensor

Moment Tensor	Beachball	Moment Tensor	Beachball
$\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$		$-\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	
$-\frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}$		$\frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}$	
$\frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}$		$\frac{1}{\sqrt{2}}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}$	
$\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$		$\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$	
$\frac{1}{\sqrt{6}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$		$\frac{1}{\sqrt{6}}\begin{pmatrix}1&0&0\\0&-2&0\\0&0&1\end{pmatrix}$	
$\frac{1}{\sqrt{6}}\begin{pmatrix}-2&0&0\\0&1&0\\0&0&1\end{pmatrix}$		$-\frac{1}{\sqrt{6}}\begin{pmatrix}-2&0&0\\0&1&0\\0&0&1\end{pmatrix}$	

explosion - implosion

vertical strike slip fault

vertical dip slip fault

45° dip thrust fault

compensated linear vector dipoles

Seismic moment M_0

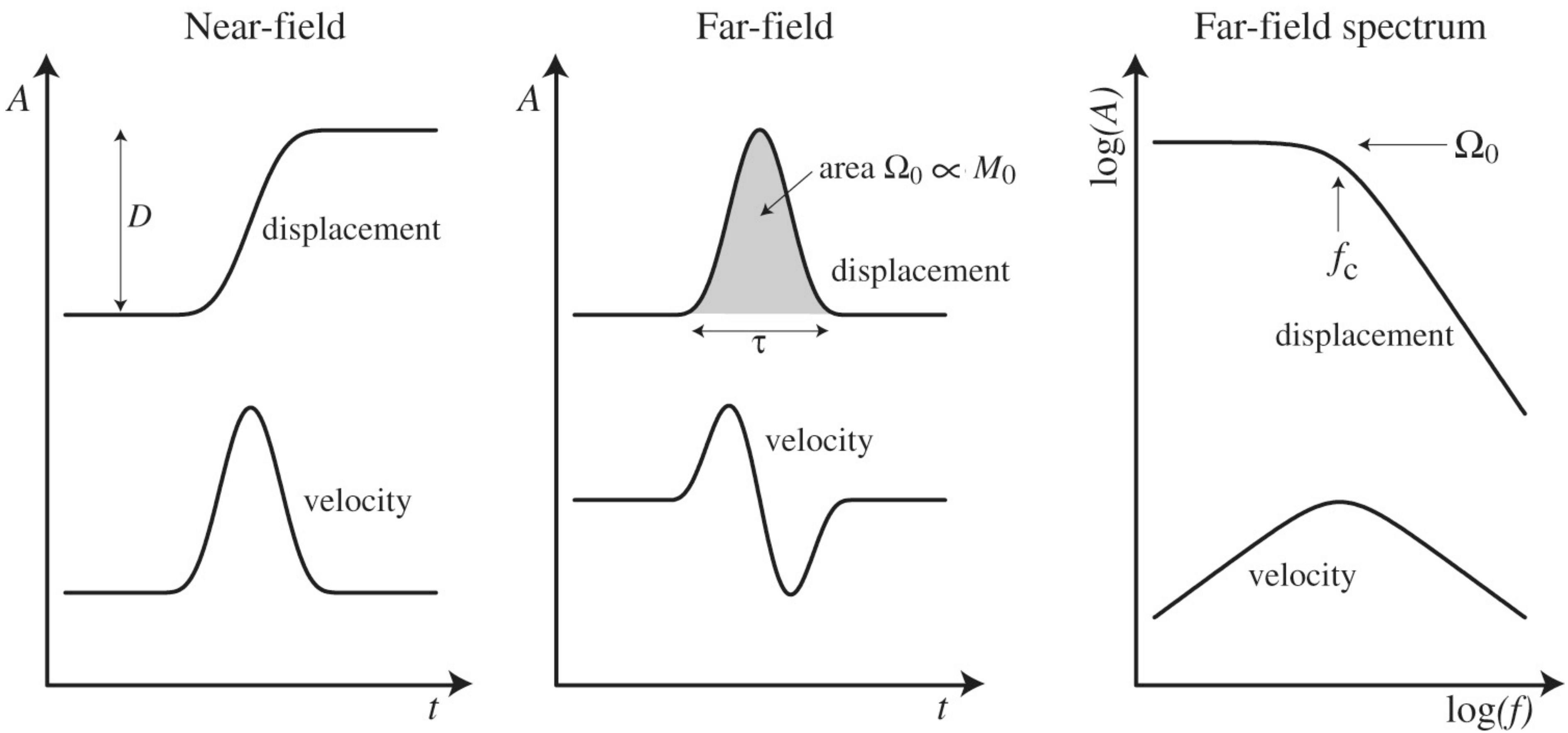
$$M_0 = \mu \langle \Delta u(t) \rangle A$$

M_0	seismic moment
m	rigidity
$<\Delta u(t)>$	average slip
A	fault area

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}, t) = & \frac{1}{4\pi\rho} \boldsymbol{A}^N \frac{1}{r^4} \int_{r/v_P}^{r/v_S} \tau M_0(t-\tau)\, d\tau \\ & + \frac{1}{4\pi\rho v_P^2} \boldsymbol{A}^{IP} \frac{1}{r^2} M_0(t - r/v_P) \\ & + \frac{1}{4\pi\rho v_S^2} \boldsymbol{A}^{IS} \frac{1}{r^2} M_0(t - r/v_S) \\ & + \frac{1}{4\pi\rho v_P^3} \boldsymbol{A}^{FP} \frac{1}{r} \dot{M}_0(t - r/v_P) \\ & + \frac{1}{4\pi\rho v_S^3} \boldsymbol{A}^{FS} \frac{1}{r} \dot{M}_0(t - r/v_S). \end{aligned}$$

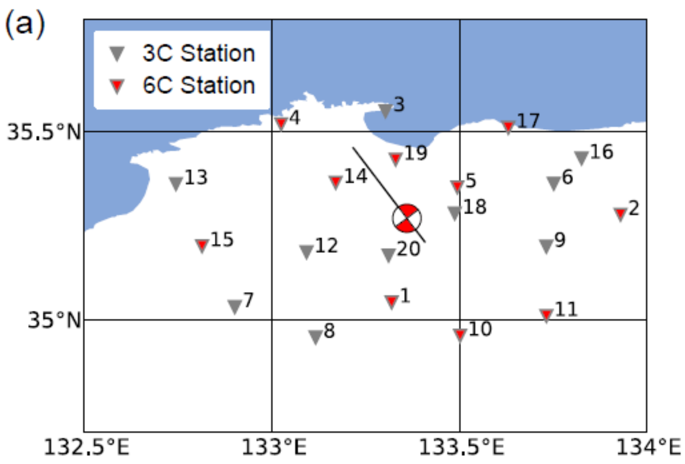
Note that the far-field displacement is proportional to the moment rate!

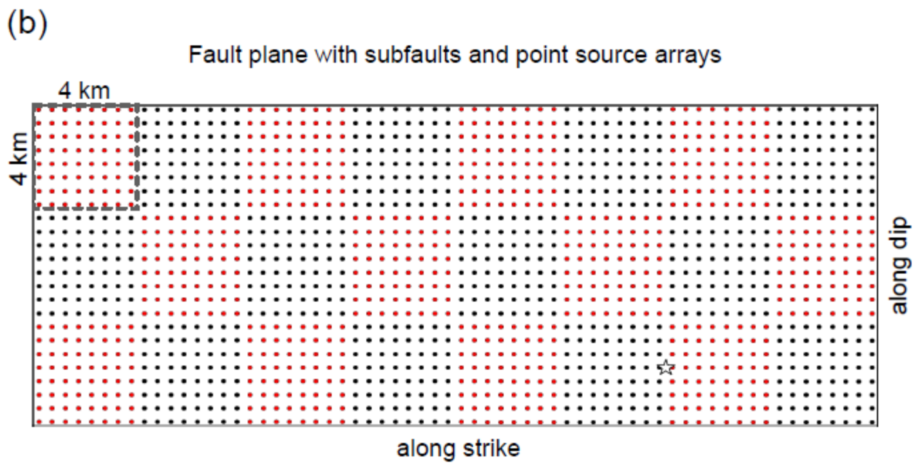
Source time function

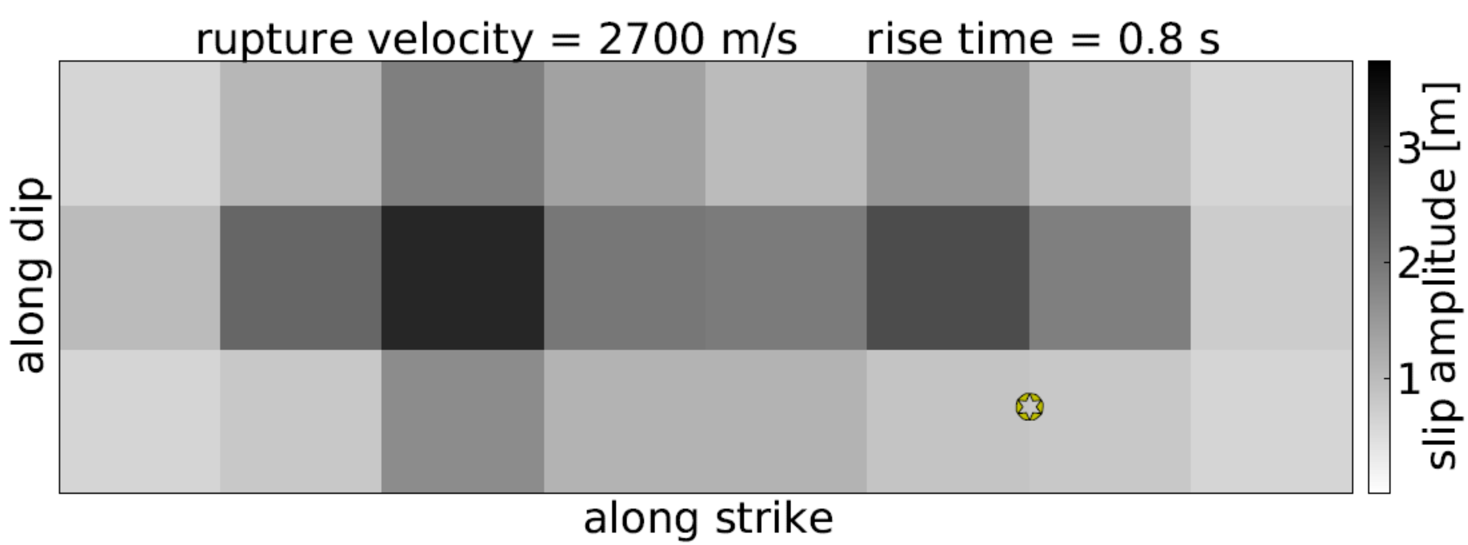


The superposition principle

Discrete representation of finite sources







Superposition principle

We allow each subfault to slip once and parameterize the slip process in terms of slip amplitude ($slip_k$), rupture velocity (c^{rup}) and rise time (R). The slip amplitude is heterogeneous across the fault plane, leading to 24 free parameters. Together with the distance between the center of subfault k and the hypocenter, the rupture velocity provides the rupture time $t_k(c^{rup})$ of subfault k. The rise time expresses the duration of the slip. Both rupture velocity and rise time are homogeneous parameters across the fault plane. Thus, we invert for 26 free parameters in total. Finally, the complete seismic response, $v_l^r(\omega)$, at station r, component l and for the circular frequency, $\omega = 2\pi f$, is computed as a linear sum of $N(= 24)$ subfault contributions

$$v_l^r(\omega) = \sum_{k=1}^{N} slip_k \exp[-i\omega t_k(c^{rup})] \, G_{kl}^r(\omega) \, S(R, \omega). \quad (2)$$

In equation (2) S represents the source function that we implemented as an ordinary ramp function. Additional details on the source function are provided in Appendix B.

space derivative) within the rupture element and name the representative approximation as $G^n_{ip,q}(x, t - \tau^n)$. Note that the above-mentioned approximations introduce frequency-dependent errors, depending also on receiver position with respect to the source position (directivity, see e.g. Spudich & Archuleta 1987). Finally, we obtain:

$$v_i(x, t) \cong \sum_{n=1}^{N} \left[\tilde{M}^n_{pq} \cdot G^n_{ip,q}(x, t - \tau^n)\right] * s^n(t) \cdot \mu^n \cdot A^n, \tag{4}$$

where A^n is the area of the nth subfault σ_n. We call the part enclosed in the brackets 'NGF' and denote it as $g^n_i(x, t - \tau^n)$. It can be calculated using a slip rate impulse with a given rupture mechanism. Finally, we obtain the basic equation for synthesis of ground motions:

$$v_i(x, t) = \sum_{n=1}^{N} g^n_i(x, t - \tau^n) * s^n_r(t)\ \mu^n\ A^n. \tag{5}$$

Note that the complete $G_{ip,q}$ tensor (see eq. 4) could be stored,

The Earth (or a numerical solver) as a linear system

Source-receiver reciprocity

The displacement field generated by a distribution of body forces and surface tractions can be synthesized using the elastodynamic Green function $G_{ij}(\boldsymbol{x}, t; \boldsymbol{x}', t')$, giving the i component of displacement at $(\boldsymbol{x}, t)$ due to a localized unit body force operating at $(\boldsymbol{x}', t')$ in the j direction. The elastodynamic Green function satisfies the Navier equation of motion for a linear elastic solid

$$\rho \frac{\partial^2}{\partial t^2} G_{ij} = \delta_{ij} \delta(\boldsymbol{x} - \boldsymbol{x}') \delta(t - t') + \frac{\partial}{\partial x_n} \left(c_{inkl} \frac{\partial}{\partial x_l} G_{kj} \right) \quad (1.10)$$

where $\delta(\ )$ is the Dirac delta function. A complete determination of G_{ij} requires meeting initial conditions (taken usually to be $G = \partial G / \partial t = 0$ for $t \leq t'$ and $\boldsymbol{x} \neq \boldsymbol{x}'$) and specified boundary conditions on the surface of the medium.

If G_{ij} satisfies homogeneous boundary conditions (i.e., zero traction or zero displacement) on S, it has the following spatiotemporal reciprocity properties

$$G_{ij}(\boldsymbol{x}, t; \boldsymbol{x}', t') = G_{ji}(\boldsymbol{x}', -t'; \boldsymbol{x}, -t). \quad (1.11)$$

Practical example

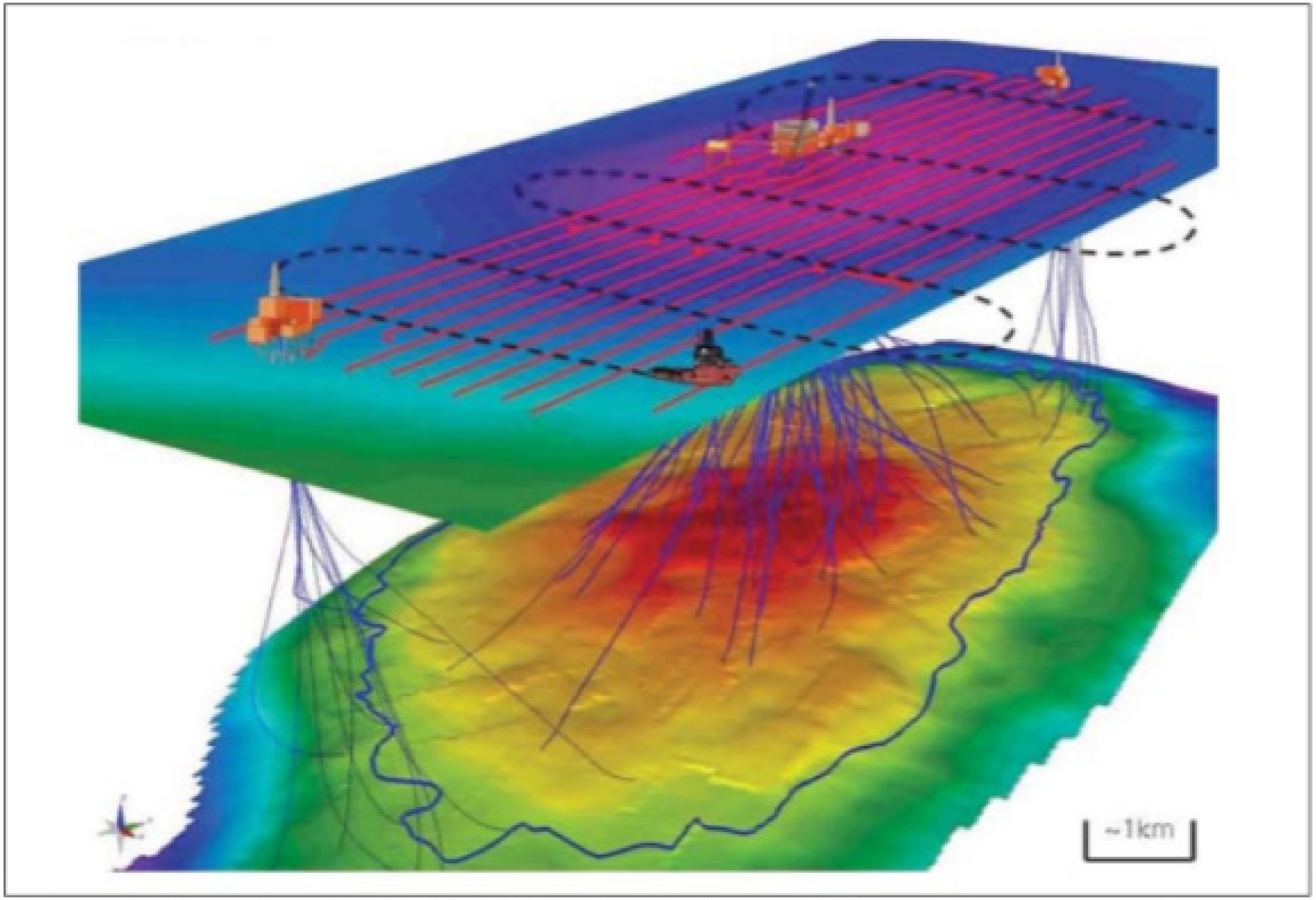
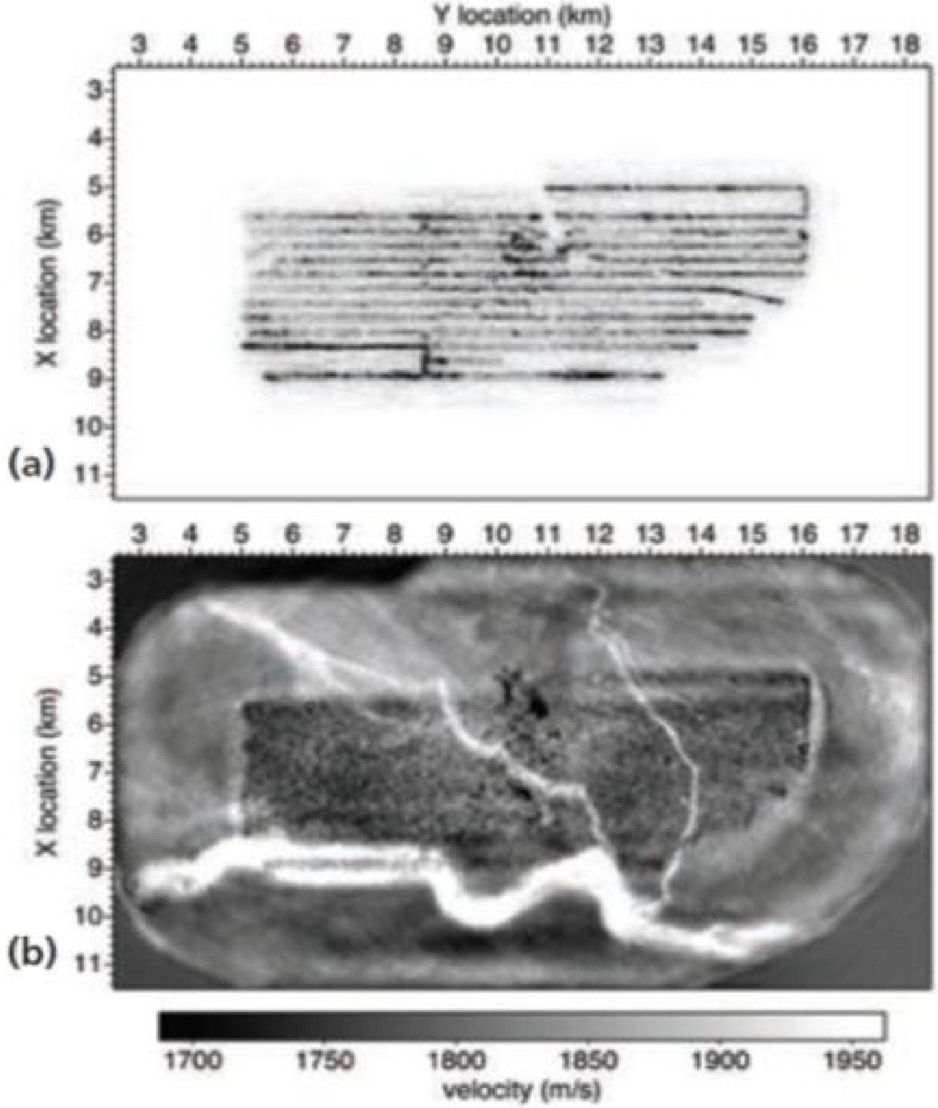


Figure 1. *Overview of Valhall Field showing the layout of the geophone array at the sea floor (red lines), the top of the reservoir, the outline of the field (dark blue line), and the wells (thin blue lines).*



Sirgue et al., 2010

Summary

To understand seismic wave propagation the following concepts need to be understood;

- The mathematical description of the deformation of an elastic 3-D object -> strain
- The forces that are at work for a given deformation and its (mostly linear!) dependence on the magnitude of deformation .> stress – strain relation
- The description of elastic modules and the various symmetry systems (-> elasticicity tensor, isotropy, transverse isotropy, hexagonal symmetry).
- The boundary condition required at the free surface (traction-free) and the consequences for wave propagation -> surface waves
- The description of seismic sources using the moment tensor concept (-> double couples, explosions)
- The origin, scale, spectrum of material heterogeneities in side the Earth (-> the reason why we need to resort to numerical methods)