Refraction seismics – the basic formulae

1. Two-layer case

We consider the case where a layer with thickness *h* and velocity v_1 is situated over a halfspace with velocity v_2. A receiver is located at a distance Δ from the source, which itself is located at the surface. What signals will we measure, if a seismic source is generating energy (e.g. an explosion)? Here we will only consider the direct waves, reflections and refractions but no take into account multiple reverberations which would be recorded in nature (but often neglected in the processing).

The geometry of the problem looks like this:

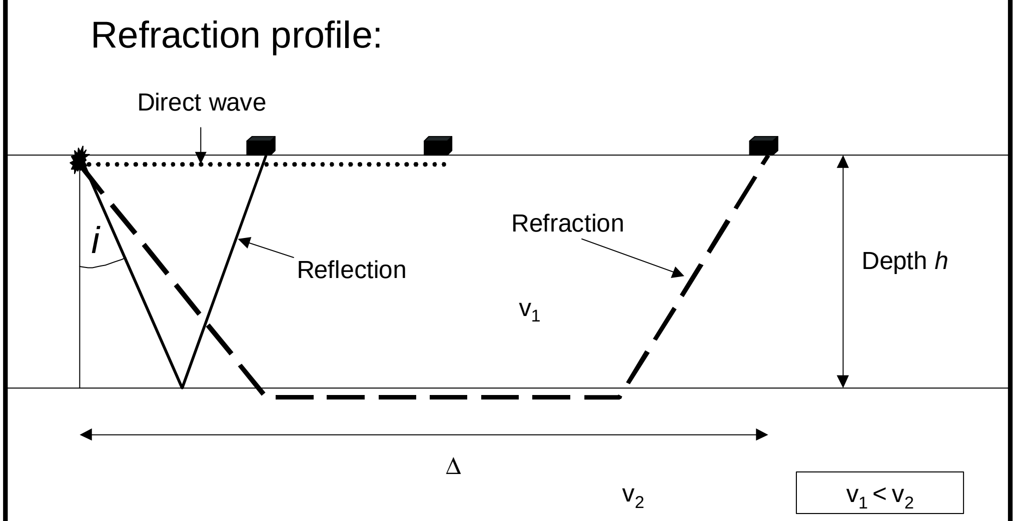

Figure 1: Geometry of reflection/refraction experiment. There are three arrivals recorded at greater distances: the direct wave, the reflection from the discontinuity at depth h and the refracted wave.

Before we try to determine the structure from observed travel times we have to understand the *forward problem*: how can we determine the travel time of the three basic rays as a function of the velocity structure and the distance from the source. The most important ingredient we need is Snell's law

$$\textbf{Snell's Law} \qquad \frac{\sin i_1}{v_1} = \frac{\sin i_2}{v_2} \qquad (1)$$

relating the incidence angle *i* in layer 1 with velocity v_1 to the transmission angle *i* in layer 2 with velocity v_2. Both angles are measured with respect to the vertical. Let us derive the arrival times for the three types separately:

1.1 The direct wave

This is the easy one! In a layered medium the direct wave travels straight along the surface with velocity v_1. At distance Δ clearly the travel time t_{dir} will be:

$$\textbf{travel time direct wave} \quad t_{dir} = \Delta / v_1 \qquad (2)$$

1.2 The reflected wave

To calculate the reflected wave we need to do a little geometry. The length of the path the ray travels in layer 1 is obviously related to the distance in a non-linear way. The travel time for the reflection is given by

$$\textbf{travel time reflected wave} \quad t_{refl} = \frac{2}{v_1}\sqrt{(\Delta/2)^2 + h^2} \qquad (3)$$

In refraction seismology this arrival is often of minor interest, as the distances are so large that the reflected wave has merged with the direct wave. Note that this has the form of a hyperbola.

1.3 The refracted wave

As we can easily see from the figure above the refracted wave needs a more involved treatment. Refracted waves correspond to energy which propagates horizontally in medium 2 with the velocity v_2. This can only happen if the emergence angle i_2 is 90°, i.e.

$$\textbf{critical angle} \quad \frac{\sin i_c}{v_1} = \frac{\sin 90°}{v_2} = \frac{1}{v_2} \Rightarrow \sin i_c = \frac{v_1}{v_2} \qquad (4)$$

where i_c is the *critical angle*. So in order to calculate the travel time we need to consider rays which impinge on the discontinuity with angle i_c. From elementary geometry it follows that the arrival time t_{refr} of the refracted wave as a function of distance Δ is given by

$$\textbf{Travel time refracted wave} \quad t_{refr} = \frac{2h\cos i_c}{v_1} + \frac{\Delta}{v_2} = t^i_{refr} + \frac{\Delta}{v_2} \qquad (5)$$

which is a straight line which crosses the time axis $\Delta=0$ at the *intercept time* t^i_{refr} and has a slope $1/v_2$.

1.4 Travel time curves – the forward problem

Now we can put things together and calculate – for a given velocity model – the arrival times and plot them in a travel-time diagram. Example:

The model parameters are:

$h=30$km
$v_1=5$km/s
$v_2=8$km/s

This could correspond to a very simple model of crust and upper mantle and the discontinuity would be the Moho. The distance at which the refracted arrival overtakes the direct arrival can be used to determine the layer depth. According to ray theory there is a minimal distance at which the refracted wave can be observed, this is called the *critical distance* (see below).

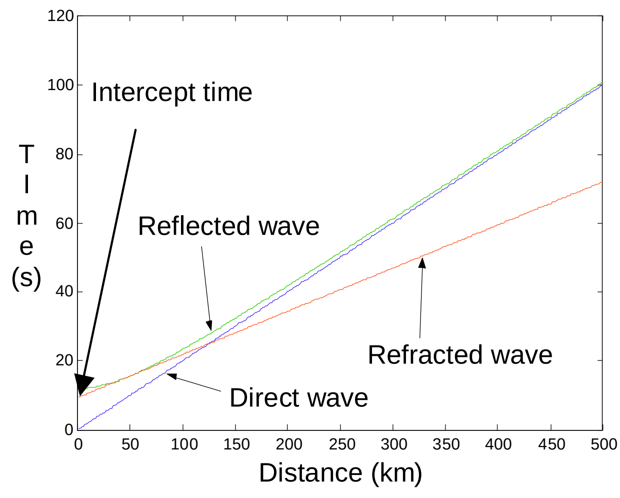

Figure 2: Travel-time diagram for the two-layer case.

1.5 Critical distance and overtaking distance

Two concepts are useful when determining the depth of the top layer. The *critical distance* is the distance at which the refracted wave is first observed according to ray theory (in real life it is observed already at smaller distances, this is due to *finite-frequency* effects which are not taken into account by standard ray theory). The critical distance Δ_c is from basic geometry

$$\textbf{critical distance} \qquad \Delta_c = 2h \tan i_c \qquad (5)$$

where the critical angle i_c is given by equation (4). If we equate the arrival time of the direct wave and the refracted wave and solve for the distance we obtain the overtaking distance. It is given by

$$\textbf{overtaking distance} \quad \Delta_u = 2h\sqrt{\frac{v_2 + v_1}{v_2 - v_1}} \qquad . \quad (6)$$

1.6 Determining the structure from travel-time diagrams: the inverse problem

The problem: determine the velocity depth model from the observed travel times (Figure 2). We proceed as follows:

a. Determine v_1 from the slope ($1/v_1$) of the direct wave.
b. Determine v_2 from the slope ($1/v_2$) of the refracted wave.
c. Calculate the critical angle from v_1 and v_2.
d. Read the intercept time t_i from the travel-time diagram.
e. Determine the depth h using equation (5), thus

$$h = \frac{v_1 t_i}{2\cos i_c} \qquad . \quad (7)$$

or

f. Read the overtaking distance from the travel-time diagram, and calculate h using equation (6).

2. Three-layer case

The three layer case is important for many realistic problems, particularly for near surface seismics, where often a low velocity weathering layer is on top of the bedrock. In principle we follow the same reasoning as before but through the additional layer the algebra is a little more involved. We have to introduce a slightly different nomenclature to take into account the different layers. The incidence angles will have two indices, the first index stands for the layer in which the angle is defined and the last index corresponds to the layer in which the ray is refracted (see Figure 3). The equation for the direct waves is of course the same as in the two-layer case. The same is true for the refraction from layer 2 but we show it to demonstrate the nomenclature.

2.1 The refraction from layer 2

The arrival time t_2 of the refraction from layer 2 is given by

$$t_2 = \frac{2h_1 \cos i_{12}}{v_1} + \frac{\Delta}{v_2} = t^{i2} + \frac{\Delta}{v_2} \qquad (8)$$

and - using the intercept time from the diagram - will allow us to determine the depth h_1 of the topmost layer.

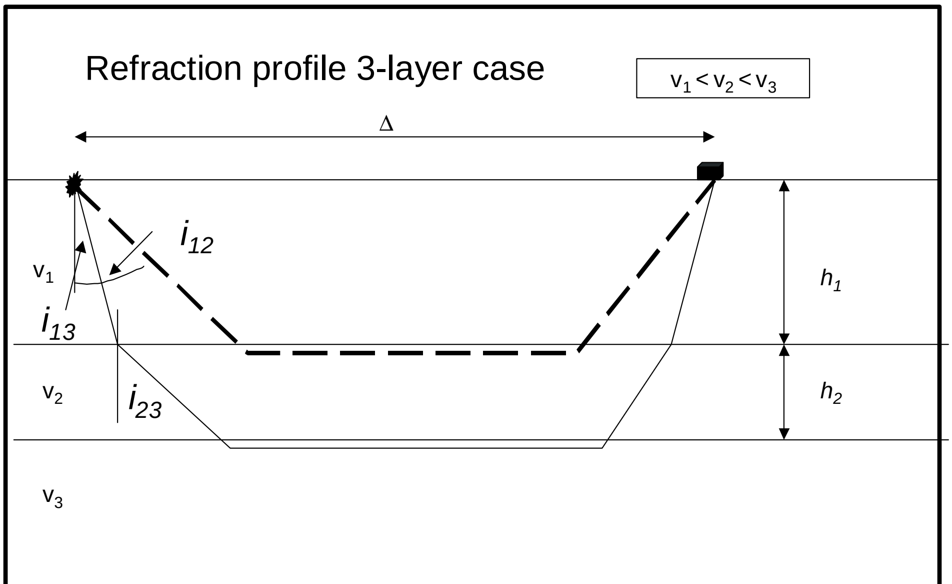

Figure 3: Geometry of 3-layer refraction experiment.

2.2 The refraction from layer 3

Due to Snell's law we have

$$\frac{\sin i_{13}}{v_1} = \frac{\sin i_{23}}{v_2} = \frac{\sin i_{33}}{v_3} = \frac{1}{v_3} \qquad (9)$$

we use this relation and basic trigonometry to derive the arrival time t_3 of the refracted wave in layer 3

$$t_3 = \underbrace{\frac{2h_1 \cos i_{13}}{v_1} + \frac{2h_2 \cos i_{23}}{v_2}}_{t^{i3}} + \frac{\Delta}{v_3} = t^{i3} + \frac{\Delta}{v_3} \qquad (10)$$

and again this is a straight line with the intercept time t^{i3} which can be read from the travel time diagram.

2.3 Determining the velocity depth model for the 3-layer case

As before our data is a diagram with the travel-times of the direct wave, the refraction from layer 2 and the refraction from layer 3 (provided we were able to read the arrivals in the seismograms). To determine the velocities and the thicknesses of layers 1 and 2 we proceed as follows:

a. Determine the velocities $v_{1\text{-}3}$ from the slopes ($1/v_{1\text{-}3}$) in the travel-time diagram.
b. Read the intercept time t^{i2} for the refraction from layer 2.
c. Determine thickness h_1 - using equation (8) such that

$$h_1 = \frac{v_1 t^{i2}}{2\cos i_{12}} \quad \text{, where} \quad i_{12} = \arcsin \frac{v_1}{v_2} \qquad (11)$$

d. Read the intercept time t^{i3} for the refraction from layer 3.
e. Calculate with the already determined values h_1 an intermediate intercept time t*

$$t^* = t^{i3} - \frac{2h_1 \cos i_{13}}{v_1} \quad \text{, where} \quad i_{13} = \arcsin\frac{v_1}{v_3} \qquad (12)$$

f. Using t^* calculate the thickness h_2 of layer 2

$$h_2 = \frac{v_2 t^*}{2\cos i_{23}} \quad \text{, where} \quad i_{23} = \arcsin\frac{v_2}{v_3} \qquad (13)$$

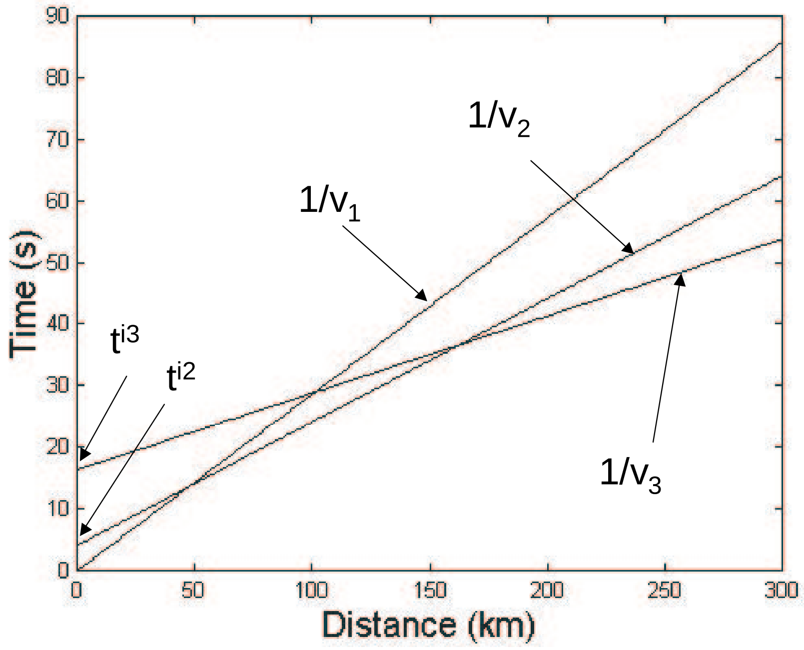

Figure 4: Travel-time diagram for the 3-layer case

In the model shown in Figure 4 the velocites are v_1=3.5km/s, v_2=5km/s, v_3=8km/s. The layer thicknesses are h_1=10km and h_2=25km.

3. Reduced time

In refraction seismology as well as in global seismology we often find travel-time diagrams where *reduced time* is used. In principle this means that the refraction arrival of interest is approximately horizontal in the travel-time diagram. This can be achieved by doing the following transformation

$$t_{red} = t - \frac{\Delta}{v_{red}} \qquad (14)$$

where v_{red} is the reduction velocity. How can we determine the real velocity from the travel-time diagrams in reduced form?

a. Choose a distance Δ_0 and read the reduced travel time t_{r0} from the diagram for the desired arrival.
b. Calculate the velocity using

$$v = \frac{\Delta_0}{t_{r0} + \frac{\Delta_0}{v_{red}} - t_i} \qquad (15)$$

where t_i is the intercept time. Note that the intercept time does not change when using reduced time!

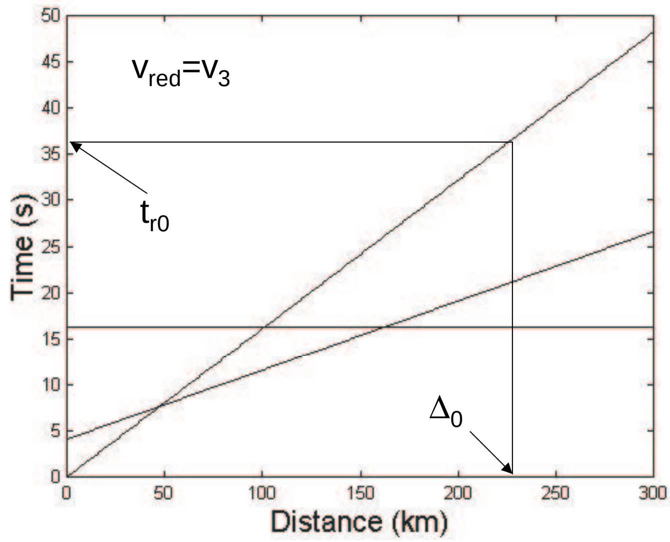

Figure 4: Travel-time diagram for the 3-layer case in reduced form for the same model as before.

To determine he velocity-depth structure from a travel-time diagram in reduced form you can - after having calculated the *real* velocities using equation (15) - follow the steps given in section 2.3.

4. Inclined 2-layer case

So far we have only considered plane layers with no structural variation along the profile. In this chapter we consider the case where a high-velocity layer is inclined with an inclination angle α (see Figure 5). The most important difference to thw previous examples (2-layer and 3-layer cases) is, that we now perform two experiments, one shooting at the near end and one shooting at the far end of the region of interest. Note that for the previous examples - due to symmetry - we would have observed the same travel time curves. For the case of an inclined layer this is no longer the case!

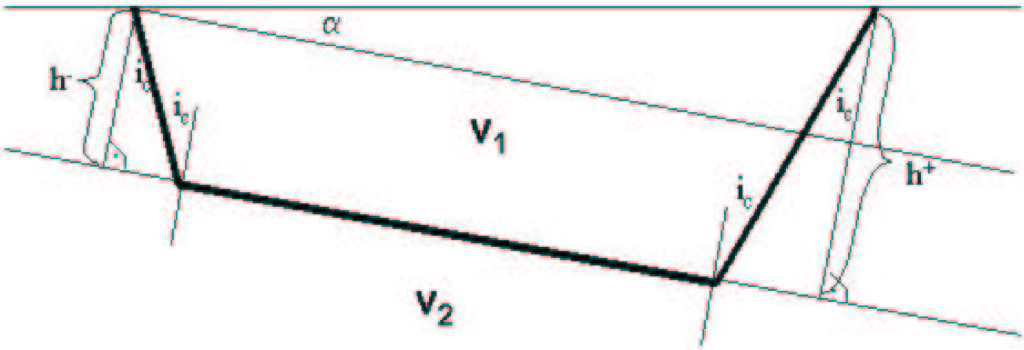

Figure 5: Refracted ray path for the inclined 2-layer case.

Let us develop the *forward problem*, i.e. calculating the travel times of the direct and refracted waves for a given model. With seismic velocities v_1 and v_2 and inclination angle α the travel time of the refracted waves are

$$t_{refr}^{-} = \frac{2h^{-}\cos i_c}{v_1} + \frac{\sin(i_c+\alpha)}{v_1}\Delta = t_i^{-} + \frac{1}{v_2^{-}}\Delta$$

$$t_{refr}^{+} = \frac{2h^{+}\cos i_c}{v_1} + \frac{\sin(i_c-\alpha)}{v_1}\Delta = t_i^{+} + \frac{1}{v_2^{+}}\Delta$$

where the (-) sign stands for the refracted arrival with **smaller** intercept time and the (+) sign for the refraction with larger intercept time, i_c is the critical angle at the interface and h^+ and h^- are defined according to Figure 5. Note that - as in all previous cases - the arrival of the direct wave is at time $t_{dir}=\Delta/v_1$. An example for the travel time curves that will be observed for a model with α=8deg, v_1=1.2km/s and v_2=4km/s is shown in Figure 6.

But how can we determine the model properties from the observed arrival times (the *inverse* problem)? Here is how you should proceed:

a. Determine the velocities v_1 and $v_2^{+/-}$ from the slopes in the travel-time diagram.
b. Use the following relations to determine α and v_2:

$$\sin(i_c+\alpha) = \frac{v_1}{v_2^{-}} \Rightarrow i_c + \alpha = \arcsin\frac{v_1}{v_2^{-}}$$

$$\sin(i_c-\alpha) = \frac{v_1}{v_2^{+}} \Rightarrow i_c - \alpha = \arcsin\frac{v_1}{v_2^{+}}$$

$$\frac{(i+\alpha)+(i-\alpha)}{2} = i \Rightarrow v_2 = \frac{v_1}{\sin i}$$
$$\frac{(i+\alpha)-(i-\alpha)}{2} = \alpha$$

c. Read the intercept times t_i^+ and t_i^- from the travel time diagram. Determine the distances from the layer interface as

$$h^- = \frac{v_1 t_i^-}{2\cos i_c}$$
$$h^+ = \frac{v_1 t_i^+}{2\cos i_c}$$

d. You can now graphically draw the layer interface by drawing circles around the profile ends with the corresponding heights $h^{+/-}$ and tangentially connecting the circles at depth.

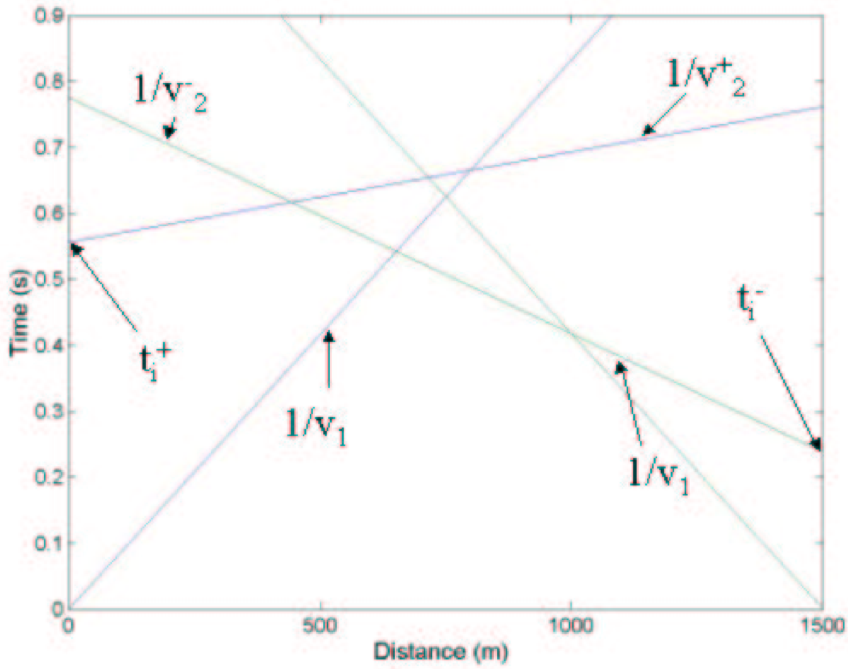

Figure 6: Travel-time diagram for the inclined-layer case for the model in the previous figure.