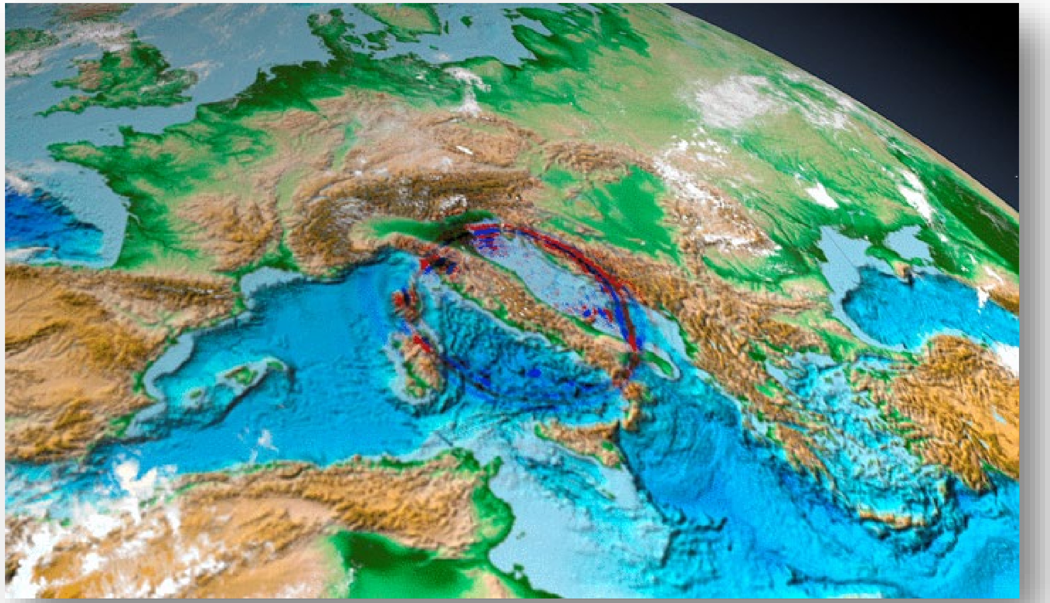
Seismic waves: A primer

- What are the **governing equations** for elastic wave propagation?
- What are the most **fundamental results** in simple media?
- How do we describe and input **seismic sources** (superposition principle)?
- What are consequences of the **reciprocity principle**?
- What **rheologies** do we need (stress-strain relation)?
- 3-D **heterogeneities** and scattering
- **Green's functions**, numerical solvers as **linear systems**



Goal: You know what to expect when running a wave simulation code!

Wave Equations

The (anisotropic) elastic wave equation (strong form)

$$\rho \partial_t^2 u_i = \partial_j (\sigma_{ij} + M_{ij}) + f_i$$

Wave equation

This is the **displacement – stress** formulation

where

$$\sigma_{ij} = c_{ijkl} \varepsilon_{kl}$$

Stress-strain relation

$$\varepsilon_{kl} = \frac{1}{2} \left(\partial_k u_l + \partial_l u_k \right)$$

Strain-displacement relation

$$\rho\partial_t^2 u_i = \partial_j(\sigma_{ij} + M_{ij}) + f_i$$

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}$$

$$\varepsilon_{kl} = \frac{1}{2}\left(\partial_k u_l + \partial_l u_k\right)$$

$\rho \rightarrow \rho(\mathbf{x})$	Mass density
$u_i \rightarrow u_i(\mathbf{x},t)$	Displacement vector
$\sigma_{ij} \rightarrow \sigma_{ij}(\mathbf{x},t)$	Stress tensor (3x3)
$M_{ij} \rightarrow M_{ij}(\mathbf{x},t)$	Moment tensor (3x3)
$f_i \rightarrow f_i(\mathbf{x},t)$	Volumetric force
$c_{ijkl} \rightarrow c_{ijkl}(\mathbf{x})$	Tensor of elastic constants (3x3x3x3)
$\varepsilon_{kl} \rightarrow \varepsilon_{kl}(\mathbf{x},t)$	Strain tensor (3x3)

3D to 1D

$$\rho \partial_t^2 u_i = \partial_j \sigma_{ij} + f_i$$

$$\rho \partial_t^2 u_x = \partial_x \sigma_{xx} + \partial_y \sigma_{xy} + \partial_z \sigma_{xz} + f_x$$

$$\rho \partial_t^2 u_y = \partial_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz} + f_y$$

$$\rho \partial_t^2 u_z = \partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz} + f_z$$

Plane wave in x-direction: $\partial_y(\cdot) = \partial_z(\cdot) = 0$!

Transverse polarization: $u_x = u_z = 0$!

$$\Rightarrow \quad \rho \partial_t^2 u_y = \partial_x \sigma_{yx} + f_y$$

1D elastic wave equation

$$\begin{aligned}\rho\partial_t^2 u_y \;=\;& \partial_x\left[\mu(\partial_x u_y + \partial_y u_x) + M_{yx}\right] \\ &+ \partial_y\left[(\lambda + 2\mu)\partial_y u_y + \lambda(\partial_x u_x + \partial_z u_z) + M_{yy}\right] \\ &+ \partial_z\left[\mu(\partial_z u_y + \partial_y u_z) + M_{yz}\right] \\ &+ f_y,\end{aligned}$$

$$\rho\partial_t^2 u_y \;=\; \partial_x(\mu\partial_x u_y) + f_y$$

1D elastic wave equation

$$\rho \ddot{u} = \partial_x (\mu \partial_x u) + f$$

This is a scalar wave equation descriptive of **transverse** motions of a string

The elastic wave equation

$$\rho \partial_t v_i = \partial_j (\sigma_{ij} + M_{ij}) + f_i$$

$$\dot{\sigma}_{ij} = c_{ijkl} \dot{\varepsilon}_{kl}$$

This is the **velocity – stress** formulation, where

$$\dot{\varepsilon}_{kl} = \frac{1}{2} \left(\partial_k v_l + \partial_l v_k \right)$$

$$v_i = \dot{u}_i = \partial_t u_i$$

$$\rho \partial_t^2 u_i = \partial_j (\sigma_{ij} + M_{ij}) + f_i$$

$$\sigma_{ij} = c_{ijkl} \varepsilon_{kl}$$

$$\varepsilon_{kl} = \frac{1}{2} \left(\partial_k u_l + \partial_l u_k \right)$$

3D acoustic wave equation

$$\ddot{p} = c^2 \Delta p + s$$

This is the constant density acoustic wave equation (sound in a liquid or gas)

$$c \rightarrow c(x)$$

P-velocity

$$p \rightarrow p(x,t)$$

Pressure

$$s \rightarrow s(x,t)$$

Sources

.

$$\Delta \rightarrow \begin{pmatrix} \partial_x^2 \\ \partial_y^2 \\ \partial_z^2 \end{pmatrix}$$

Laplace Operator

This is equation is still tremendously important in exploration seismics!

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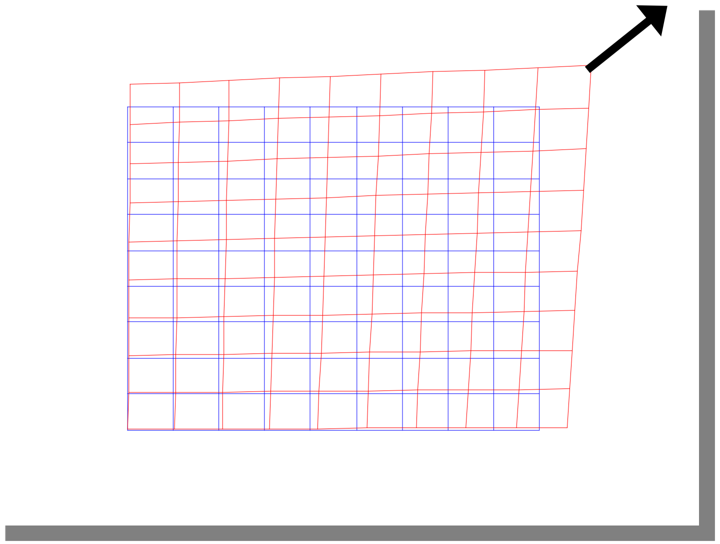
Rheologies

Stress and strain

To first order the Earth 's crust deforms like an elastic body when the deformation (strain) is small.

In other words, if the force that causes the deformation is stopped the rock will go back to its original form.



The change in shape (i.e., the deformation) is called strain, the forces that cause this strain are called stresses.

Stress-strain relation

The relation between stress and strain in general is described by the tensor of elastic constants c_{ijkl}

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}$$

Generalised Hooke's Law

From the symmetry of the stress and strain tensor and a thermodynamic condition if follows that the maximum number if independent constants of c_{ijkl} is 21. In an isotropic body, where the properties do not depend on direction the relation reduces to

$$\sigma_{ij} = \lambda\Theta\delta_{ij} + 2\mu\varepsilon_{ij}$$

Hooke's Law

where l and m are the Lame parameters, q is the dilatation and d_{ij} is the Kronecker delta.

$$\Theta\delta_{ij} = \varepsilon_{kk}\delta_{ij} = \left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right)\delta_{ij}$$

Other rheologies (not further explored in this course)

Viscoelasticity
- the loss of energy due to internal friction
- possibly frequency-dependent
- different for P and S waves (why?)
- described by Q
- Not easy to implement numerically for time-domain methods

Porosity
- Effects of pore space (empty, filled, partially filled) on stress-strain
- Frequency-dependent effects
- Additional wave types (slow P wave)
- Highly relevant for reservoir wave propagation

Plasticity
- permanent deformation due to changes in the material as a function of deformation or stress
- resulting from (micro-) damage to the rock mass
- often caused by damage on a crystallographic scale
- important close to the earthquake source
- not well constrained by observations

Seismic Waves

Consequences of the equations of motion

$$\rho \ddot{u}_i = f_i + \partial_j \sigma_{ij}$$

What are the solutions to this equation? At first we look at infinite homogeneous isotropic media, then:

P−waves

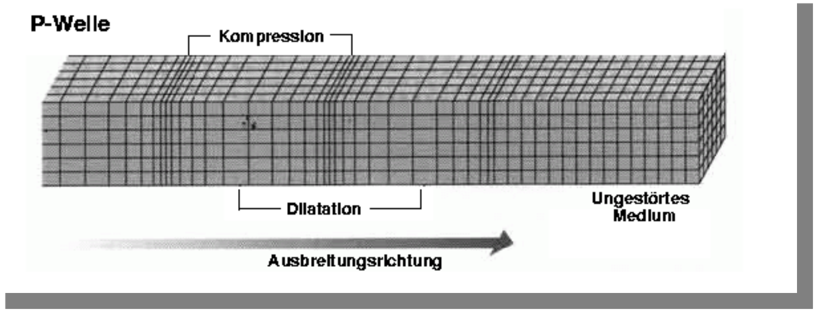
$$v_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

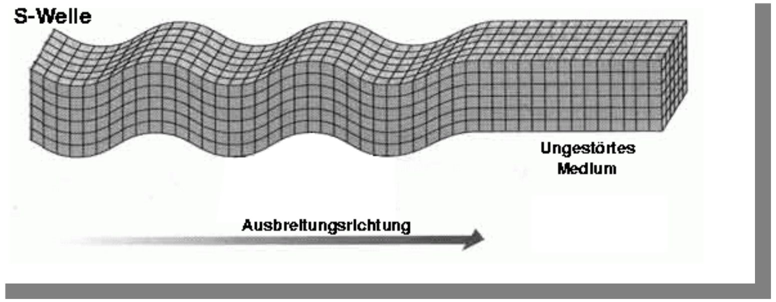
approximately

$$v_p = \sqrt{3} v_s$$

S−waves

$$v_s = \sqrt{\frac{\mu}{\rho}}$$





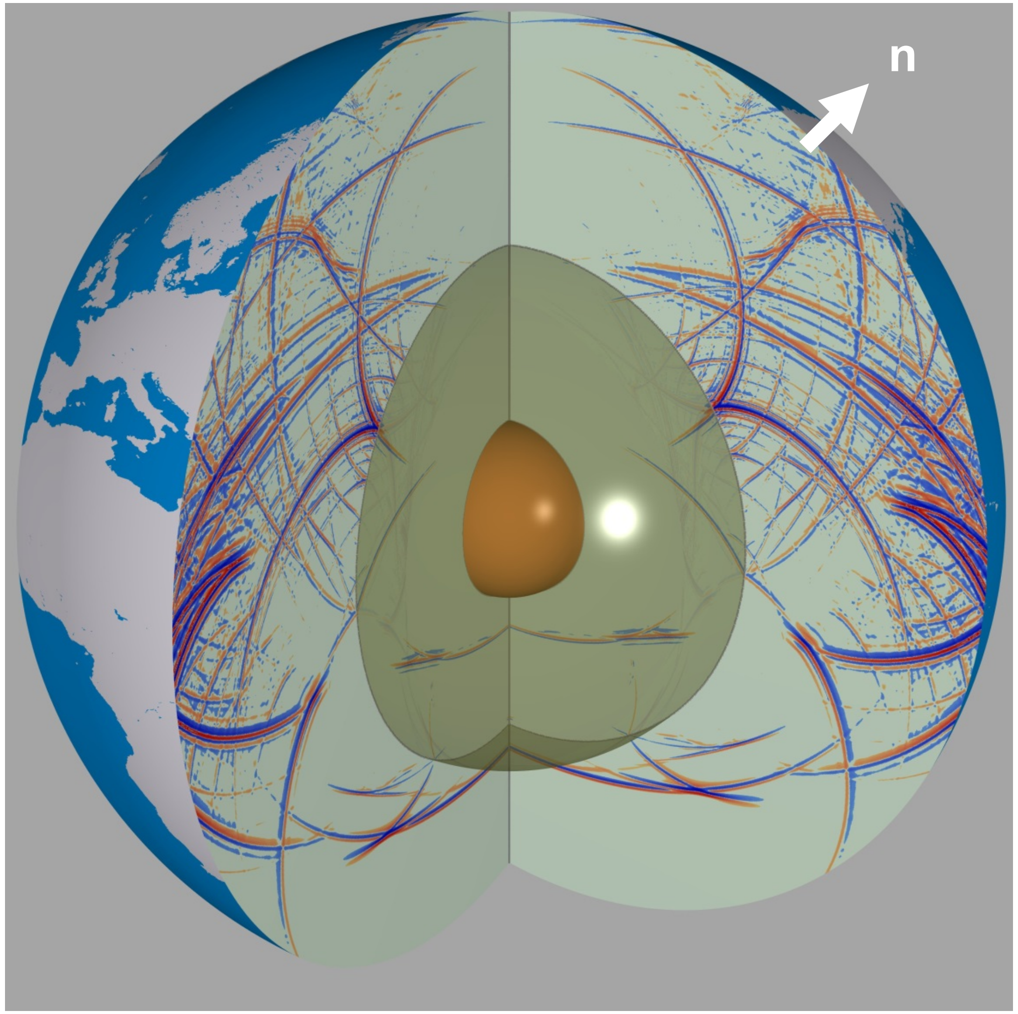
Boundary conditions: external and internal interfaces

Traction is zero perpendicular to free surface (needs special attention with most numerical methods)

$$t_{' j} = \sigma_{ij} n_j = 0$$

At internal interfaces we speak of *welded contact* Normal tractions are continuous (they are usually not directly implemented, except fluid-solid)

$$\boldsymbol{\sigma}_1 \hat{\mathbf{n}} = \boldsymbol{\sigma}_2 \hat{\mathbf{n}}$$



Seismic wave types

Surface waves waves

Love waves – transversely polarized – superposition of SH waves in layered media

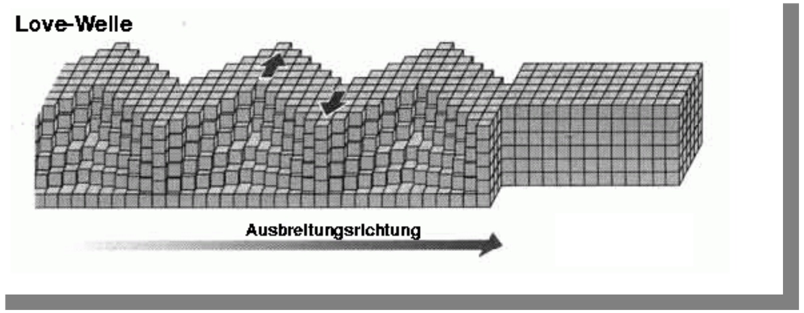
Non-existing in half space

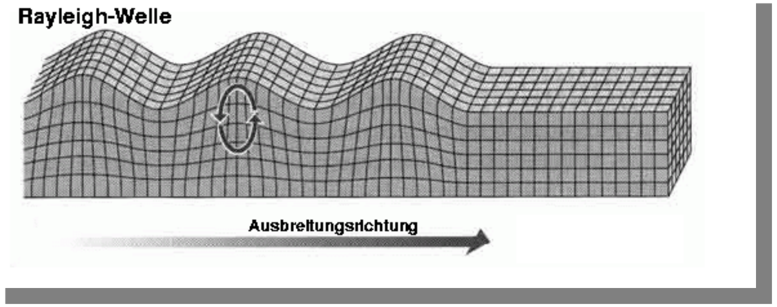
Always dispersive in layered media

Rayleigh waves – polarized in the plane through source and receiver – superposition of P and SV waves

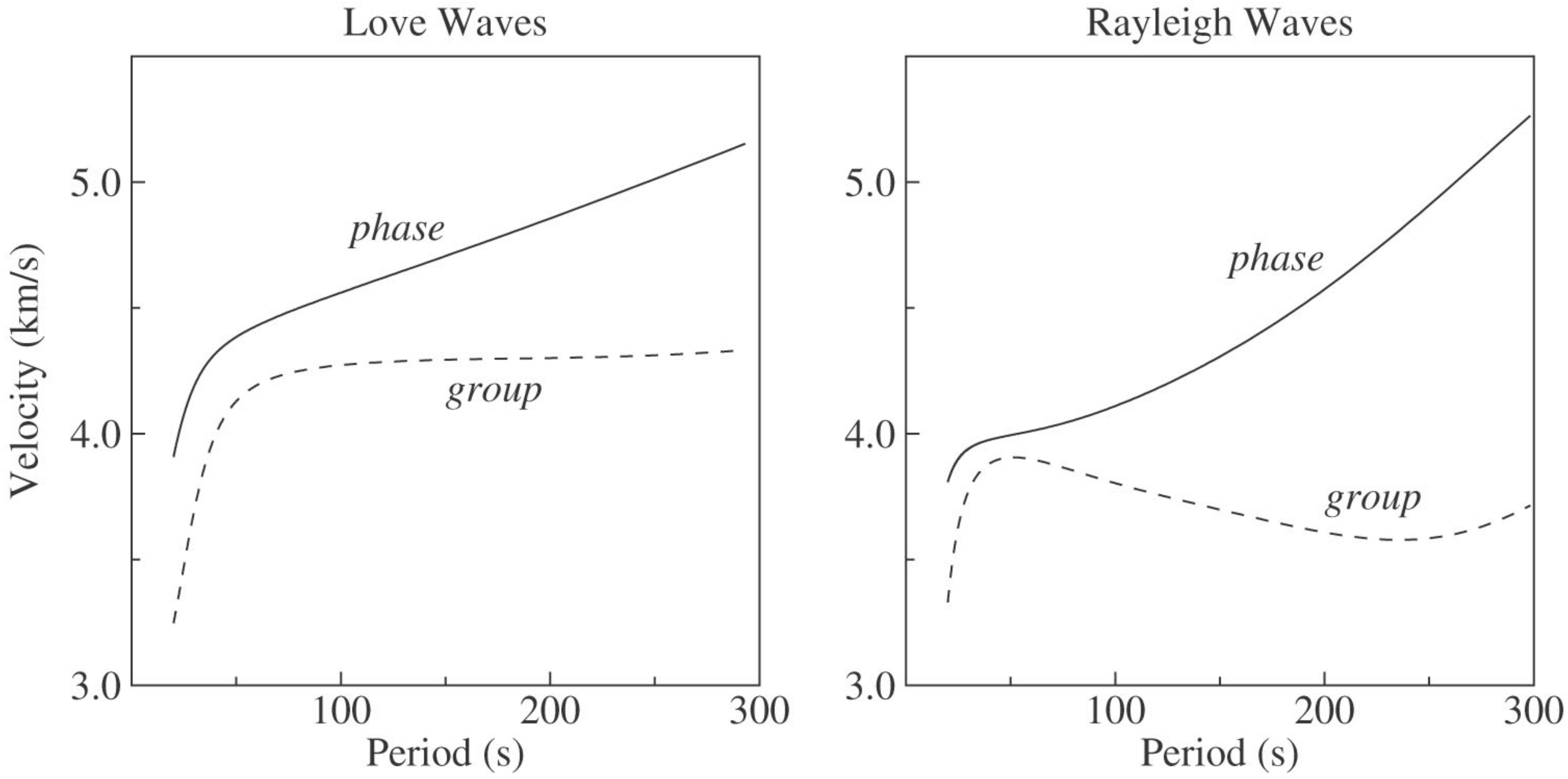
Non-dispersive in half space

Dispersive in layered media

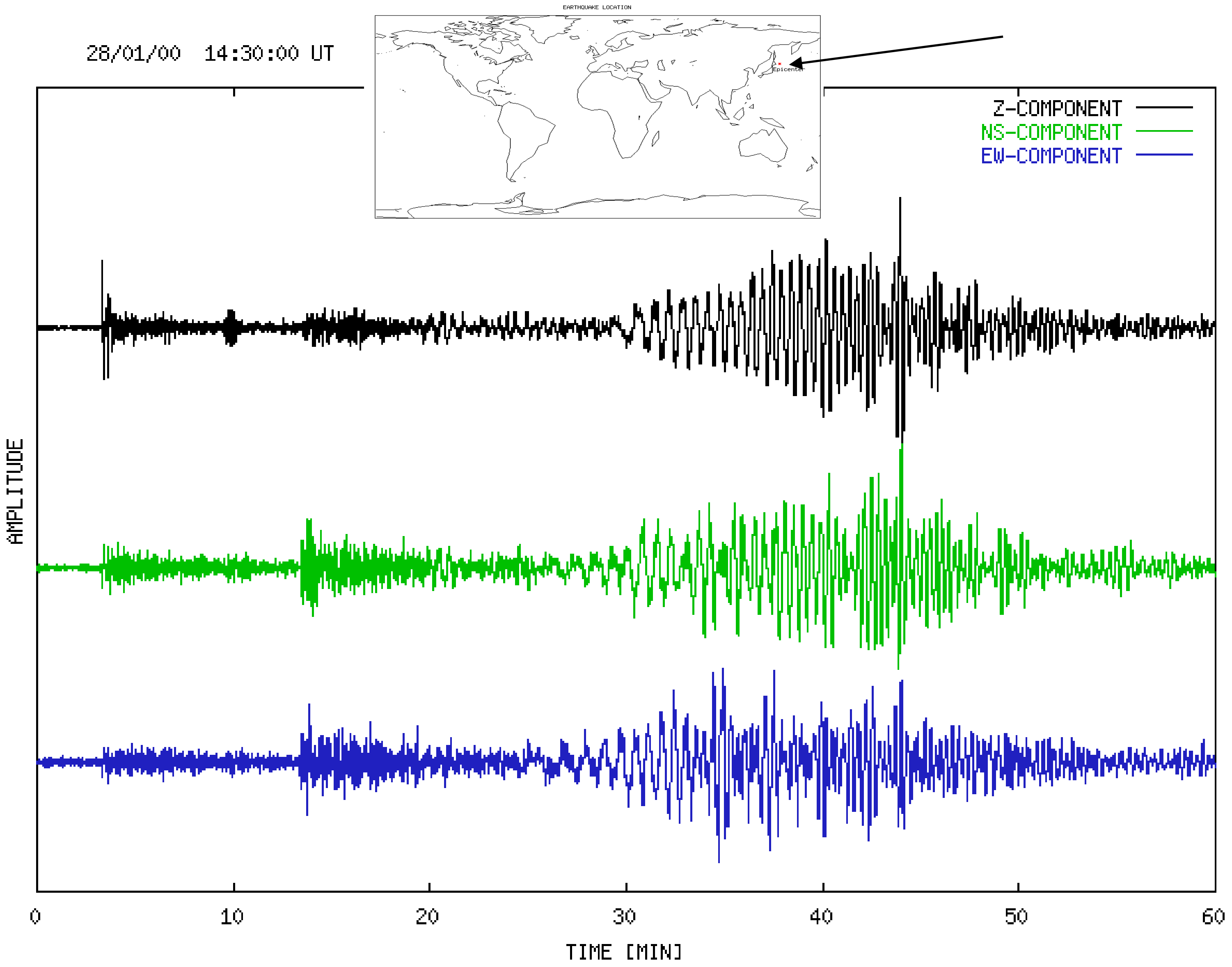




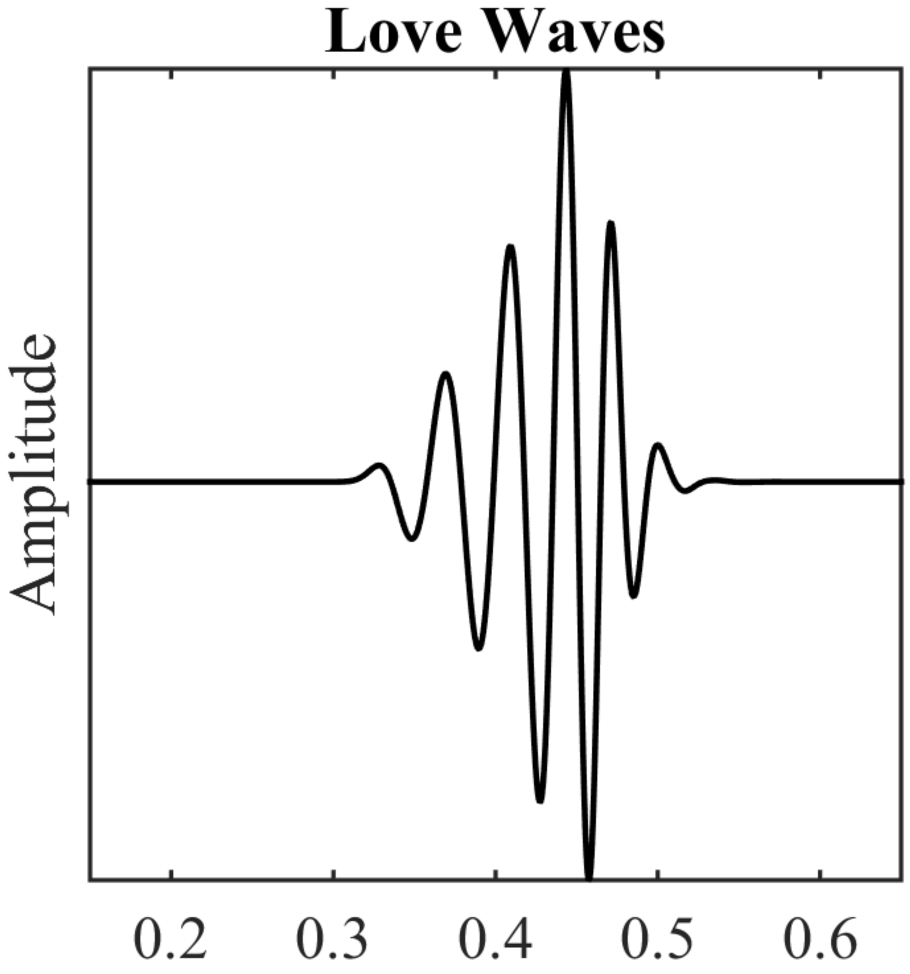
Surface wave dispersion

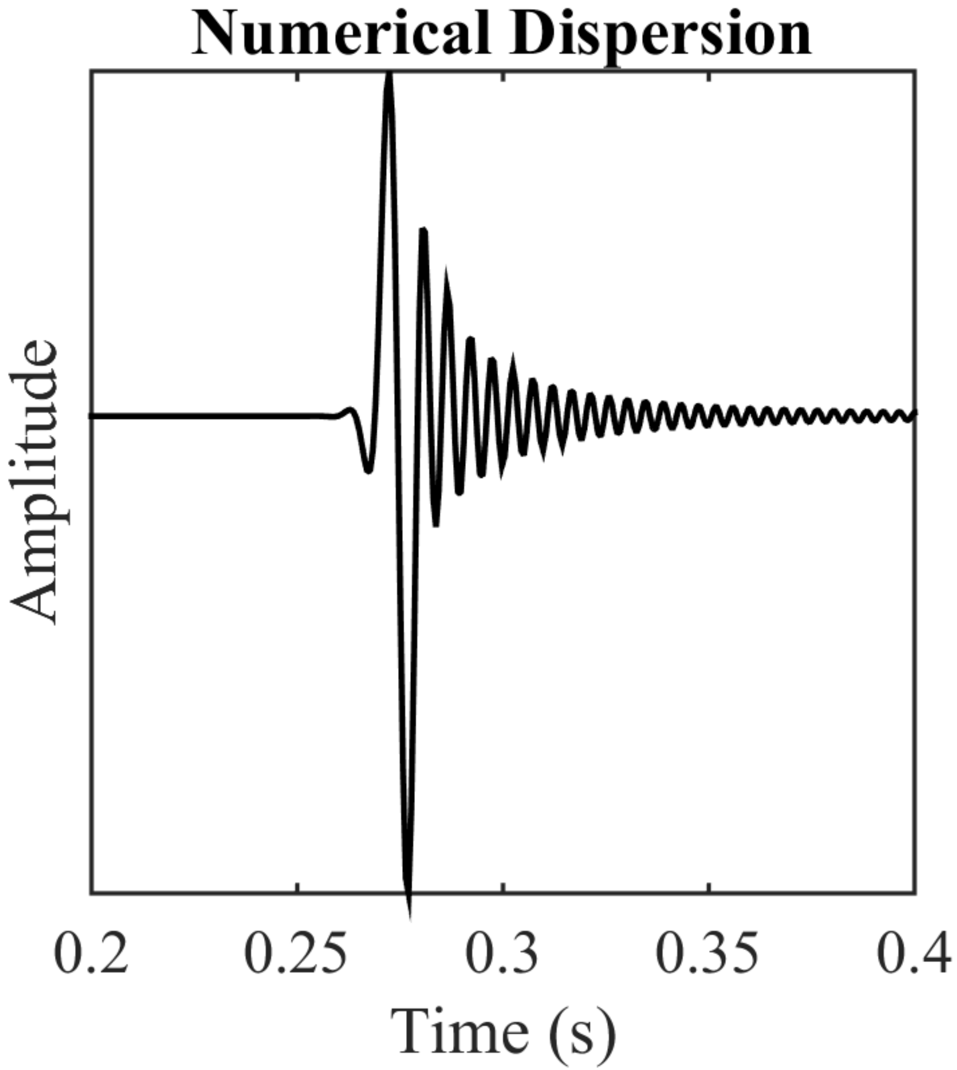


Data Example



Real vs. numerial dispersion





Surface waves summary

- Elastic surface waves (Love and Rayleigh) in nature generally show **dispersive behavior** (later we will see that there is also dispersive behaviour due to numerical effects!)

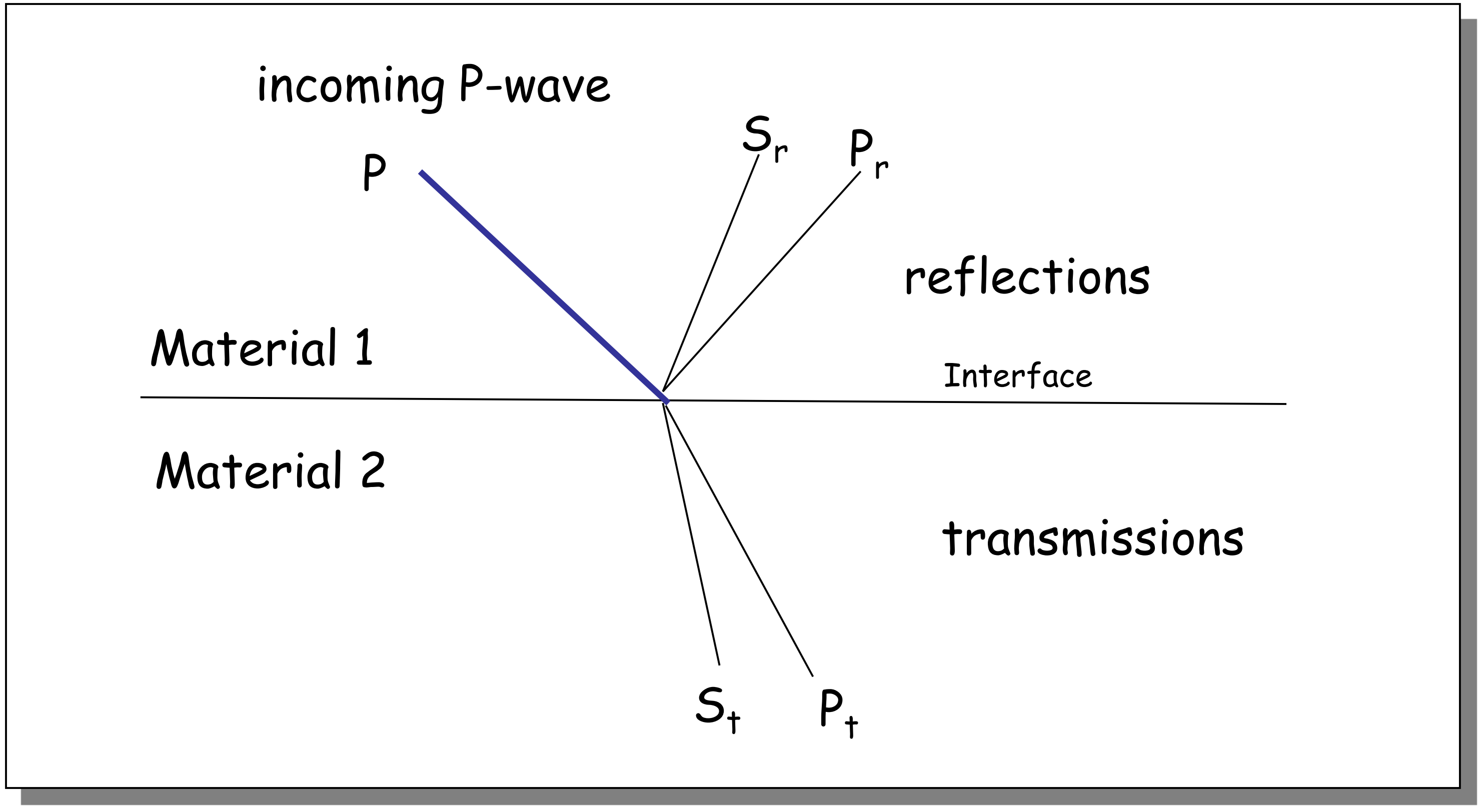
- Surface waves are a consequence of the **free-surface boundary condition**. We thus might expect that – when using numerical approximations there might be differences concerning the accurate implementation of this boundary condition.

- The accurate simulation of surface waves plays a dominant role in **global and regional (continental scale) seismology** and is usually not so important in exploration geophysics.

Reflection, Transmission

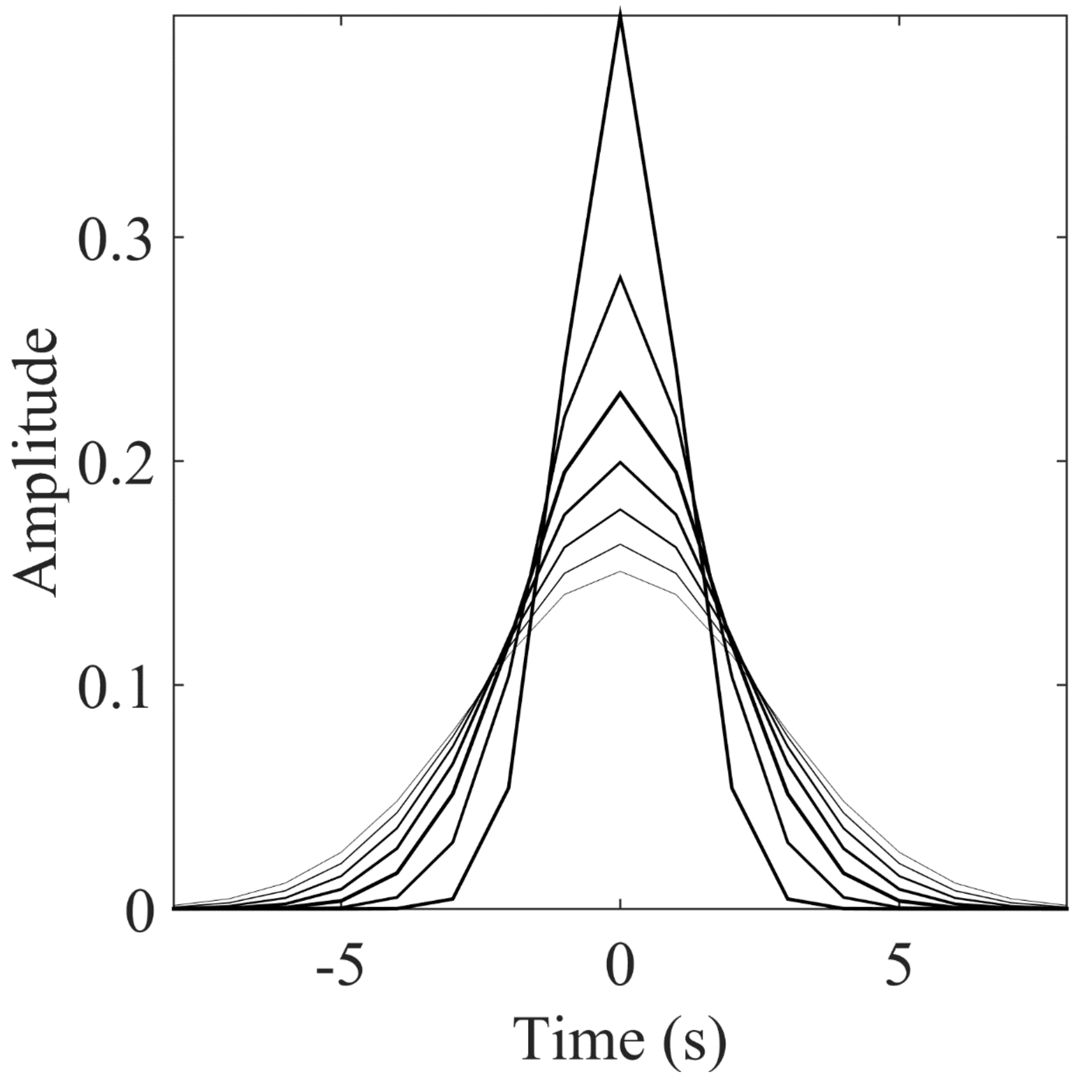
Reflection and transmission at boundaries
oblique incidence - conversion

P waves can be **converted** to S waves and vice versa. This creates a quite complex behavior of wave amplitudes and wave forms at interfaces. This behavior can be used to constrain the properties of the material interface.



Analytical solutions

„delta“-generating function



Spatial (or temporal) source function

$$\delta_{bc}(x) = \begin{cases} 1/dx & |x| \leq dx/2 \\ 0 & \text{elsewhere} \end{cases}$$

bc stands for boxcar

Analytical solutions for acoustic wave equation (Green‘s function)

1D	2D	3D
$\frac{1}{2c} H(t - \frac{\lvert r \rvert}{c})$	$\frac{1}{2\pi c^2} \frac{H(t - \frac{\lvert r \rvert}{c})}{\sqrt{t^2 - \frac{r^2}{c^2}}}$	$\frac{1}{4\pi c^2 r} \delta(t - r/c)$
$r = x$	$r = \sqrt{x^2 + y^2}$	$r = \sqrt{x^2 + y^2 + z^2}$

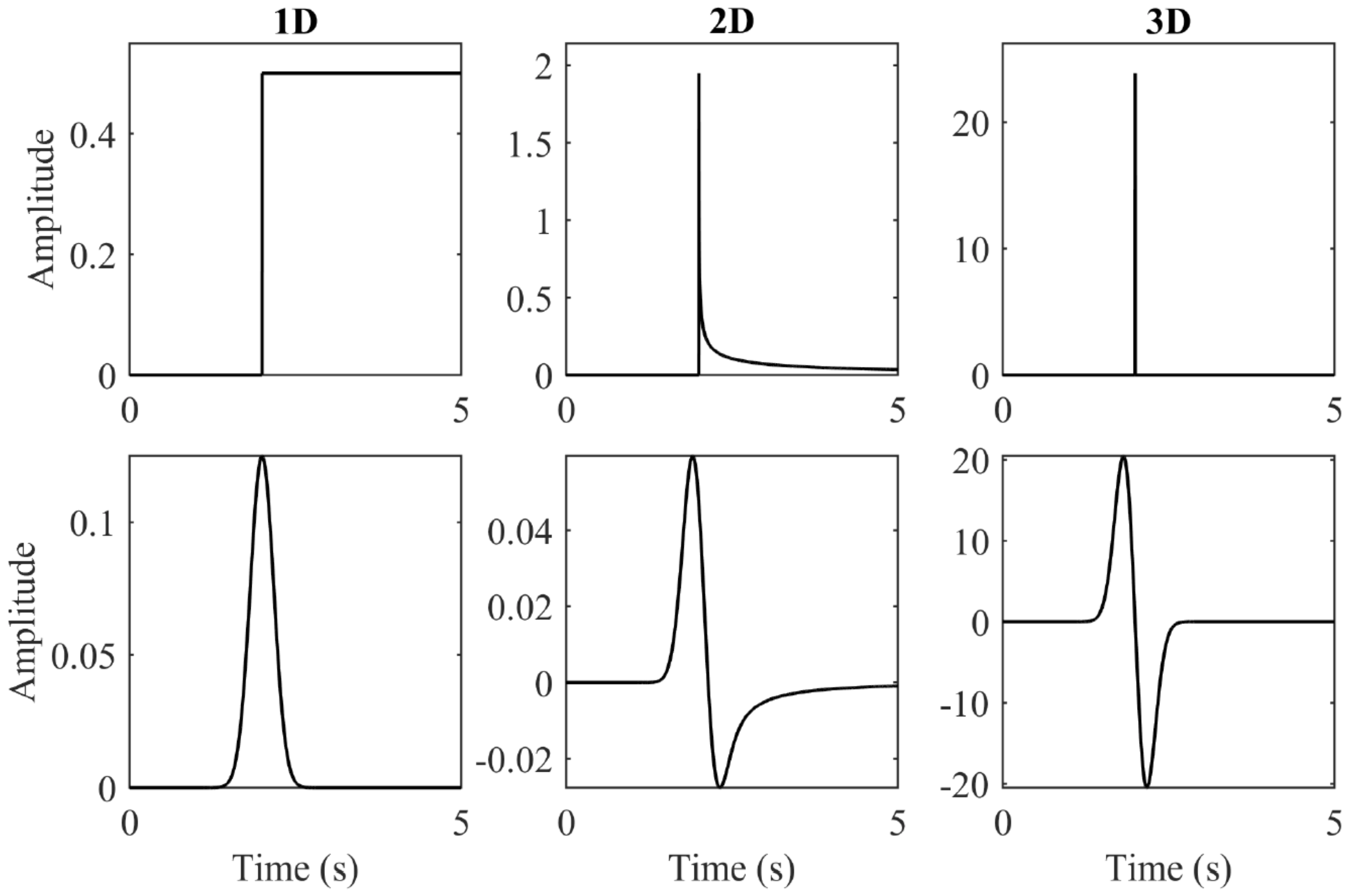


Fig. 2.9 Analytical solutions to the scalar wave equation. **Top**: Green's functions in 1D, 2D, and 3D. **Bottom**: Green's functions obtained after convolution with the 1st derivative of a Gaussian with 1Hz dominant frequency (see text for details). Note that the source time function is centred around $t = 0$.

Seismic sources

Radiation from a point double-couple source

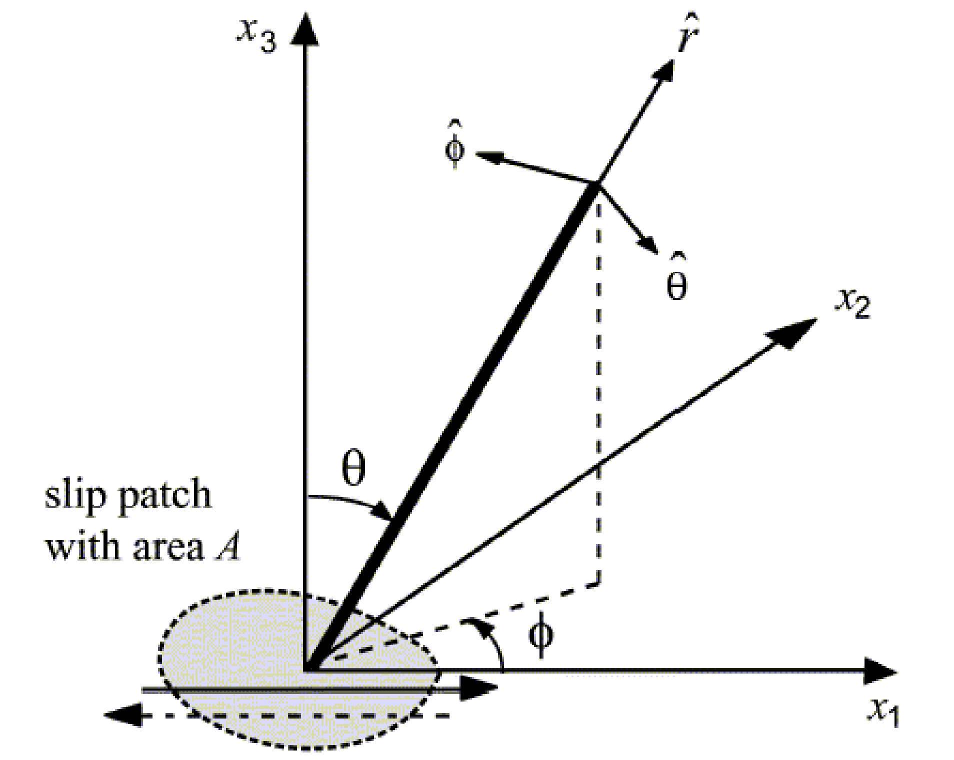


FIGURE 5 Cartesian and polar coordinate systems for analysis of radiation by a slip patch with area A and average slip $\langle \Delta u(t) \rangle$.

Geometry we use to express the seismic wavefield radiated by point double-couple source with area A and slip Du

Here the fault plane is the x_1x_2-plane and the slip is in x_1-direction. Which stress components are affected?

Radiation from a point source (M_{zx}=M_{xz}=M_0)

$$
\begin{aligned}
\boldsymbol{u}(\boldsymbol{x}, t) = {} & \frac{1}{4\pi\rho} \boldsymbol{A}^N \frac{1}{r^4} \int_{r/v_P}^{r/v_S} \tau M_0(t-\tau)\, d\tau \\
& + \frac{1}{4\pi\rho v_P^2} \boldsymbol{A}^{IP} \frac{1}{r^2} M_0(t - r/v_P) \\
& + \frac{1}{4\pi\rho v_S^2} \boldsymbol{A}^{IS} \frac{1}{r^2} M_0(t - r/v_S) \\
& + \frac{1}{4\pi\rho v_P^3} \boldsymbol{A}^{FP} \frac{1}{r} \dot{M}_0(t - r/v_P) \\
& + \frac{1}{4\pi\rho v_S^3} \boldsymbol{A}^{FS} \frac{1}{r} \dot{M}_0(t - r/v_S).
\end{aligned}
$$

… one of the most important results of seismology!
… Let's have a closer look …

u ground displacement as a function of space and time
r density
r distance from source
V_s shear velocity
V_p P-velocity
N near field
IP/S intermediate field
FP/S far field
M_0 seismic moment

$$
\begin{aligned}
\boldsymbol{A}^N &= 9 \sin 2\theta \cos\phi \hat{\boldsymbol{r}} - 6(\cos 2\theta \cos\phi \hat{\boldsymbol{\theta}} - \cos\theta \sin\phi \hat{\boldsymbol{\phi}}), \\
\boldsymbol{A}^{IP} &= 4 \sin 2\theta \cos\phi \hat{\boldsymbol{r}} - 2(\cos 2\theta \cos\phi \hat{\boldsymbol{\theta}} - \cos\theta \sin\phi \hat{\boldsymbol{\phi}}), \\
\boldsymbol{A}^{IS} &= -3 \sin 2\theta \cos\phi \hat{\boldsymbol{r}} + 3(\cos 2\theta \cos\phi \hat{\boldsymbol{\theta}} - \cos\theta \sin\phi \hat{\boldsymbol{\phi}}), \\
\boldsymbol{A}^{FP} &= \sin 2\theta \cos\phi \hat{\boldsymbol{r}}, \\
\boldsymbol{A}^{FS} &= \cos 2\theta \cos\phi \hat{\boldsymbol{\theta}} - \cos\theta \sin\phi \hat{\boldsymbol{\phi}},
\end{aligned}
$$

Radiation from a point source ($M_{zx}=M_{xz}=M_0$)

Near field term contains the static deformation

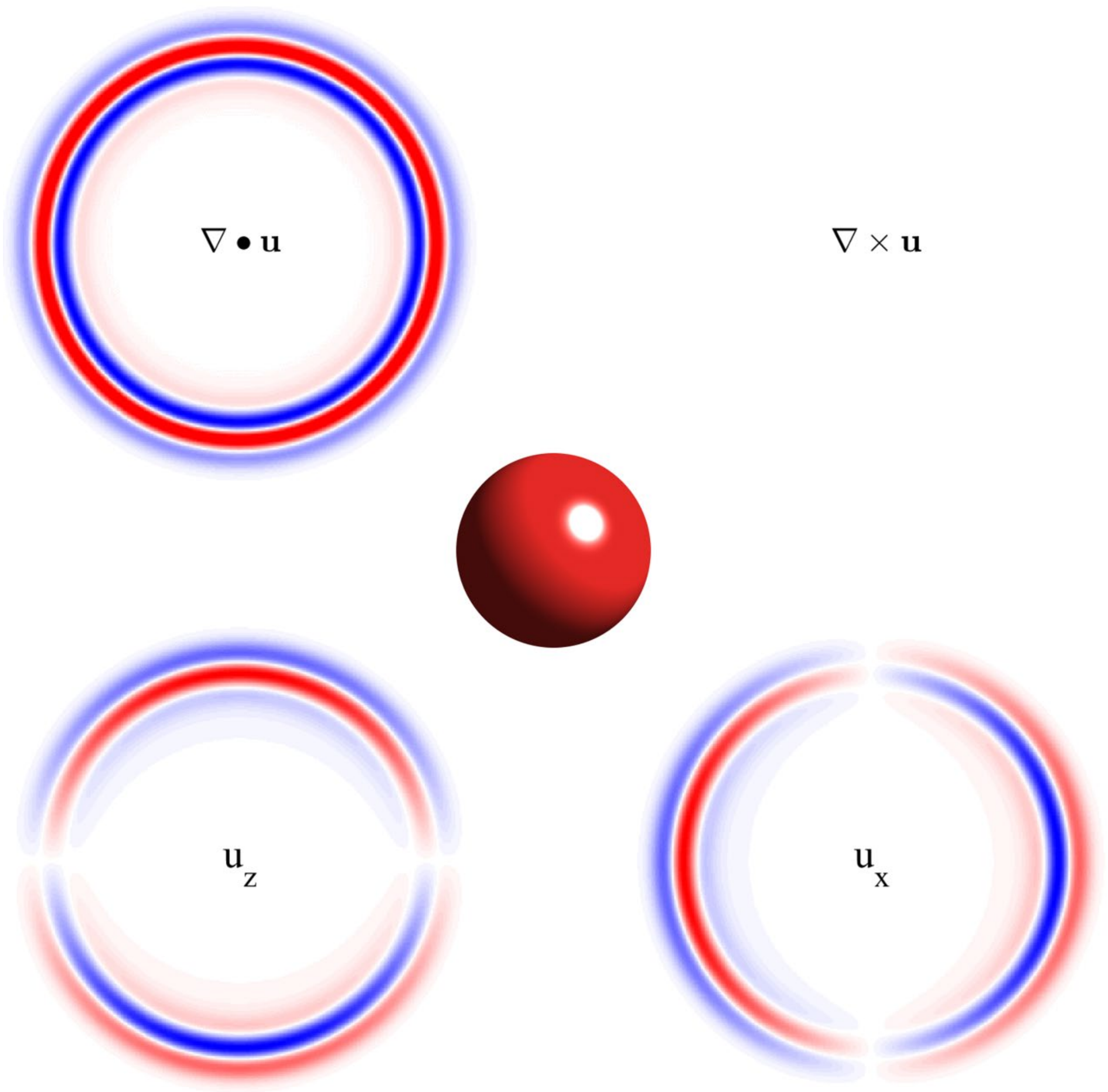
Intermediate terms

Far field terms: the main ingredient for source inversion, ray theory, etc.

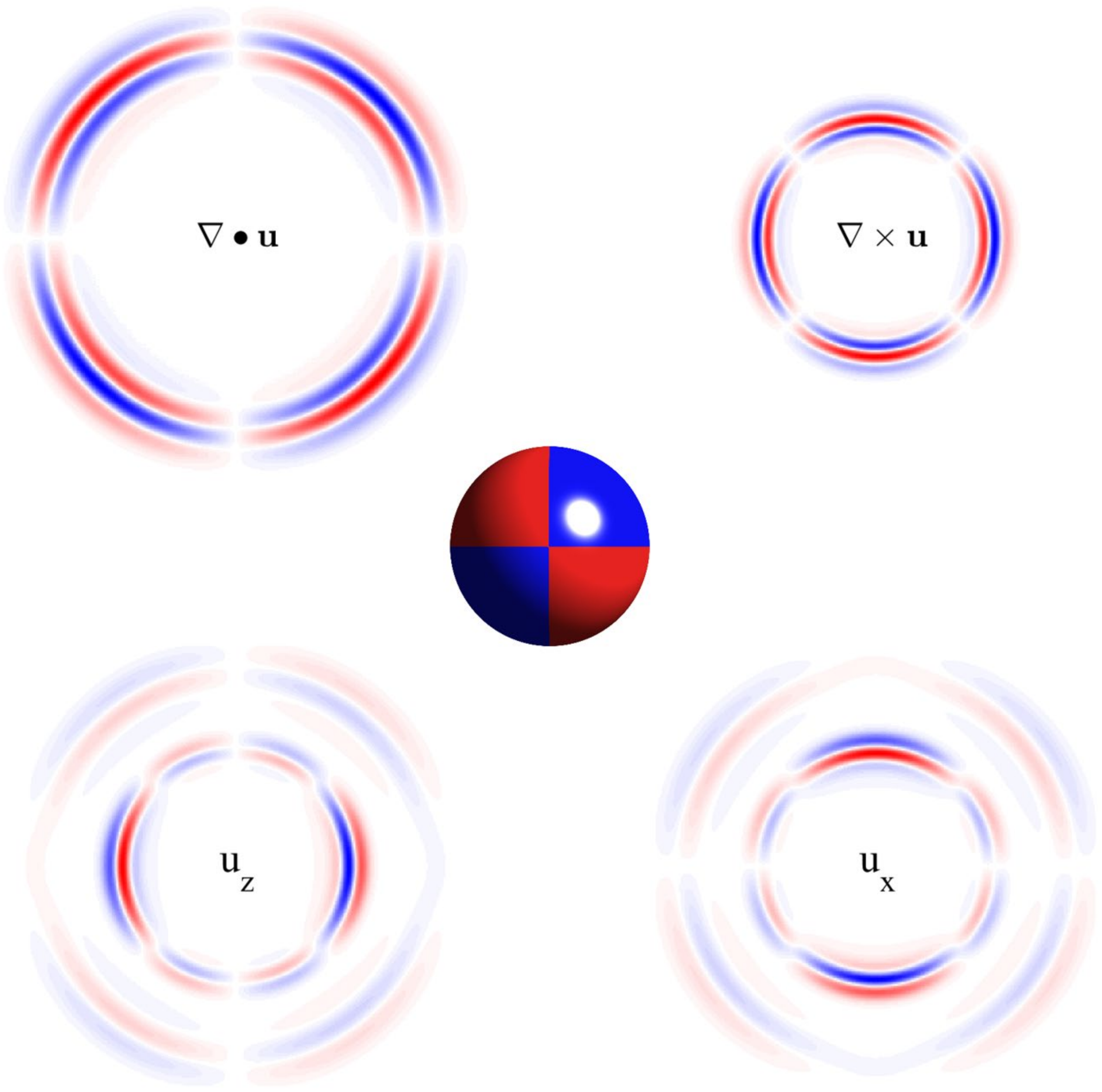
$$
\begin{aligned}
\boldsymbol{u}(\boldsymbol{x},t) = {} & \frac{1}{4\pi\rho}\boldsymbol{A}^N \frac{1}{r^4}\int_{r/v_P}^{r/v_S} \tau M_0(t-\tau)\,d\tau \\
& + \frac{1}{4\pi\rho v_P^2}\boldsymbol{A}^{IP}\frac{1}{r^2}M_0(t-r/v_P) \\
& + \frac{1}{4\pi\rho v_S^2}\boldsymbol{A}^{IS}\frac{1}{r^2}M_0(t-r/v_S) \\
& + \frac{1}{4\pi\rho v_P^3}\boldsymbol{A}^{FP}\frac{1}{r}\dot{M}_0(t-r/v_P) \\
& + \frac{1}{4\pi\rho v_S^3}\boldsymbol{A}^{FS}\frac{1}{r}\dot{M}_0(t-r/v_S).
\end{aligned}
$$

Elastic waves 2D

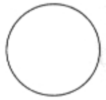
Explosion



Double couple



Beachballs and moment tensor

Moment Tensor	Beachball	Moment Tensor	Beachball
$\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$		$-\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	
$-\frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}$		$\frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}$	
$\frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}$		$\frac{1}{\sqrt{2}}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}$	
$\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$		$\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$	
$\frac{1}{\sqrt{6}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$		$\frac{1}{\sqrt{6}}\begin{pmatrix}1&0&0\\0&-2&0\\0&0&1\end{pmatrix}$	
$\frac{1}{\sqrt{6}}\begin{pmatrix}-2&0&0\\0&1&0\\0&0&1\end{pmatrix}$		$-\frac{1}{\sqrt{6}}\begin{pmatrix}-2&0&0\\0&1&0\\0&0&1\end{pmatrix}$	

explosion – implosion

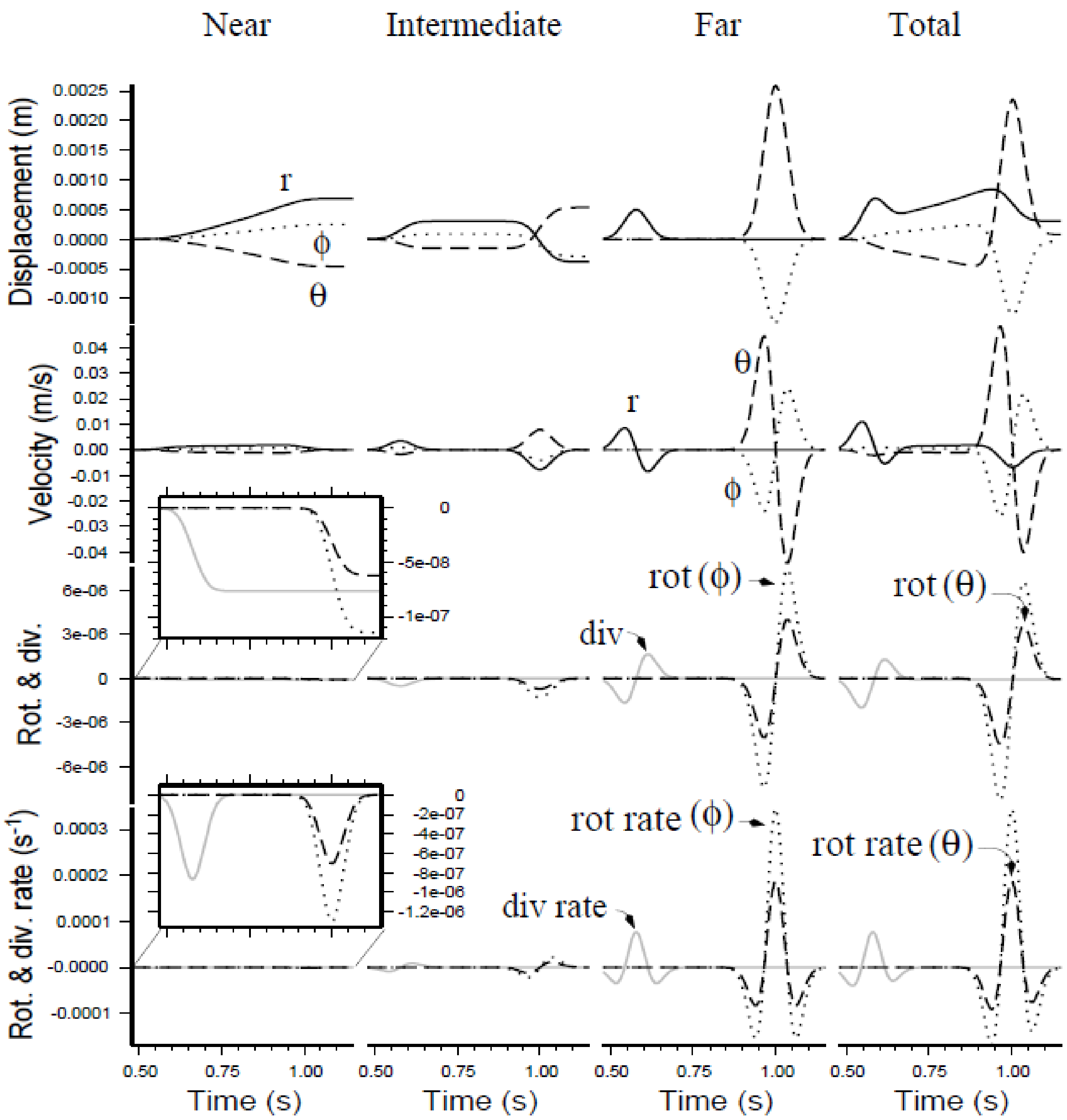
vertical strike slip fault

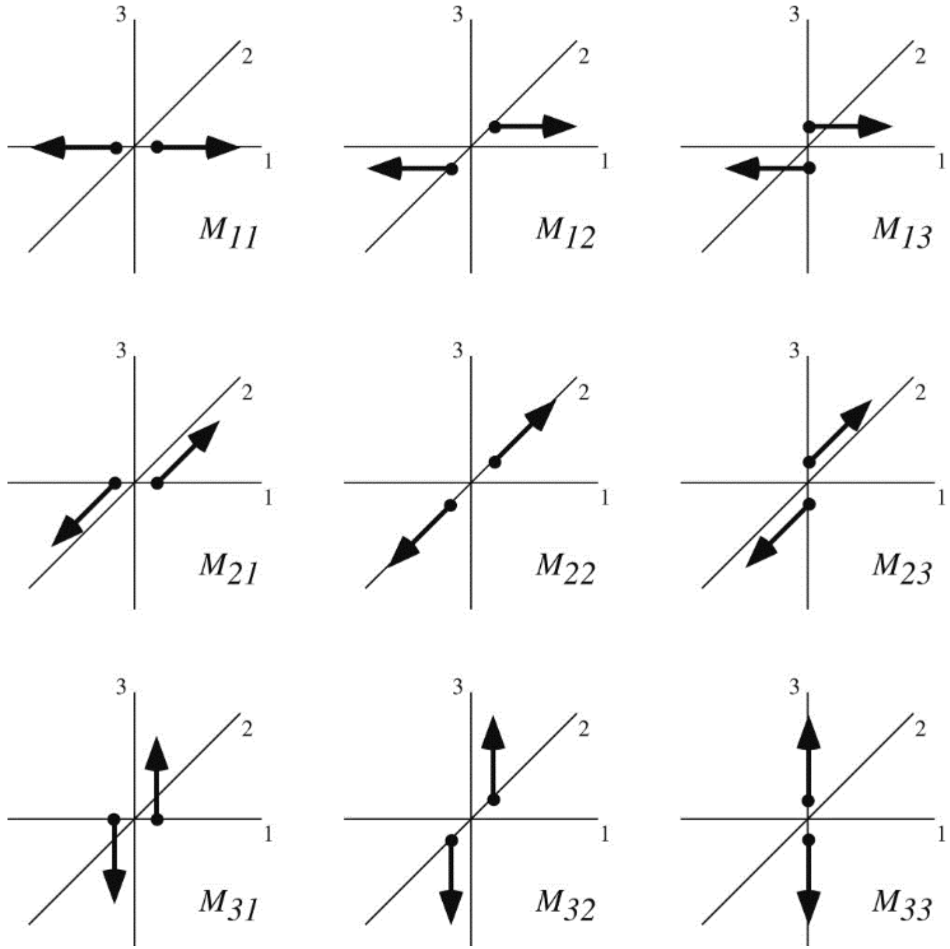
vertical dip slip fault

45° dip thrust fault

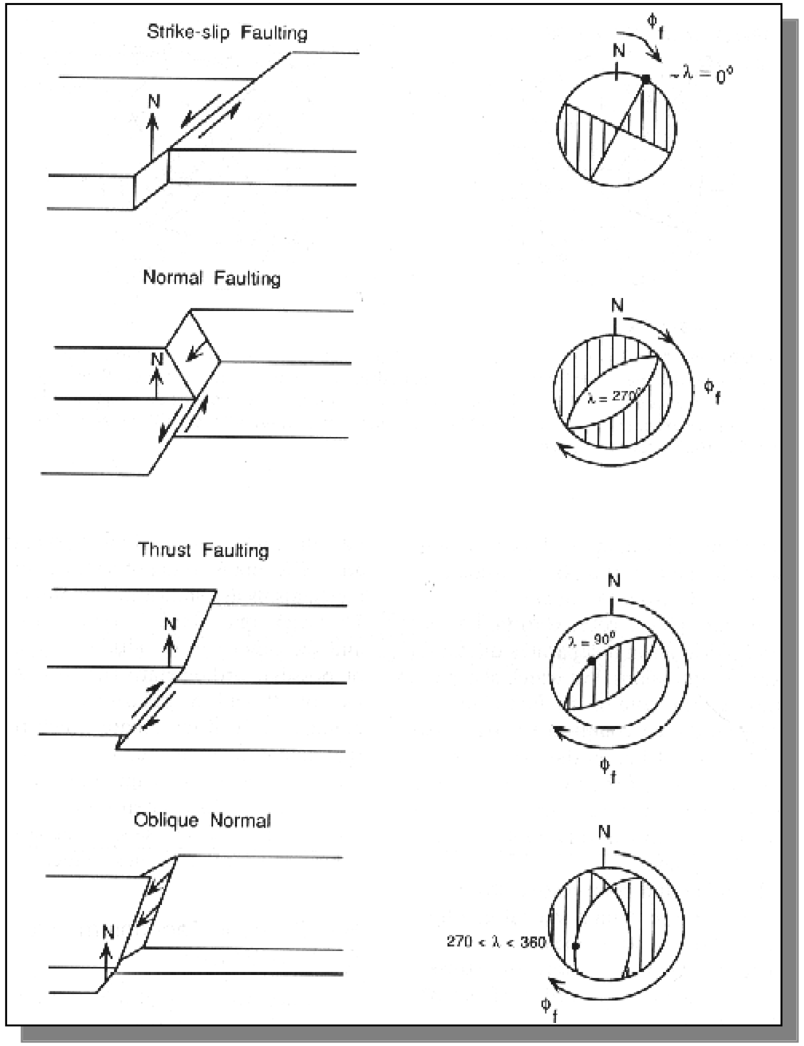
compensated linear vector dipoles

Translation, Divergence, Rotation, and all that (M4, 3km away)





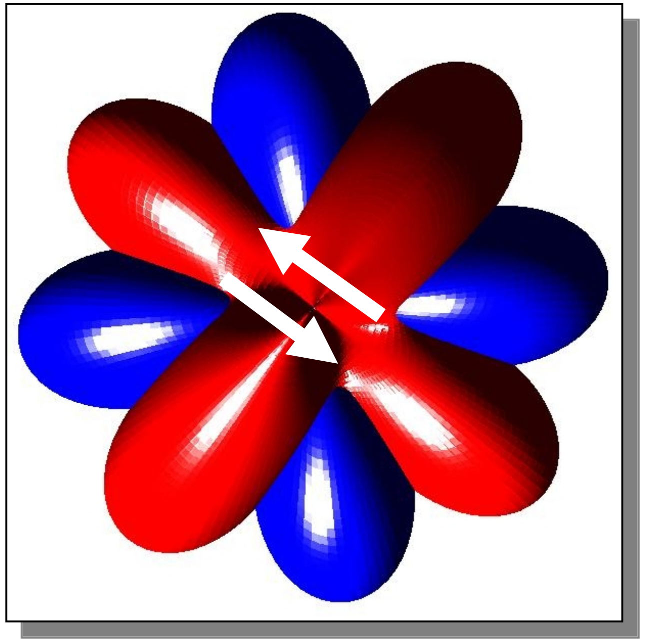
Source mechanisms

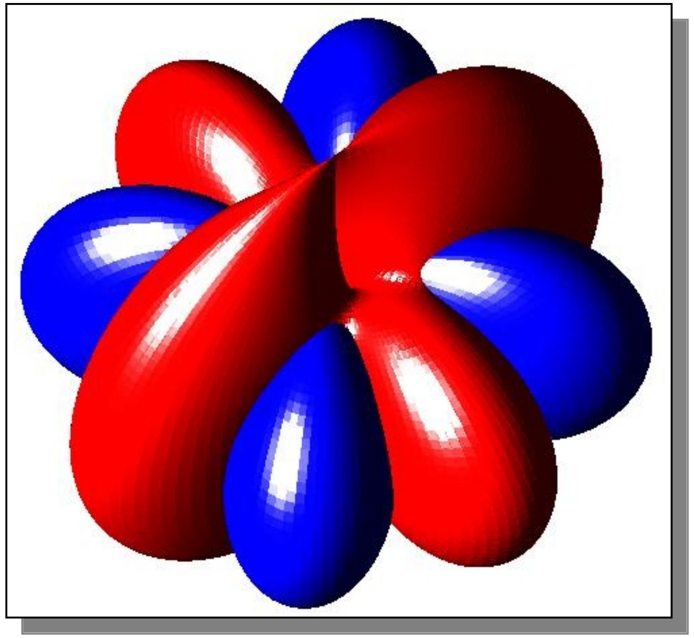


Basic fault types and their appearance in the focal mechanisms. Dark regions indicate compressional P-wave motion.

Radiation patterns of a double couple point sources

Far field P – blue
Far field S – red





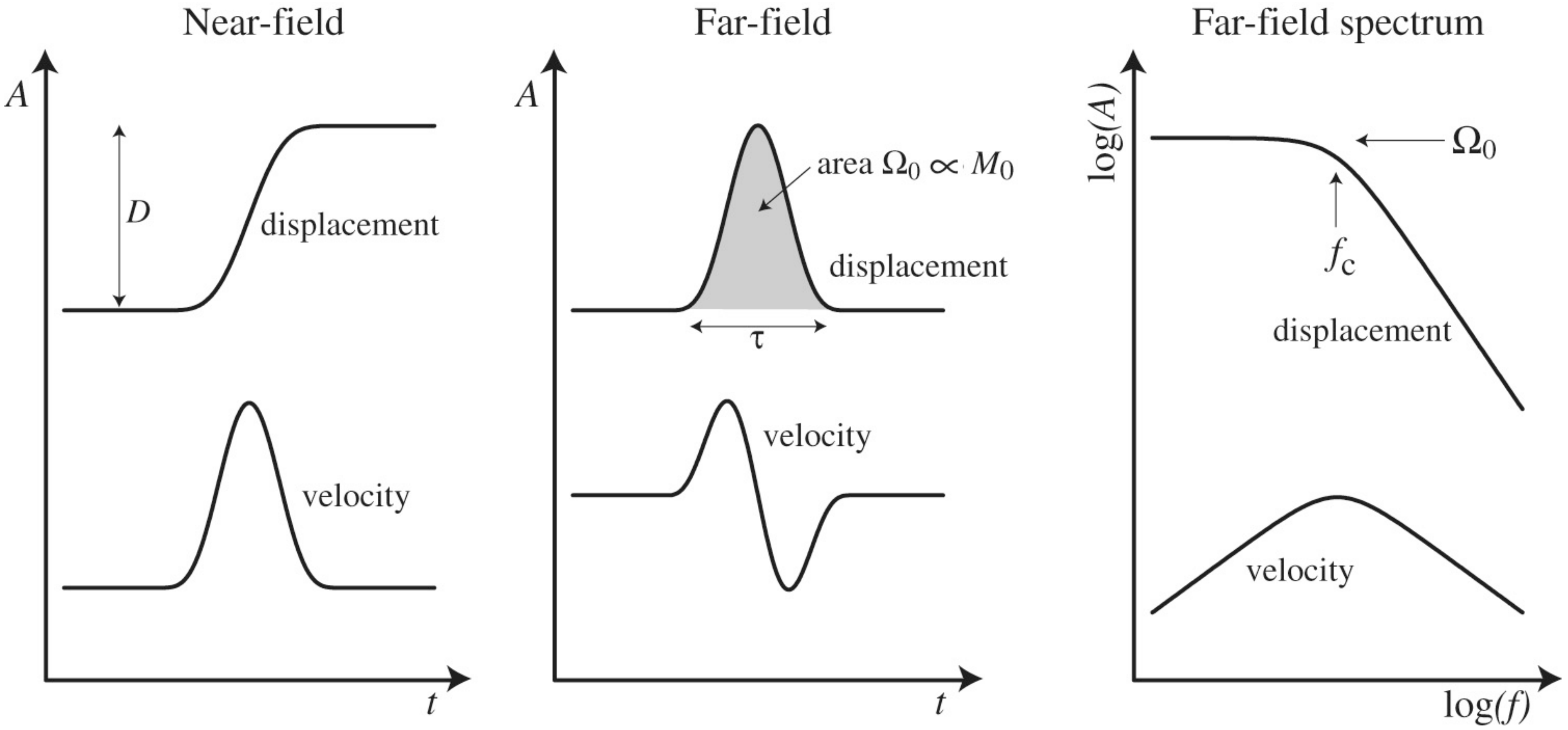
$$M_0 = \mu \langle \Delta u(t) \rangle A$$

M_0	seismic moment
m	rigidity
$<\Delta u(t)>$	average slip
A	fault area

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}, t) = & \frac{1}{4\pi\rho} \boldsymbol{A}^N \frac{1}{r^4} \int_{r/v_P}^{r/v_S} \tau M_0(t-\tau)\, d\tau \\ & + \frac{1}{4\pi\rho v_P^2} \boldsymbol{A}^{IP} \frac{1}{r^2} M_0(t - r/v_P) \\ & + \frac{1}{4\pi\rho v_S^2} \boldsymbol{A}^{IS} \frac{1}{r^2} M_0(t - r/v_S) \\ & + \frac{1}{4\pi\rho v_P^3} \boldsymbol{A}^{FP} \frac{1}{r} \dot{M}_0(t - r/v_P) \\ & + \frac{1}{4\pi\rho v_S^3} \boldsymbol{A}^{FS} \frac{1}{r} \dot{M}_0(t - r/v_S). \end{aligned}$$

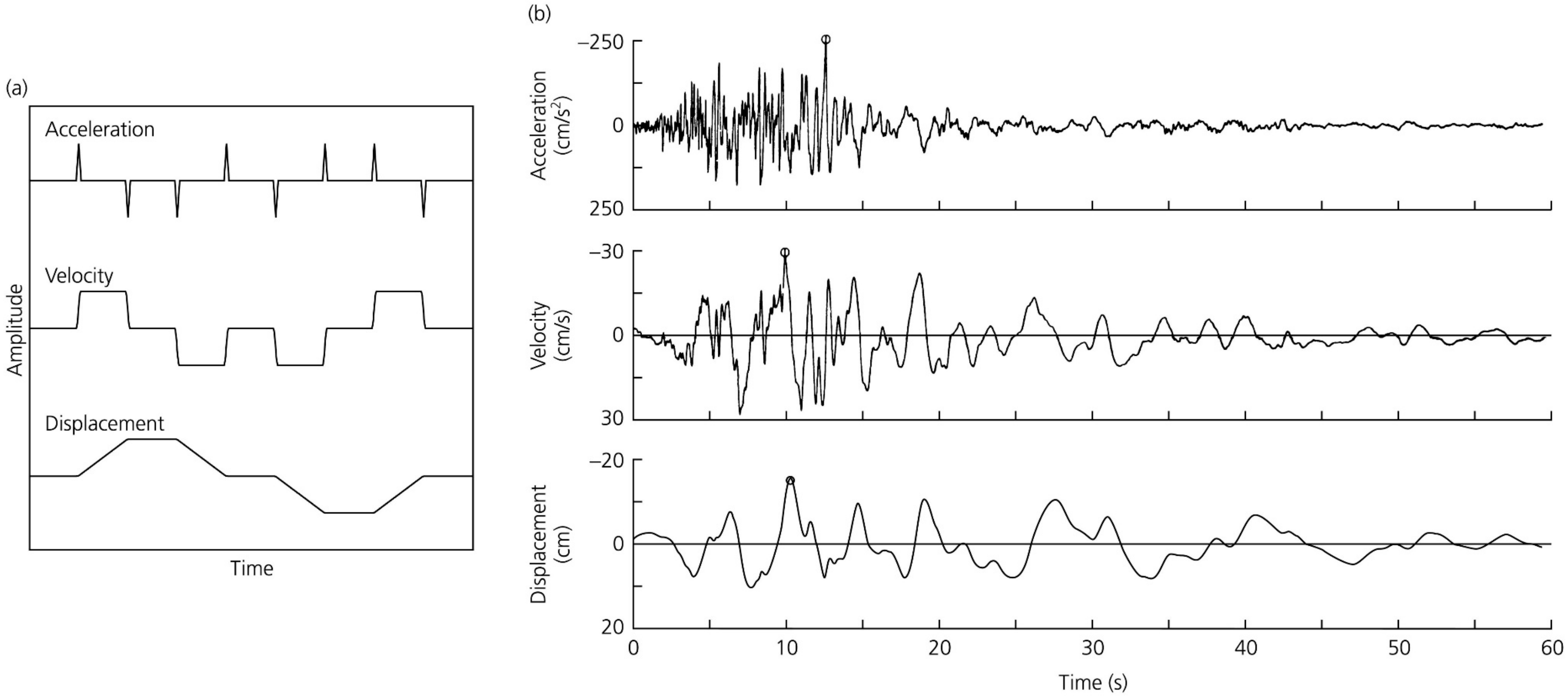
Note that the far-field displacement is proportional to the moment rate!

Source time function



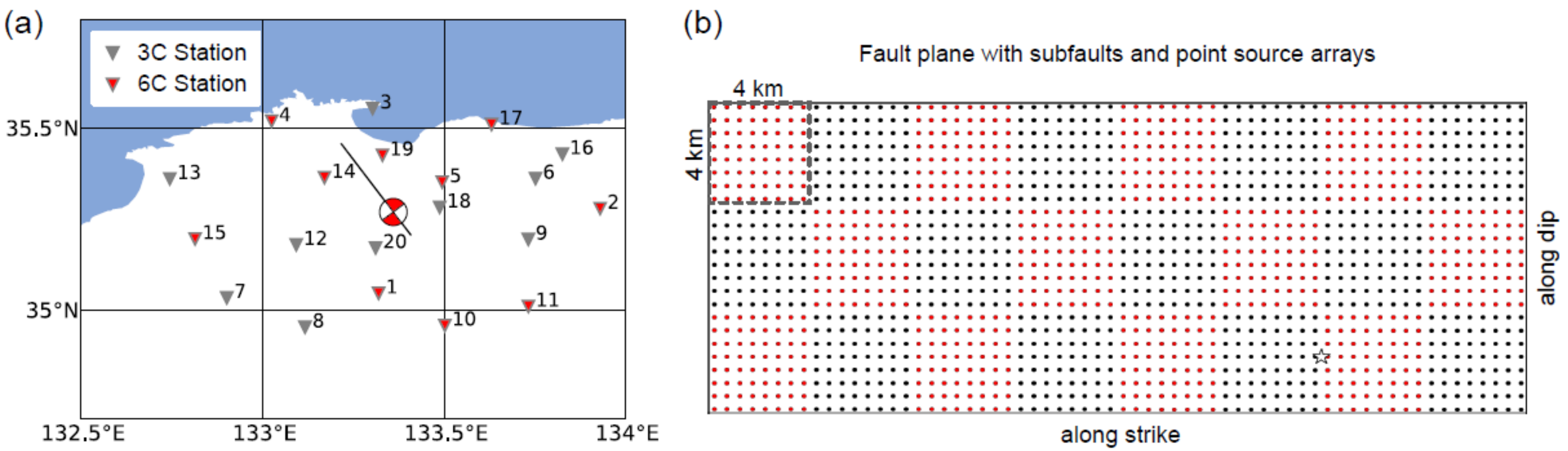
Displacement, Velocity, Acceleration

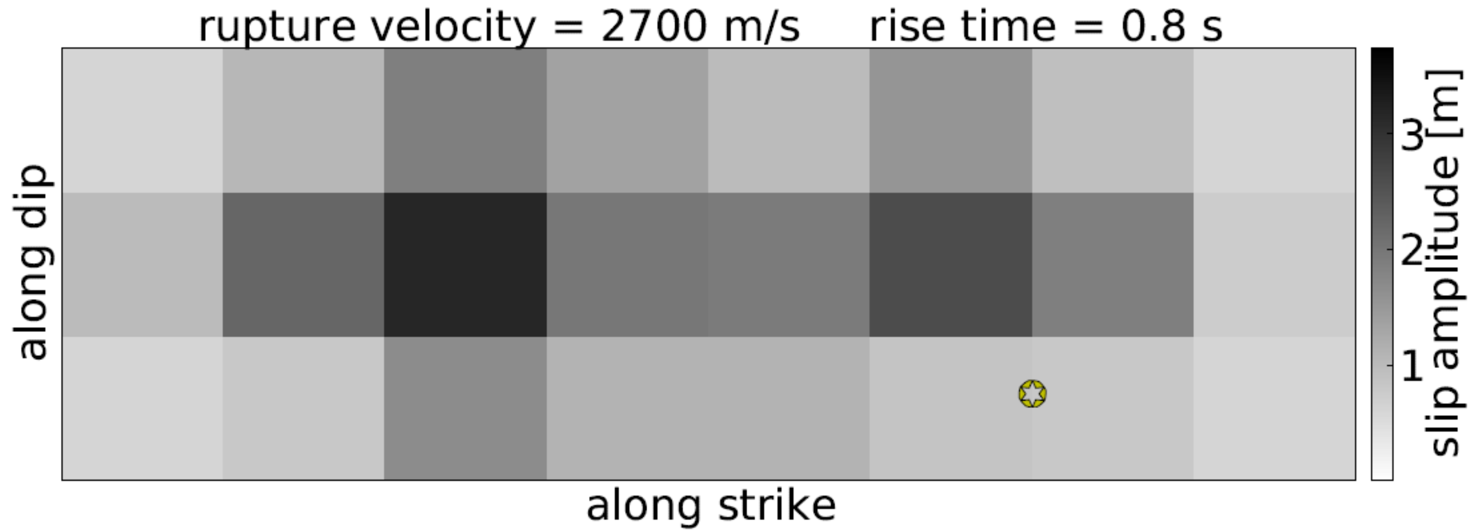
Figure 6.6-14: Relation between displacement, velocity, and acceleration in the time domain.



The superposition principle

Discrete representation of finite sources





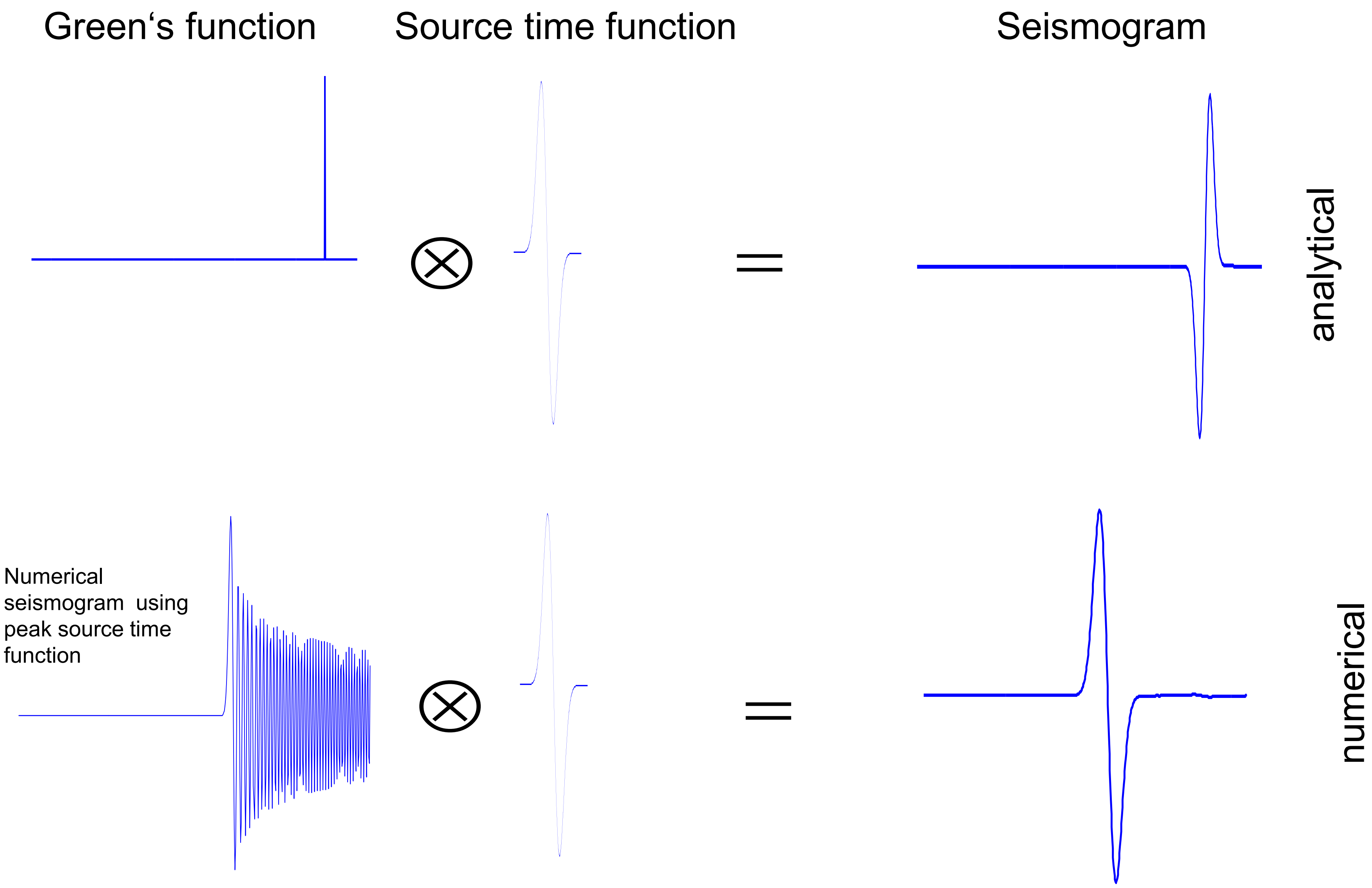
Superposition principle

We allow each subfault to slip once and parameterize the slip process in terms of slip amplitude ($slip_k$), rupture velocity (c^{rup}) and rise time (R). The slip amplitude is heterogeneous across the fault plane, leading to 24 free parameters. Together with the distance between the center of subfault k and the hypocenter, the rupture velocity provides the rupture time $t_k(c^{rup})$ of subfault k. The rise time expresses the duration of the slip. Both rupture velocity and rise time are homogeneous parameters across the fault plane. Thus, we invert for 26 free parameters in total. Finally, the complete seismic response, $v_l^r(\omega)$, at station r, component l and for the circular frequency, $\omega = 2\pi f$, is computed as a linear sum of $N(= 24)$ subfault contributions

$$v_l^r(\omega) = \sum_{k=1}^{N} slip_k \exp[-i\omega\, t_k(c^{rup})]\, G_{kl}^r(\omega)\, S(R,\omega). \qquad (2)$$

In equation (2) S represents the source function that we implemented as an ordinary ramp function. Additional details on the source function are provided in Appendix B.

The Earth (or a numerical solver) as a linear system


Green‘s function
Source time function
Seismogram
analytical
Numerical seismogram using peak source time function
numerical

Source-receiver reciprocity

[redacted]

The displacement field generated by a distribution of body forces and surface tractions can be synthesized using the elastodynamic Green function $G_{ij}(\boldsymbol{x}, t; \boldsymbol{x}', t')$, giving the i component of displacement at $(\boldsymbol{x}, t)$ due to a localized unit body force operating at $(\boldsymbol{x}', t')$ in the j direction. The elastodynamic Green function satisfies the Navier equation of motion for a linear elastic solid

$$\rho \frac{\partial^2}{\partial t^2} G_{ij} = \delta_{ij}\delta(\boldsymbol{x} - \boldsymbol{x}')\delta(t - t') + \frac{\partial}{\partial x_n}\left(c_{inkl} \frac{\partial}{\partial x_l} G_{kj}\right) \quad (1.10)$$

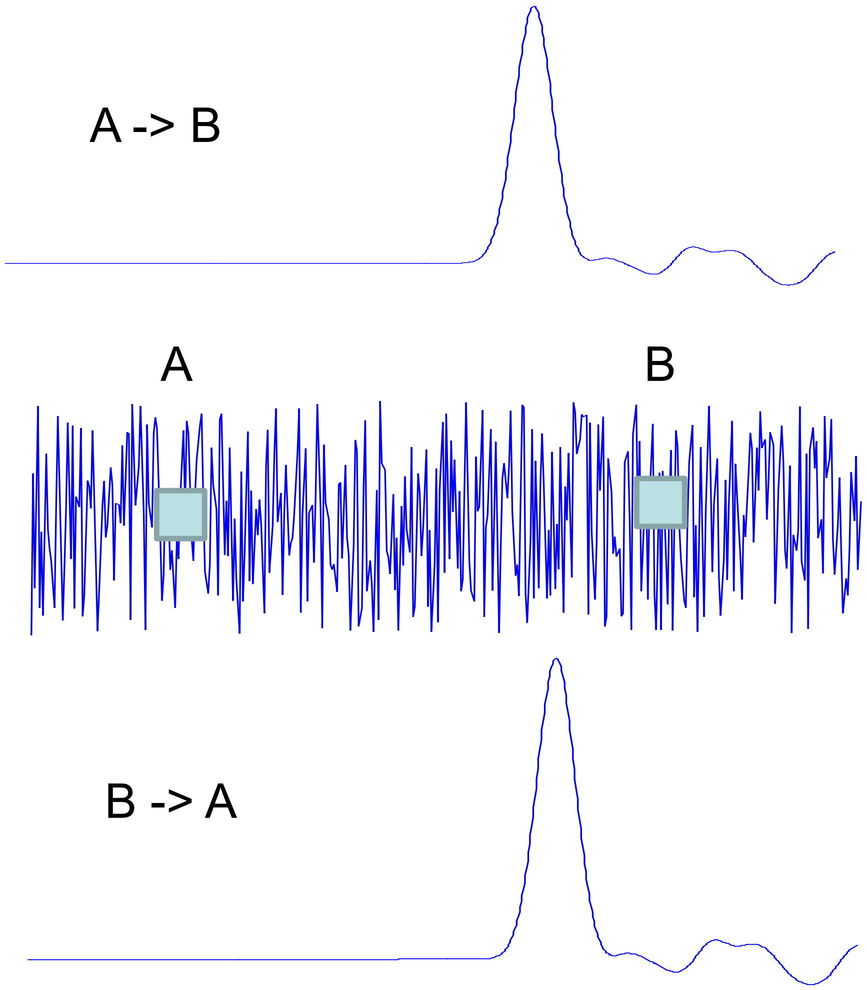
where $\delta(\,)$ is the Dirac delta function. A complete determination of G_{ij} requires meeting initial conditions (taken usually to be $G = \partial G/\partial t = 0$ for $t \leq t'$ and $\boldsymbol{x} \neq \boldsymbol{x}'$) and specified boundary conditions on the surface of the medium.

If G_{ij} satisfies homogeneous boundary conditions (i.e., zero traction or zero displacement) on S, it has the following spatiotemporal reciprocity properties

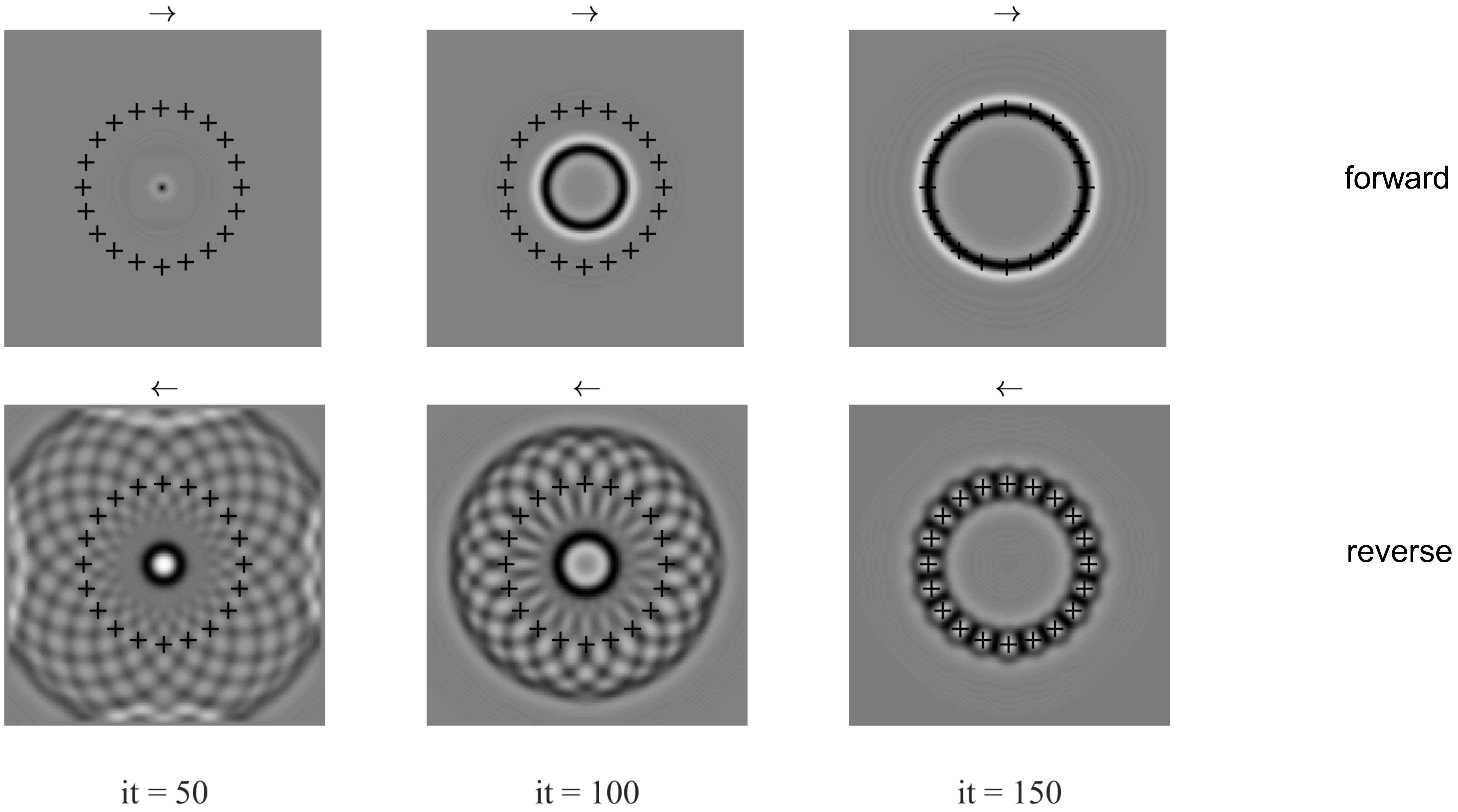
$$G_{ij}(\boldsymbol{x}, t; \boldsymbol{x}', t') = G_{ji}(\boldsymbol{x}', -t'; \boldsymbol{x}, -t). \qquad (1.11)$$

In other words

Seismogram
through
random model



Time reversal – reverse acoustics



Practical example – Valhall active experiment

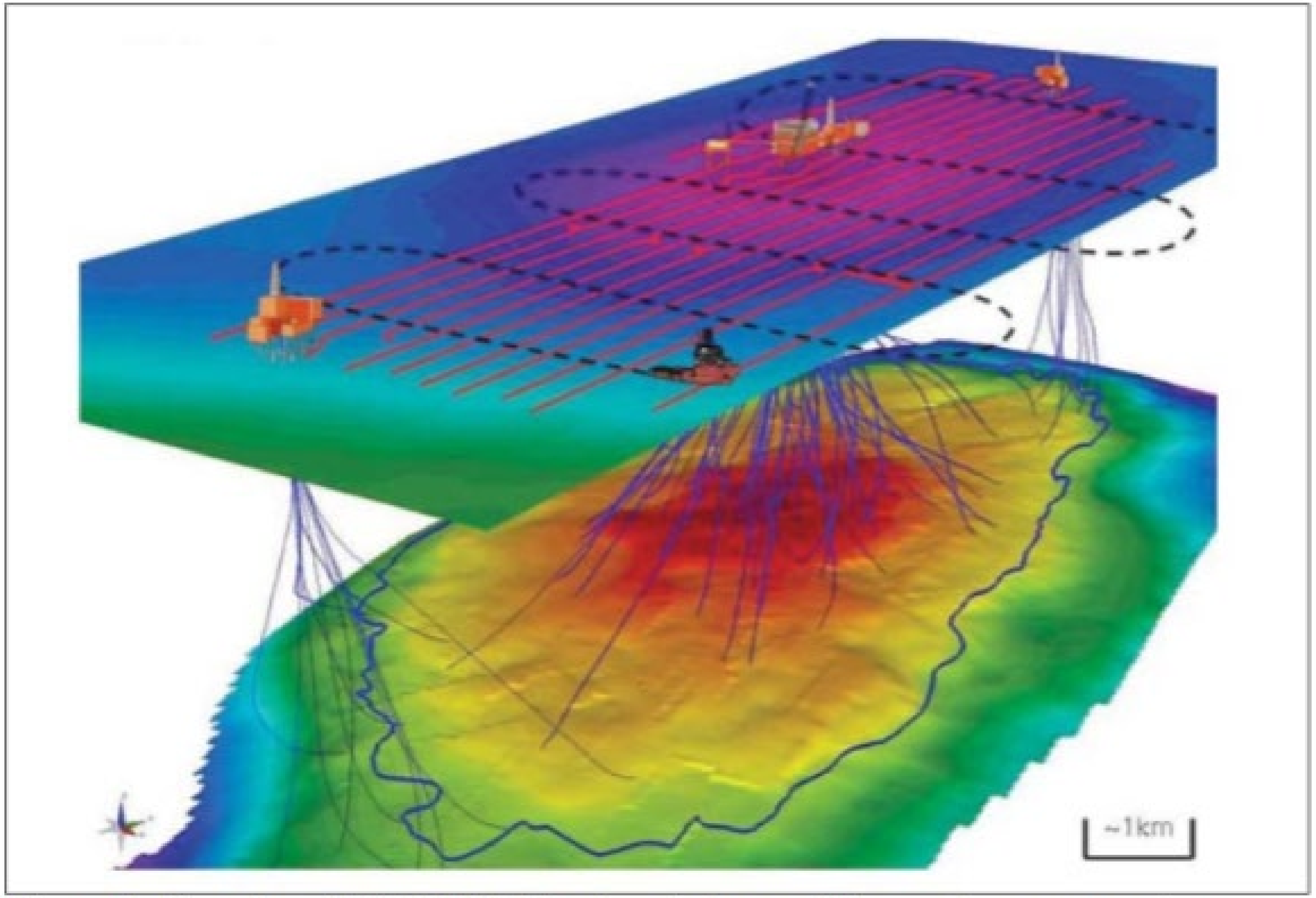
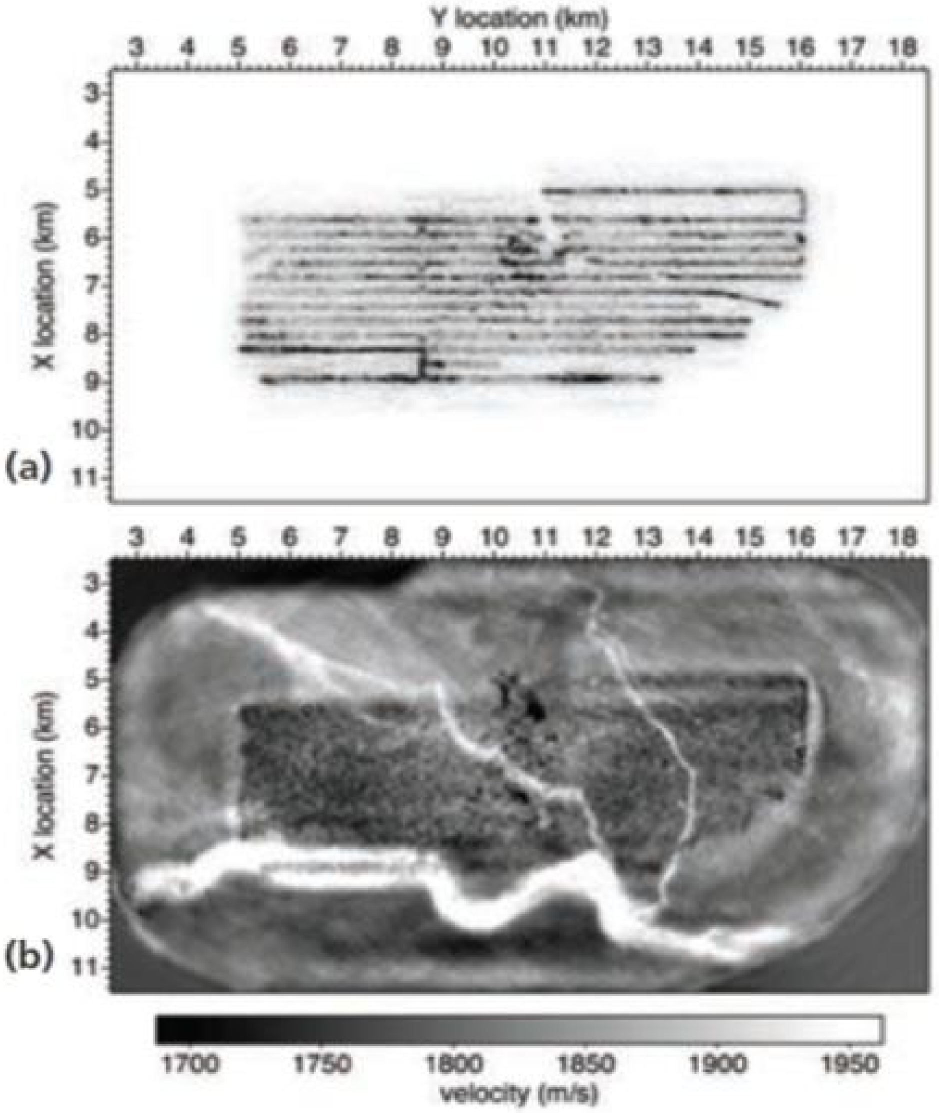


Figure 1. *Overview of Valhall Field showing the layout of the geophone array at the sea floor (red lines), the top of the reservoir, the outline of the field (dark blue line), and the wells (thin blue lines).*



Sirgue et al., 2010

Summary

To understand seismic wave propagation the following concepts need to be understood:

- The mathematical description of the **deformation** of an elastic 3-D object -> strain
- The forces that are at work for a given deformation and its (mostly linear!) dependence on the magnitude of deformation > **stress – strain relation**
- The description of elastic modules and the various symmetry systems (-> **elasticicity tensor, isotropy, transverse isotropy, hexagonal symmetry**).
- The boundary condition required at the **free surface (traction-free)** and the consequences for wave propagation -> surface waves
- The description of seismic sources using the **moment tensor** concept (-> double couples, explosions)
- The origin, scale, spectrum of material heterogeneities in side the Earth (-> the reason why we need to resort to numerical methods)