Bulletin of the Seismological Society of America, Vol. 99, No. 2B, pp. 1137–1146, May 2009, doi: 10.1785/0120080160

Spinors and Twistors in the Description of Rotational Seismic Waves and Spin and Twist Solitons

by Eugeniusz Majewski

Abstract A noncommutative (anti-) self-dual Yang–Mills theory as a source of multisoliton solutions of nonlinear wave equations was applied to the description of rotational seismic waves that are excited in the earthquake source. Spinors and twistors are used to describe spin and twist solitons branching off dispersion curves for rotational seismic waves. Complex physical structures are adopted to describe spin and twist effects resulting from the presence of translational and rotational defects in elastic rocks. A seismic space is also assumed to have a complex structure. An earthquake source zone is modeled by a set of equations for interacting fields that is mathematically similar to the noncommutative (anti-) self-dual Yang–Mills equations. Some similarities between dislocations and strings are emphasized, for example, those that exist between surface defects and D-branes in string theories. Dislocations and disclinations are treated as sources for seismic spin and twist fields. By symmetry reduction various soliton equations for seismic spin and twist solitons can be obtained from the set of earthquake source zone equations, which is similar to the noncommutative (anti-) self-dual Yang–Mills equations by symmetry reduction.

Introduction

There is a lot of observational evidence concerning rotational motions on the Earth's surface excited by earthquakes. From observations of rotations of tombstones, chimneys, small pyramids of wooden blocks, St. Bruno obelisks, stone lanterns, vases, and many other objects on the Earth's surface, we can infer the existence of rotational motions in earthquake focal zones. Rotational motions are present in many other natural phenomena and processes. Beyond earthquakes (see Majewski, 2006a), some other tectonic, volcanic, mining, and land sliding events can be sources of rotational motions as well. It is observed that rotational motions (spin and twist) in earthquake sources naturally excite rotational seismic waves and solitons (Majewski, 2008a). The key idea of this article consists in extending the description of rotational seismic waves into a nonlinear regime and employing results known from Yang-Mills theory and its relation to soliton theory. The goal of this article is to describe rotational seismic waves and spin and twist solitons excited in earthquake sources. To model an earthquake source zone, we apply a set of equations that is mathematically similar to the noncommutative (anti-) self-dual Yang–Mills (NC ASDYM) equations (Majewski, 2008b). The NC version of ASDYM theory was adopted here in order to avoid some singularities. It is also connected with complexification of the space-time. We need such a complex space-time to describe rotational seismic fields that are modeled here as complex fields. It results from an analogy between complex electromagnetic fields and complex rotational fields. The NC ASDYM equations are interesting for a number of reasons, and we will discuss the NC ASDYM theory in some detail. We consider a nonlinear elastic medium with translational and rotational defects in the form of dislocations and disclinations and surface defects. Such a medium allows the excitation and propagation of nonlinear rotational seismic waves and solitons. The NC version of the ASDYM-like equation has an extremely rich structure and can describe various physical fields as, for example, seismic spin and twist fields. These equations can also describe interactions between the fields. We concentrate mostly on describing the seismic spin and twist waves and solitons. The considered version of the NC ASDYM-like equation is treated here as a master equation that can be reduced to some soliton equations for spin and twist solitons. It is well known in the literature that the Bäcklund transformations and solution generating techniques in the form of Atiyah–Ward ansatz solutions in terms of quasi determinants based on a symmetry reduction of the NC ASDYM equation in supergravity allows one to derive a plethora of NC soliton equations. We adopt this technique to formulate differential equations for rotational seismic waves and solitons. As a result, we obtain nonlinear equations describing rotational seismic waves propagating in the elastic Earth modeled as a nonlinear elastic medium with translational and rotational defects. We can see that the influence of hidden symmetries in the NC ASDYM equations is visible in affecting the form of NC solitons. The NC ASDYM theory provides us with the formalism that is essential in the description of families of soliton equations. The derived nonlinear equations reveal the interplay between the nonlinearity and dispersion. In the formulation of the NC ASDYM theory, we use spinors and twistors. We should emphasize that, in spite of the semantic association, there is no direct association of spin and twist waves with spinors and twistors; thus, there is no association: spin wave—spinors, twist wave—twistors. However, it is noteworthy that the NC ASDYM theory is part of a twistor string theory. We point out some similarities between dislocations and strings and between D-branes from string theories and surface defects. For some wavelength to grain size ratios, the seismic waves propagate as seismic solitary waves or seismic solitons. We distinguish two kinds of rotational seismic waves: (1) rotational longitudinal waves, that is, *PR* waves; and (2) rotational shear waves, that is, *SR* waves. Rotational seismic waves propagate faster in solid rocks and much slower in fractured media along tectonic faults. The slow rotational tectonic waves propagate along the fractured tectonic fault with a speed of about 1 km/day. It has been observed that these waves may have a form of rotational seismic solitons and that they can trigger earthquakes (Majewski, 2008a). Because of the fact that solitons can propagate without any loss of energy, these waves are extremely important carriers of seismic energy.

Newman and Penrose (1962), Penrose (1968, 1983), and Penrose and Rindler (1986) applied twistors to describe massless spinning particles and light rays. They pointed out that twistors can describe twisted photons, nonlinear structure of graviton, or charges for massless spin-3/2 fields (gravitino). Because of the well-known particle-soliton duality, it seems reasonable to relate twistors with spin and twist solitons. We adopt the twistor quantization theory developed by Penrose and Rindler (1986) and employ twistors to describe spin and twist solitons, that is, quanta of spin and twist energy (see Majewski, 2006a,b,c,d,e). Twistors can also describe seismic rays.

Spin and Twist Motions and Waves Excited in Earthquake Sources

The existence of rotational motions excited by earthquakes was proposed by Teisseyre (1973, 1974). He constructed the first rotational seismogram (Teisseyre, 1974) in an indirect way by using an azimuthal array of horizontal seismographs. Droste and Teisseyre (1976) derived rotational seismograms of rock bursts from a nearby coal mine in Upper Silesia in Poland. Later on, Teisseyre *et al.* (2003) were recording rotational seismic waves of different events in the very near field. For the first time the rotational seismic motions were directly recorded (Stedman *et al.*, 1995; Stedman, 1997) using a ring laser in Christchurch, New Zealand, and in Wetzell, Germany (Igel *et al.*, 2005; Cochard *et al.*, 2006; Schreiber *et al.*, 2006). Moreover, Takeo (2006) was measuring rotational motions in a near field. Measurements of seismic spin and twist waves were conducted by Teisseyre and Suchcicki (2006) and Teisseyre (2007).

Most earthquakes occur under a certain high level of confined pressure. The constitutive law during an earthquake is controlled by a macroscopic property of the fault such as macroscopic roughness of the fault, thickness of the fault gouge layer, geometry of the fault, the macroscopic change of the fault strength, and so on.

Torques generated by force couples that yield spin motions in an earthquake source are illustrated in Figure 1. An earthquake fault surface is formed as a shear surface during an earthquake. As the result of a motion of tectonic plates along the fault surface in opposite directions, two force couples generate spin motions. The force couple S_1 is perpendicular to the force couple S_2. Because of the confine pressure acting in the direction of the compression forces, the action of the force couple S_1 is enforced. Eventually, the spins S_1 and S_2 become sources of seismic spin waves and spin solitons (see Majewski, 2006b).

Because of strong inhomogeneities of rocks in the fault zone, the shear direction and with it the directions of the prin-

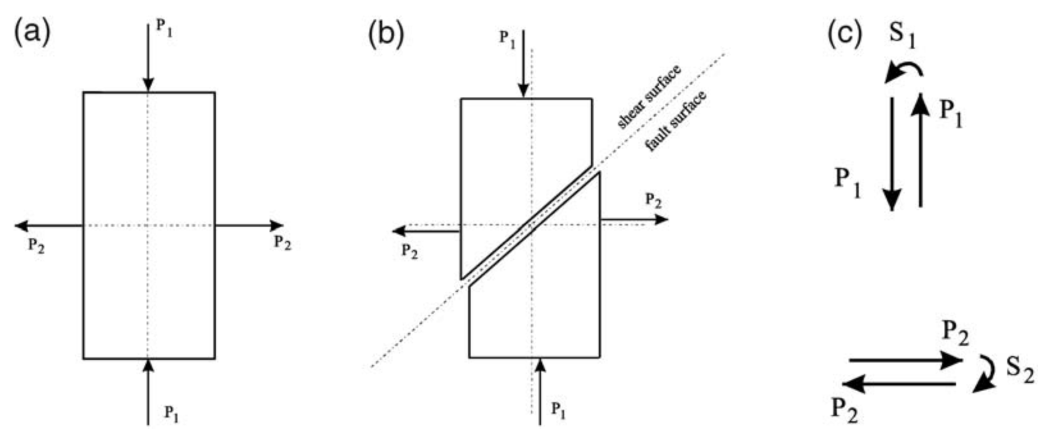

Figure 1. Torques generated by force couples that yield spin motions in the earthquake source. (a) Before an earthquake, (b) during an earthquake, and (c) two force couples generating spin motions.

cipal shear stress axes are changing around the prevailing shear stress directions (see Teisseyre, 2009; Teisseyre and Górski, 2009). These changes have a twist nature (see Fig. 2), and they are sources of seismic twist waves and twist solitons.

The Nonlinear Rotational Seismic-Wave Theory: Application of the (Anti-) Self-Dual Yang–Mills Fields

We propose a nonlinear rotational seismic-wave theory to describe processes in the earthquake source zone. A mathematical formalism of the ASDYM is adopted here. Rotational seismic fields (spin and twist) are described in the language of the Yang–Mills theory. The mathematical formulation of the ASDYM theory is ideal for our purposes because it can describe integrable systems; thus, this is exactly what we need in seismology. This formalism is well suited for nonlinear seismic waves and describes a systematic approach to deriving soliton equations. In particular, the ASDYM theory was frequently applied to describe chiral fields, that is, rotational fields. In addition, spin coefficients play an important part in gravitational radiation (Newman and Penrose, 1962). In particular, spinors and twistors are very convenient tools in our analysis of spin and twist motions in rotational seismology.

We assume that the seismic space is the four-dimensional space $\mathbb{R}^4$ characterized by the metric (see Lechtenfeld and Popov, 2007)

$$ds^2 = \mathbf{g}_{\mu\nu} dx^\mu dx^\nu = \det(dx^{AA'}) = dx^{11'}dx^{22'} - dx^{21'}dx^{12'}, \quad (1)$$

where $\mathbf{g}_{\mu\nu}$ is the metric tensor and μ, $\nu = 0$, 1, 2, and 3 are space-time indices and $A = 1$, 2 and $A' = 1'$, $2'$ are spinor indices.

We have a Yang–Mills field defined by

$$\Re_{\mu\nu} = [\nabla_\mu, \nabla_\nu] = \mathbf{V}_{\nu,\mu} - \mathbf{V}_{\mu,\nu} + [\mathbf{V}_\mu, \mathbf{V}_\nu], \quad (2)$$

where $\Re_{\mu\nu}$ is the rotational field strength tensor and $\mathbf{V}^i_\mu$ represents a gauge field (Mason *et al.*, 1995).

We postulate the field to be self-dual

$$\Re_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\kappa\rho}\Re^{\kappa\rho} \qquad \text{or} \qquad \Re = *\Re, \quad (3)$$

where $\epsilon_{\mu\nu\kappa\rho}$ is a completely antisymmetric tensor.

Here, the gauge field $\mathbf{V}^i_\mu$ is a rotational field. Its upper index is related to its Lie algebraic nature, while its lower index is a space-time one. This field is real in the special unitary group SU(2) Lie algebra, because a proper choice of a basis in this algebra allows us to make these coefficients real. We should emphasize that we try to use different formalisms connected with complexification of the space-time. We need a complex space-time in order to describe a complex rotational seismic field $\omega_{\mu\nu}$. We should add that we want the Yang–Mills field to be defined in a NC space-time in order to avoid some singularities (Hamanaka, 2006). In a gauge theory the potentials $\mathbf{V}^i_\mu$ can be extended into a complex space with the complex coordinates y, $\tilde{y}$, z, and $\tilde{z}$. Now, the self-duality equations (3) and (4) can be expressed as

$$\Re_{y\tilde{y}} - \Re_{z\tilde{z}} = 0, \qquad \Re_{yz} = \Re_{\tilde{y}\tilde{z}} = 0. \quad (4)$$

Equation (4) in spinorial notation reads as

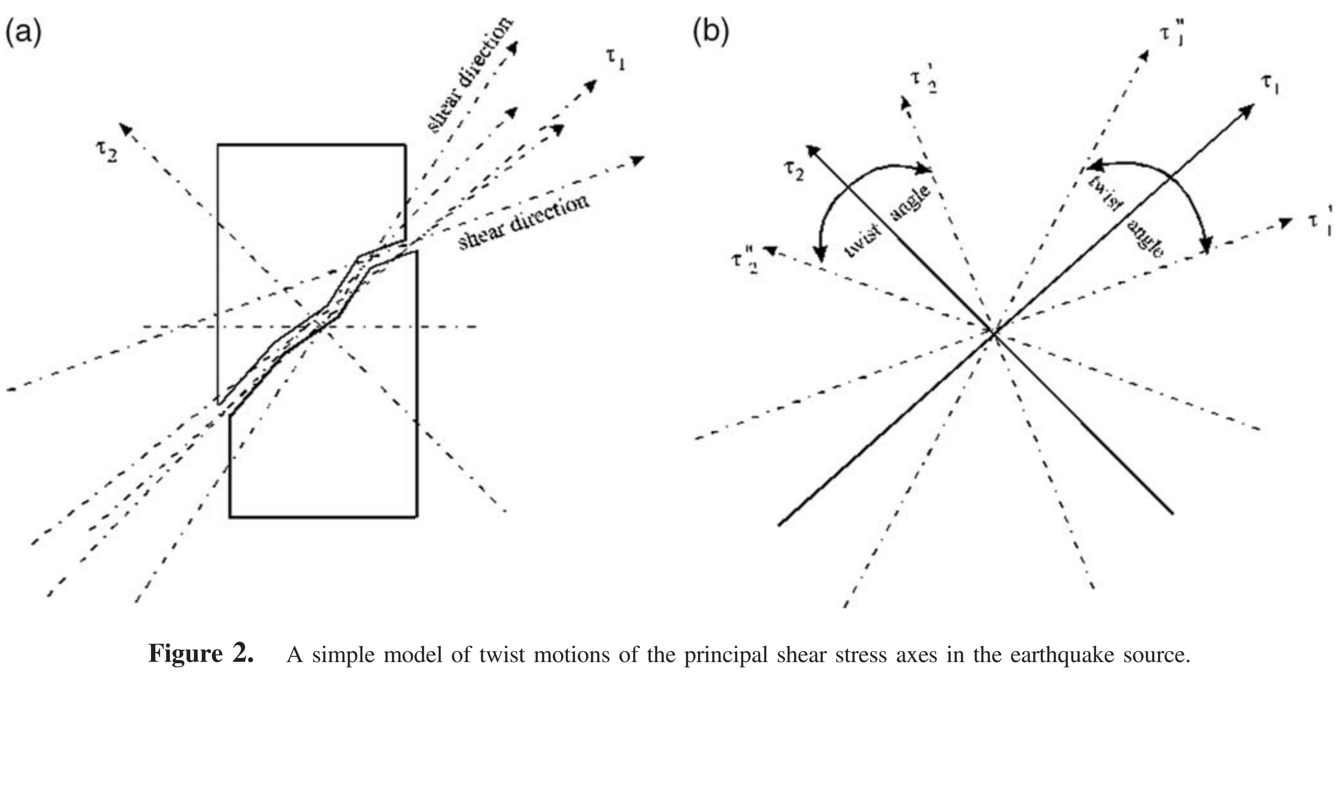

Figure 2. A simple model of twist motions of the principal shear stress axes in the earthquake source.

$$[\nabla_{AA'}, \nabla_{BB'}] = \Re_{AA'BB'} = \epsilon_{A'B'}\Re_{AB} + \epsilon_{AB}\Re_{A'B'}, \quad (5)$$

where $\Re_{AB}$ and $\Re_{A'B'}$ denote a self-dual and an anti-self-dual part of the rotational gauge field strength, respectively.

From the second equation in (4) it follows that the rotational gauge potentials $\mathbf{V}_y, \mathbf{V}_z(\mathbf{V}_{\tilde{y}}, \mathbf{V}_{\tilde{z}})$ can be treated as a pure gauges for fixed $\tilde{y}, \tilde{z}(y, z)$. Now, one is in a position to conclude that two complex matrices $\mathbf{C}$ and $\tilde{\mathbf{C}}$ exist that satisfy the following conditions:

$$\begin{aligned} \mathbf{V}_y &= \mathbf{C}^{-1}\mathbf{C}_{,y}, & \mathbf{V}_{\tilde{y}} &= \tilde{\mathbf{C}}^{-1}\tilde{\mathbf{C}}_{,\tilde{y}}, \\ \mathbf{V}_z &= \mathbf{C}^{-1}\mathbf{C}_{,z}, & \mathbf{V}_{\tilde{z}} &= \tilde{\mathbf{C}}^{-1}\tilde{\mathbf{C}}_{,\tilde{z}}. \end{aligned} \quad (6)$$

If we denote $\mathbf{J} = \mathbf{C}\tilde{\mathbf{C}}^{-1}$, then the last from the previous relationships can express self-dual Yang–Mills fields and takes the form

$$(\mathbf{J}^{-1}\mathbf{J}_{,y})_{\tilde{y}} - (\mathbf{J}^{-1}\mathbf{J}_{,z})_{,\tilde{z}} = 0. \quad (7)$$

Let us introduce the following spinors:

$$\Re_{AB} = -\frac{1}{2}\epsilon^{A'B'}\Re_{AA',BB'} \qquad \Re_{A'B'} = -\frac{1}{2}\epsilon^{AB}\Re_{AA',BB'}. \quad (8)$$

Now, we use a rotational gauge potential $V_{AA'}$ and the spinorial coordinate system to obtain the following expression:

$$\begin{aligned} \Re_{AA',BB'} &= V_{BB',AA'} - V_{AA',BB'} + [V_{AA'}, V_{BB'}] \\ &= \epsilon_{A'B'}\Re_{AB} + \epsilon_{AB}\Re_{A'B'}. \end{aligned} \quad (9)$$

Because we consider a seismic-wave excitation in rock materials with defects, it would be useful to comment about some similarities between defect theory and string theory. Let us look at a structure of linear and surface defects. We can observe that a forest of dislocations is similar to strings. The dislocations are anchored in surface defects that are similar to D-branes in a string theory (see Fig. 3).

Starting from an analogy to a complex electromagnetic field

$$\mathbf{C} = \boldsymbol{B} + i\boldsymbol{E}, \quad (10)$$

where $\boldsymbol{B}$ and $\boldsymbol{E}$ are the magnetic and electric fields, respectively, Teisseyre *et al.* (2006a) (see also Boratyński and Teisseyre, 2006) considered a complex rotational field that can be expressed as

$$\boldsymbol{\omega}_{\mu\nu} = \boldsymbol{\omega}_{[\mu\nu]} + i\boldsymbol{\omega}_{(\mu\nu)}, \quad (11)$$

where $\boldsymbol{\omega}_{[\mu\nu]}$ and $\boldsymbol{\omega}_{(\mu\nu)}$ are the spin and twist components, respectively. So, the spin and twist components are treated as the antisymmetric and symmetric tensor components, respectively. In order to express the rotational fields as complex fields, we will use the Moyal star product.

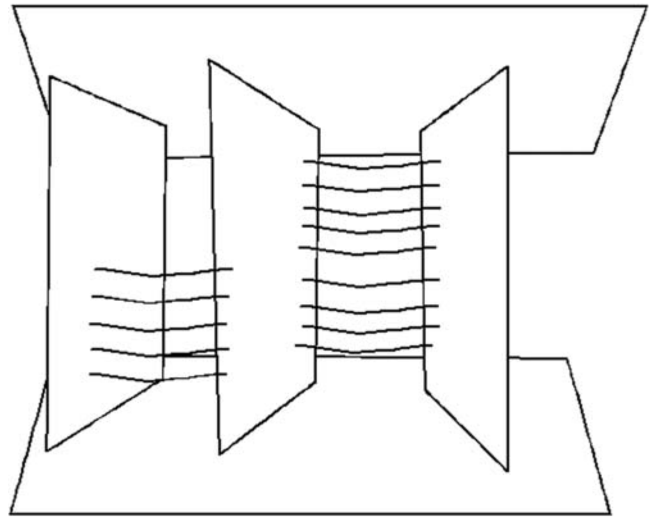

Figure 3. Structure of linear and surface defects. A forest of dislocations is similar to strings. The dislocations are anchored in surface defects that are similar to D-branes in string theory.

Deformation of Fields by the Moyal (Star) Product

The Moyal star product is defined by

$$f(x) * g(x) = \zeta^{\frac{i}{2}\Phi^{\mu\nu}\frac{\partial}{\partial\zeta^{\mu}}\frac{\partial}{\partial\eta^{\nu}}} f(x+\zeta)g(x+\eta)|_{\zeta=\eta=0}. \quad (12)$$

In virtue of this definition, the rotational gauge fields have to be complex. The NC Yang–Mills action is invariant under the gauge transformations as

$$\mathbf{V}^{g}_{\mu} = \boldsymbol{g} * \mathbf{V}_{\mu} * \boldsymbol{g}^{-1}_{*} - \boldsymbol{g}_{,u} * \boldsymbol{g}^{-1}_{*}, \quad (13)$$

where $\boldsymbol{g}^{-1}_{*}$ is the inverse of $\boldsymbol{g}$ with respect to the star product

$$\boldsymbol{g} * \boldsymbol{g}^{-1}_{*} = \boldsymbol{g}^{-1}_{*} * \boldsymbol{g} = 1. \quad (14)$$

The contributions of the terms $i\boldsymbol{\Phi}^{\mu\nu}$ in the star product transform the rotational gauge fields into complex fields. Only conditions such as $\mathbf{V}^{\dagger}_{\mu} = -\mathbf{V}_{\mu}$ could be preserved under gauge transformations provided that $\boldsymbol{g}$ is unitary: $\boldsymbol{g}^{\dagger} * \boldsymbol{g} = \boldsymbol{g} * \boldsymbol{g}^{\dagger} = 1$. It is not possible to restrict $\mathbf{V}_{\mu}$ to be real or imaginary to get the orthogonal or symplectic gauge groups as these properties are not preserved by the star product.

We postulate (see Majewski, 2008c) that the curvature associated with this rotational gauge field can be related to the disclination density $\boldsymbol{\Theta}_{\mu\nu}$ in the following way:

$$\mathbf{R}^{a}_{\mu\nu b} = \boldsymbol{\Gamma}^{a}_{\nu b,\mu} - \boldsymbol{\Gamma}^{a}_{\mu b,\nu} + \boldsymbol{\Gamma}^{a}_{\mu c}\boldsymbol{\Gamma}^{c}_{\nu b} - \boldsymbol{\Gamma}^{a}_{\nu c}\boldsymbol{\Gamma}^{c}_{\mu b}, \quad (15)$$

$$\boldsymbol{\Theta}^{\mu\nu} = \frac{1}{4}\boldsymbol{g}^{-1}\epsilon^{\mu pq}\epsilon^{\nu rs}\mathbf{R}_{pqrs}. \quad (16)$$

The curvature tensor can be expressed in terms of the rotational tensors as follows:

$$\mathbf{R}_{ijlk} = -2(\boldsymbol{\omega}_{[j|lk|,i]} + \boldsymbol{g}^{rs}\boldsymbol{\omega}_{[i|kr|}\boldsymbol{\omega}_{j]ls}). \tag{17}$$

This equation can be expressed in a spinor basis as (see Majewski, 2008c)

$$\begin{aligned} &X_{IJLK}\varepsilon_{I'J'}\varepsilon_{L'K'} + \Xi_{IJL'K'}\varepsilon_{I'J'}\varepsilon_{LK} + \tilde{\Xi}_{I'J'LK}\varepsilon_{IJ}\varepsilon_{L'K'} \\ &\quad + \tilde{X}_{I'J'L'K'}\varepsilon_{IJ}\varepsilon_{LK} = -2(\boldsymbol{\omega}_{[j|lk|,i]} + \boldsymbol{g}^{rs}\boldsymbol{\omega}_{[i|kr|}\boldsymbol{\omega}_{j]ls}), \end{aligned} \tag{18}$$

where X and Ξ are curvature spinors, the tilde denotes a complex conjugation, and ε_{AB} denotes the symplectic form or the fundamental spinor. Because of the disclination—curvature analogy (see Majewski, 2008c), we refer to the aforementioned curvature spinors as to disclination spinors.

To keep things simple, we will use an operator formulation of the noncommutativity approach. In order to describe the deformed product, we use a commutator of coordinates in the following form:

$$[x^\mu, x^\nu]_* = x^\mu * x^\nu - x^\nu * x^\mu = i\boldsymbol{\Phi}^{\mu\nu}(x), \tag{19}$$

where $\boldsymbol{\Phi}^{\mu\nu}(x)$ is an antisymmetric matrix and has the following form:

$$\boldsymbol{\Phi}^{\mu\nu} = \begin{bmatrix} 0 & \Phi^1 & 0 & 0 \\ -\Phi^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Phi^2 \\ 0 & 0 & -\Phi^2 & 0 \end{bmatrix}. \tag{20}$$

We should add that NC spaces are characterized by the noncommutativity of the spatial coordinates expressed by equation (20). This equation resembles the canonical commutation formula $[x, p] = i\hbar$ in quantum physics and yields a space-space uncertainty relation. Thus, the various singularities that exist on commutative spaces could resolve on NC spaces. This is one of the important features of NC theories and helps understand some new physical objects, for example, visible Dirac-like strings, fluxons, and instantons. These instantons derive their existence mainly from the resolution of small instanton singularities of the complete instanton moduli space. The solitons in NC spaces are sometimes so accessible that we can calculate many physical quantities, such as the soliton energy, soliton width, soliton speed, some fluctuations around the soliton configuration, and many more. This is also due to the properties of NC spaces that the singular configuration becomes smooth enough and suitable for practical calculations. In this article, we discuss NC solitons with applications to earthquake dynamics (Majewski, 2008b).

Derivation of the NC ASDYM-Like Equation from a System of Linear Operator Equations

The NC ASDYM-like equations for rotational fields in the earthquake source zone can be formulated based on the following procedure given by Gilson *et al.* (2009) and Hamanaka (2005, 2006) in a context of supergravity. We start from the following system of linear operator equations:

$$\mathfrak{M} * \boldsymbol{\Psi} = (D_y - \lambda D_{\tilde{z}}) * \boldsymbol{\Psi} = 0, \tag{21}$$

$$\mathfrak{A} * \boldsymbol{\Psi} = (D_z - \lambda D_{\tilde{y}}) * \boldsymbol{\Psi} = 0, \tag{22}$$

where $\mathfrak{M}$ and $\mathfrak{A}$ are operators, D_z denotes a covariant derivative, and λ is a spectral parameter.

We can illustrate the previous system of equations with the following example:

$$\begin{pmatrix} \tilde{z} & y \\ \tilde{y} & z \end{pmatrix} = 2^{-1/2}\begin{pmatrix} t + \mathbf{i}x^1 & x^2 - \mathbf{i}x^3 \\ x^2 + \mathbf{i}x^3 & t - \mathbf{i}x^1 \end{pmatrix}. \tag{23}$$

Because of space limitation, we cannot give more details concerning all the steps in the derivations of the NC ASDYM-like equation. We have only been able to sketch the main ideas and concepts. Now, the compatibility condition for the system (21) and (22) takes the form

$$\begin{aligned} [\mathfrak{M} * \mathfrak{A}]_* &= [D_y, D_z]_* + \lambda([D_z, D_{\tilde{z}}]_* - [D_y, D_{\tilde{y}}]_*) \\ &\quad + \lambda^2[D_{\tilde{z}}, D_{\tilde{y}}]_* = 0; \end{aligned} \tag{24}$$

thus,

$$\mathfrak{R}_{\mu\nu} = \mathbf{V}_{\nu,\mu} - \mathbf{V}_{\mu,\nu} + [\mathbf{V}_\mu, \mathbf{V}_\nu]_*, \tag{25}$$

$$\mathfrak{R}_{y\tilde{y}} - \mathfrak{R}_{z\tilde{z}} = 0 \to [D_y, D_{\tilde{y}}]_* = [D_z, D_{\tilde{z}}]_*, \tag{26}$$

$$\mathfrak{R}_{zy} = [D_z, D_y]_* = 0, \tag{27}$$

and, finally, we obtain the NC ASDYM equation for rotational motions derived from the system (21) and (22) in the following form:

$$\mathfrak{R}_{\tilde{z}\tilde{y}} = [D_{\tilde{z}}, D_{\tilde{y}}]_* = 0. \tag{28}$$

Rotational Seismic Soliton Equations Resulting from the Noncommutative (Anti-) Self-Dual Yang–Mills-Like Equation (NC ASDYM)

We consider NC seismic solitons. This exposition will pioneer a new approach to seismic waves. We have adopted the NC ASDYM-like equations to model rotational processes occurring in the earthquake source zone. Our main interest is focused on modeling rotational seismic-wave processes. It was first conjectured by Ward (1986) that in both commutative and NC cases the ASDYM equations after symmetry reductions lead to integrable equations with soliton solutions. After applying the Bäcklund transformations and solution generating techniques in the form of Atiyah–Ward ansatz solutions in terms of quasi-determinants to the NC ASDYM-like equation describing rotational seismic waves, we can obtain a plethora of rotational seismic soliton equations, for example, the sine-Gordon equation (SG), Klein–Gordon equation (KG), Korteweg–de Vries equation (KdV), modified KdV equation (mKdV), perturbed KdV equation (pKdV), Kadomtsev–Petviashvili equation (KP), two Boussinesq equations, doubly dispersive equation (DDE), nonlinear Schrödinger equation (NLS), and Gross–Pitaevskiy equation (GP).

Now, we will adopt for the rotational seismic field the procedure from an exposition given by Gilson *et al.* (2009) and Hamanaka (2006) in a context of supergravity. We will consider a reduction of the NC ASDYM-like equations to the NC KdV equation. First, we consider a dimensional reduction and gauge fixing

$$(y, \tilde{y}, z, \tilde{z}) \to (t, x) = (y + \tilde{y}, z), \tag{29}$$

where x, y, and z are the spatial coordinates and $\tilde{y}$ and $\tilde{z}$ are their complex conjugates.

The reduced NC ASDYM-like equation for rotational potentials is as follows:

$$[\mathbf{V}_y, \mathbf{V}_{\tilde{z}}]_* = 0, \tag{30}$$

$$\mathbf{V}'_y - \mathbf{V}'_{\tilde{y}} + [\mathbf{V}_z, \mathbf{V}_{\tilde{z}}]_* = [\mathbf{V}_y, \mathbf{V}_{\tilde{y}}]_*, \tag{31}$$

$$\mathbf{V}'_{\tilde{z}} - \dot{\mathbf{V}}_y + [\mathbf{V}_y, \mathbf{V}_z]_* = 0. \tag{32}$$

Denoting that $\phi = \phi(x, t)$ and $\phi' = \frac{\partial \phi}{\partial x}$, where x, y, and z are the spatial coordinates and $\phi_{,x}$ are the components of the angular displacement, we consider the following reduction:

$$\mathbf{V}_y = \begin{pmatrix} \phi & -1 \\ \phi' + \phi * \phi & -\phi \end{pmatrix}, \qquad \mathbf{V}_{\tilde{y}} = 0, \tag{33}$$

$$\begin{aligned} \mathbf{V}_z &= \begin{pmatrix} \frac{1}{2}\phi'' + \phi' * \phi & -\phi' \\ \beta(\phi''', \phi'', \phi', \phi) & -\frac{1}{2}\phi'' - \phi * \phi' \end{pmatrix}, \\ \mathbf{V}_{\tilde{z}} &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{34}$$

After some algebraic calculations and bearing in mind that

$$\boldsymbol{\omega} = 2\phi' = 2\frac{\partial \phi}{\partial x} \quad \text{and} \quad [t, x] = i\Phi(x), \tag{35}$$

we can obtain the NC KdV equation for rotational seismic waves

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \frac{1}{4}\frac{\partial^3 \boldsymbol{\omega}}{\partial x^3} - \frac{3}{4}\left(\frac{\partial \boldsymbol{\omega}}{\partial x} * \boldsymbol{\omega} + \boldsymbol{\omega} * \frac{\partial \boldsymbol{\omega}}{\partial x}\right) = 0 \tag{36}$$

that has N-soliton solutions in the form

$$\boldsymbol{\omega} = 2\frac{\partial}{\partial x}\sum_{n=1}^{N}\left(\frac{\partial \mathbf{Q}_n}{\partial x}\right) * \mathbf{Q}_n^{-1}, \tag{37}$$

where $\mathbf{Q}_n = |\mathbf{Q}(\beta_1, \ldots, \beta_n)|_{n,n}$ is a quasi-determinant of the Wronskian (Hamanaka 2005,2006; Gilson *et al.*, 2009), and

$$\beta_n = \exp\gamma(x, \eta_n) + C_n \exp[-\gamma(x, \eta_n)], \tag{38}$$

where $\gamma(x, \eta_n) = x\eta + t\eta^3$.

For comparison, let us recall here a commutative KdV equation for seismic twist solitons in the form (Majewski, 2006b)

$$\frac{\partial \boldsymbol{\omega}_{(w)}}{\partial t} + \frac{1}{4}\frac{\partial^3 \boldsymbol{\omega}_{(w)}}{\partial x^3} + \frac{3}{2}\boldsymbol{\omega}_{(w)}\frac{\partial \boldsymbol{\omega}_{(w)}}{\partial x} = 0. \tag{39}$$

From this we learn something interesting. Using the aforementioned solution generating technique, we can obtain the following NC KP equation for seismic spin solitons:

$$\begin{aligned} &\frac{\partial \boldsymbol{\omega}_{[s]}}{\partial t} + \frac{1}{4}\frac{\partial^4 \boldsymbol{\omega}_{[s]}}{\partial x^4} + \frac{3}{4}\left(\frac{\partial \boldsymbol{\omega}_{[s]}}{\partial x} * \boldsymbol{\omega}_{[s]} + \boldsymbol{\omega}_{[s]} * \frac{\partial \boldsymbol{\omega}_{[s]}}{\partial x}\right) \\ &\quad = -\frac{3}{4}\left[\boldsymbol{\omega}, \partial_x^{-1}\left(\frac{\partial \boldsymbol{\omega}_{[s]}}{\partial y}\right)\right]_* - \frac{3}{4}\partial_x^{-1}\left(\frac{\partial^2 \boldsymbol{\omega}_{[s]}}{\partial y^2}\right). \end{aligned} \tag{40}$$

The NC Burgers equation for seismic twist solitons can be expressed as follows:

$$\begin{aligned} &\frac{\partial \boldsymbol{\omega}_{(w)}}{\partial t} - C_1\frac{\partial^2 \boldsymbol{\omega}_{(w)}}{\partial x^2} + (C_1 - C_2 + 1)\frac{\partial \boldsymbol{\omega}_{(w)}}{\partial x} * \boldsymbol{\omega}_{(w)} \\ &\quad = (C_1 + C_2 - 1)\boldsymbol{\omega}_{(w)} * \frac{\partial \boldsymbol{\omega}_{(w)}}{\partial x}, \end{aligned} \tag{41}$$

where C_1 and C_2 are constants.

The NC Calogero–Bogoyavlanskii–Schiff equation for seismic spin solitons takes the form

$$\begin{aligned} &\frac{\partial \boldsymbol{\omega}_{[s]}}{\partial t} + \frac{1}{4}\frac{\partial^3 \boldsymbol{\omega}_{[s]}}{\partial x^2 \partial y} + \frac{1}{2}\left(\frac{\partial \boldsymbol{\omega}_{[s]}}{\partial y} * \boldsymbol{\omega}_{[s]} + \boldsymbol{\omega}_{[s]} * \frac{\partial \boldsymbol{\omega}_{[s]}}{\partial y}\right) \\ &\quad = -\frac{1}{4}\frac{\partial \boldsymbol{\omega}_{[s]}}{\partial x} * \partial_x^{-1}\left(\frac{\partial \boldsymbol{\omega}_{[s]}}{\partial y}\right) - \frac{1}{4}\partial_x^{-1}\left(\frac{\partial^2 \omega_{[s]}}{\partial y^2}\right) * \frac{\partial \boldsymbol{\omega}_{[s]}}{\partial x} \\ &\quad\quad - \frac{1}{4}\left\{\boldsymbol{\omega}_{[s]}, \partial_x^{-1}\left[\boldsymbol{\omega}_{[s]}, \partial_x^{-1}\left(\frac{\partial \boldsymbol{\omega}_{[s]}}{\partial y}\right)\right]_*\right\}_*. \end{aligned} \tag{42}$$

The NC approach to the ASDYM-like equations is very promising because it provides us with a plethora of seismic soliton equations (illustrated in Fig. 4) and allows us to avoid some singularities (Hamanaka, 2005, 2006; Gilson *et al.*, 2009).

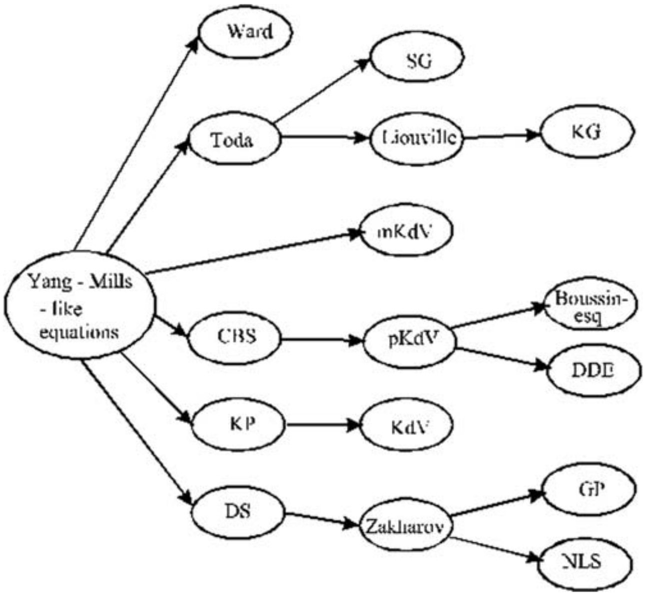

Figure 4. The NC ASDYM-like equations for the earthquake source region and resulting rotational seismic soliton equations (modified from Hamanaka, 2006).

Rotational Seismic Solitons Branching off Dispersion Curves

Now, we present our results in the wavenumber *w* versus the wave vector **k** graph. They are depicted in Figure 5. We plotted the dispersion curves of the seismic *P* and *S* waves. In addition, we drew the dispersion curves of two rotational waves, namely the rotational longitudinal wave *PR* and the rotational transverse wave *SR*. The diagram illustrates how rotational seismic solitons branch off the dispersion curves corresponding to rotational seismic waves.

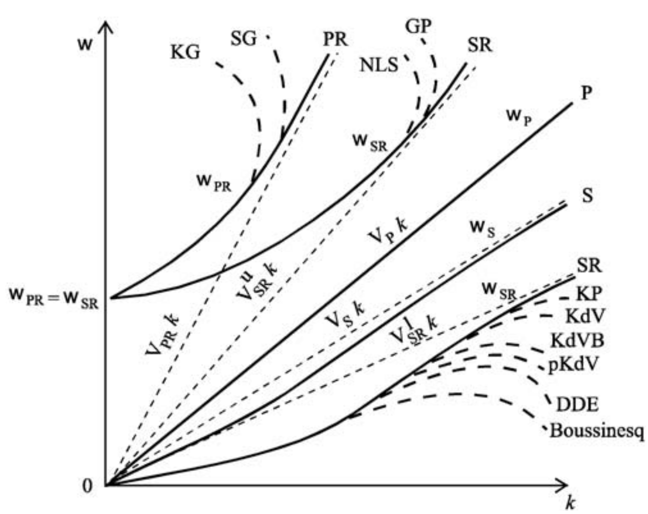

Figure 5. Rotational seismic solitons branching off dispersion curves for rotational seismic *PR* and *SR* waves in an elastic Earth with defects.

The Nonlinear Schrödinger Equation in Terms of Twistors Applied to Seismic Spin and Twist Solitons

Following Majewski (2008d), we consider a soliton wave function $\hat{\Psi}$, which is a complex function of twistor Z^α. We assume that the soliton wave function does not depend on the complex conjugate $\tilde{Z}_\alpha$; thus, we have

$$\frac{\partial \hat{\Psi}}{\partial \tilde{Z}_\alpha} = 0. \tag{43}$$

In such a case, the complex conjugate $\tilde{Z}_\alpha$ can play a part of a differentiation operator acting on the soliton wave function:

$$[\tilde{Z}_\alpha]\hat{\Psi} \rightarrow -\hbar \frac{\partial \hat{\Psi}}{\partial Z^\alpha}. \tag{44}$$

The twistor Z^α can be also treated as a multiplication operator, that is,

$$[Z^\alpha]\hat{\Psi} \rightarrow Z^\alpha \hat{\Psi}. \tag{45}$$

The twistor and its complex conjugate Z^α and $\tilde{Z}_\alpha$, respectively, can exchange their roles. In such a situation, the soliton wave function $\hat{\Psi}$ will be a function of the complex conjugate $\tilde{Z}_\alpha$. We can denote $\tilde{Z}_\alpha$ as its dual twistor W_α and Z^α as $\tilde{W}^\alpha$, then the soliton wave function $\hat{\Psi}$ will be a function of the dual twistor W_α. Thus, the dual multiplication and differentiation operators $[W_\alpha]$ and $[\tilde{W}^\alpha]$, respectively, act on the soliton wave function according to the formulas

$$[W_\alpha]\hat{\Psi} \rightarrow W_\alpha \hat{\Psi} \tag{46}$$

and

$$[\tilde{W}^\alpha]\hat{\Psi} \rightarrow \hbar \frac{\partial \hat{\Psi}}{\partial W_\alpha}. \tag{47}$$

By analogy to these quantum physics considerations, we are in a position to propose the following NLS for seismic spin or twist amplitudes in terms of twistors:

$$iA \frac{\partial \tilde{\Psi}}{\partial \tau} = -v^2 \frac{\partial^2 \tilde{\Psi}}{\partial W_\alpha^2} + B|\tilde{\Psi}|^2 \tilde{\Psi}, \tag{48}$$

where τ is time, A and B are constant coefficients, and v is the seismic-wave speed. Solutions to the previous equation can describe seismic spin or twist Schrödinger's solitons. These solitons are spin or twist pulses. If we apply it to an earthquake fault, the equation can describe the so-called tectonic solitons, that is, the rotational seismic solitons that can be excited by past earthquake processes and may propagate slowly along the earthquake fault to trigger new earthquakes.

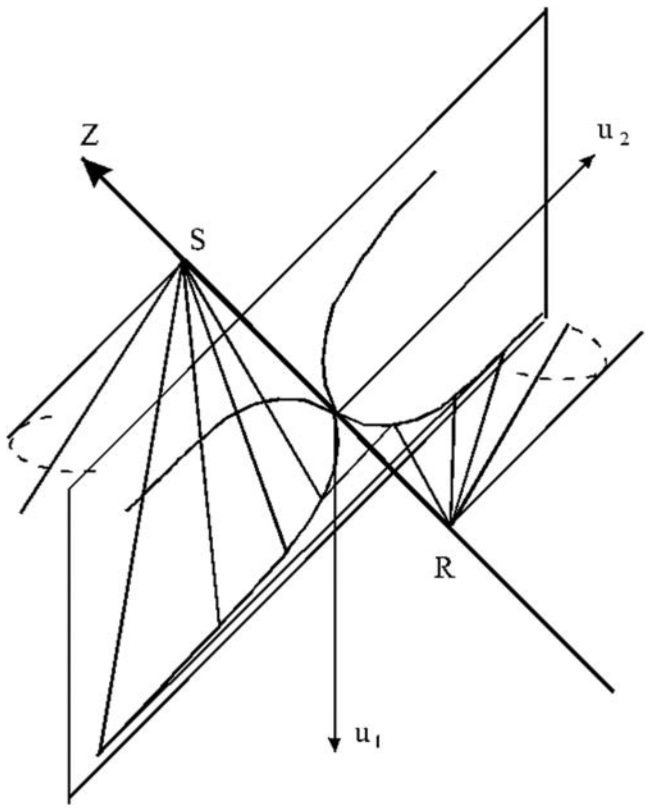

Figure 6. Illustration of rotational seismic-wave propagation in a twistor space-time (Majewski, 2008d). The line denoted by Z represents a twistor that is referred to as a worldline or a seismic ray. The time cone at point R is a null future cone that contains future events. The time cone at point S is a null past cone that contains past events (modified from Belinski and Verdaguer, 2001).

Application of a Robinson Congruence to Visualize the Motion of the Rotational Seismic Solitons

Figure 6 depicts rotational seismic-wave propagation in a twistor space-time. A twistor Z is described here as a worldline or a seismic ray. An inspiration for Penrose (1968, 1983) who was seeking a visualization of Ivor Robinson congruence, was a special geometrical construction for finding solutions to Maxwell's free-space equations. Maxwell's electromagnetic fields are nonsingular, shear free, geodetic, and twisting. Thus, Maxwell's fields are suitable to describe rotations. Robinson's approach consists in constructing his solutions on a twisting shear-free congruence of light rays (family of null lines). Such a family of null straight lines (similar to seismic rays) whose tangent directions constitute this field is defined as the Robinson congruence (see Penrose and Rindler, 1984).

Figure 7 depicts a motion of the seismic twist soliton in space-time. It shows three time slices of the tangents to a twisting family of circles at time $t = 0$, $t = t_1$, and $t = t_2$. The figure is based on Penrose geometrical visualization of a Robinson congruence (Penrose and Rindler, 1986; Majewski, 2008d).

Conclusions

We presented briefly a nonlinear seismic-wave theory and attempted to derive seismic spin and twist solitons. The NC ASDYM equations were adopted to describe processes in the earthquake source zone. The NC ASDYM fields were identified as rotational fields, for example, spin and twist fields. The main conclusion here is that spinors can be used successfully to describe seismic spin and twist solitons in twistor spaces. Moreover, the twistor spaces can be

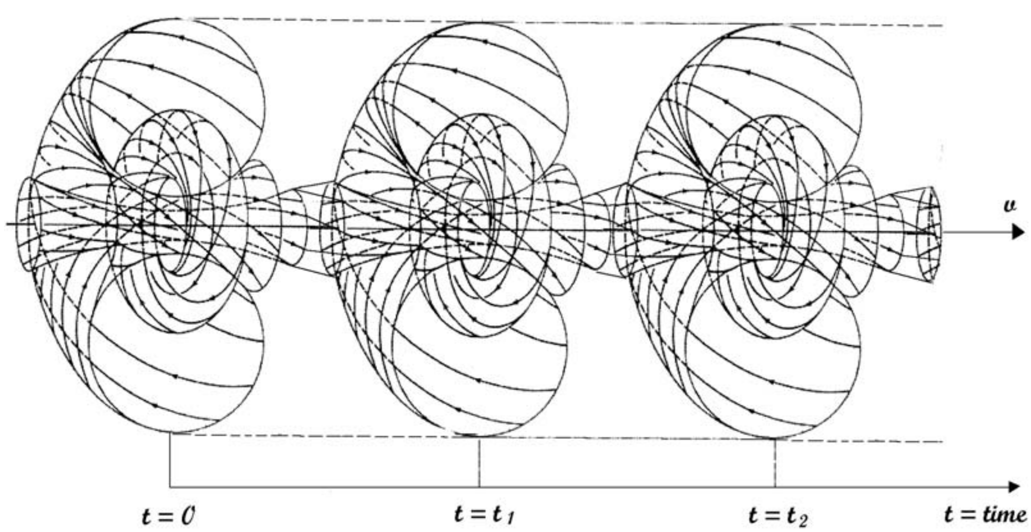

Figure 7. Illustration of the motion of the seismic twist soliton in space-time. The first time slice of the tangents to a twisting family of linked circles is visualized at time $t = 0$; the second time slice is shown at time $t = t_1$; the third time slice is shown at time $t = t_2$; the twist configuration moves to the right along the time axis with velocity v. The figure is a modification of Penrose geometrical visualization of a Robinson congruence (modified from Penrose and Rindler, 1986).

helpful to characterize motions and interactions of spin and twist solitons. Using the formalisms of the Yang–Mills theory and twistor theory, we can get much deeper insight into rotational seismic-wave processes and spin and twist solitons. In addition, starting from a Penrose twistor quantization theory, we proposed a NLS for seismic spin and twist amplitudes in terms of twistors.

On a final note, twistor theory has a complex structure, whose complete content is not yet investigated. It is mathematical structure but deeply embedded in physics. It incorporates and connects crucial principles of physics. Undoubtedly there are many unsolved problems including seismic spin entropy, spin entropy current, spin density, and the propagation of spin and twist solitons in curved twistor spaces that need to be analyzed.

Data and Resources

All data used in this article came from published sources listed in the references.

Acknowledgments

The support by the International Working Group for Rotational Seismology is greatly acknowledged. Valuable comments and helpful suggestions from two anonymous reviewers are appreciated.

References

Belinski, V., and E. Verdaguer (2001). *Gravitational Solitons*, Cambridge U Press, Cambridge.

Boratyński, W., and R. Teisseyre (2006). Fault dynamics and related radiation, in *Earthquake Source Asymmetry, Structural Media and Rotation Effects*, R. Teisseyre, M. Takeo, and E. Majewski (Editors), Springer, Berlin, 77–89.

Cochard, A., H. Igel, B. Schuberth, W. Suryanto, A. Velikoseltsev, U. Schreiber, J. Wassermann, F. Scherbaum, and D. Vollmer (2006). Rotational motions in seismology: theory, observation, simulation, in *Earthquake Source Asymmetry, Structural Media and Rotation Effects*, R. Teisseyre, M. Takeo, and E. Majewski (Editors), Springer, Berlin, 391–411.

Droste, Z., and R. Teisseyre (1976). Rotational and displacemental components of ground motion as deduced from data of the azimuth system of seismograph, *Publ. Inst. Geophys. Pol. Acad. Sci.* **97,** 157–167.

Gilson, C. R., M. Hamanaka, and J. J. C. Nimmo (2009). Bäcklund transformations for noncommutative and anti-self-dual Yang–Mills equations, *Glas. Math. J.* arXiv:0709.2069v2

Hamanaka, M. (2005). On reduction of noncommutative anti-self-dual Yang–Mills equations, *Phys. Lett.* **B625,** 324–332, hep-th/0507112v4, doi 10.1016/j.physletb.2005.08.077.

Hamanaka, M. (2006). NC Ward's conjecture and integrable systems, *Nucl. Phys.* **B741,** 368.

Igel, H., K. U. Schreiber, A. Flaws, B. Schuberth, A. Velikoseltsev, and A. Cochard (2005). Rotational motions induced by the M 8.1 Tokachi-oki earthquake, September 25, 2003, *Geophys. Res. Lett.* **32,** L08309, doi 10.1029/2004GL022336.

Lechtenfeld, O., and A. D. Popov (2007). Noncommutative solitons in a supersymmetric chiral model in $2 + 1$ dimensions, *J. High Energy Phys.* **6,** no. 65, doi 10.1088/1126-6708/2007/06/065, arXiv: 0704.0530v2 [hep-th].

Majewski, E. (2006a). Soliton physics, in *Earthquake Source Asymmetry, Structural Media and Rotation Effects*, R. Teisseyre, M. Takeo, and E. Majewski (Editors), Springer, Berlin, 113–128.

Majewski, E. (2006b). Seismic rotation waves: spin and twist solitons, in *Earthquake Source Asymmetry, Structural Media and Rotation Effects*, R. Teisseyre, M. Takeo, and E. Majewski (Editors), Springer, Berlin, 255–272.

Majewski, E. (2006c). Seismic rotation waves in the continuum with nonlinear microstructure, in *Earthquake Source Asymmetry, Structural Media and Rotation Effects*, R. Teisseyre, M. Takeo, and E. Majewski (Editors), Springer, Berlin, 293–300.

Majewski, E. (2006d). Tectonic solitons propagating along the fault, in *Earthquake Source Asymmetry, Structural Media and Rotation Effects*, R. Teisseyre, M. Takeo, and E. Majewski (Editors), Springer, Berlin, 301–309.

Majewski, E. (2006e). Complexity of rotation soliton propagation, in *Earthquake Source Asymmetry, Structural Media and Rotation Effects*, R. Teisseyre, M. Takeo, and E. Majewski (Editors), Springer, Berlin, 255–272.

Majewski, E. (2008a). Fracture physics based on a soliton approach, in *Physics of Asymmetric Continuum: Extreme and Fracture Processes—Earthquake Rotation and Soliton Waves*, R. Teisseyre, H. Nagahama, and E. Majewski (Editors), Springer, Berlin, 193–208.

Majewski, E. (2008b). Canonical approach to asymmetric continua, in *Physics of Asymmetric Continuum: Extreme and Fracture Processes—Earthquake Rotation and Soliton Waves*, R. Teisseyre, H. Nagahama, and E. Majewski (Editors), Springer, Berlin, 209–218.

Majewski, E. (2008c). Spinors and torsion in a Riemann-Cartan approach to elasticity with a continuous defect distribution and analogies to the Einstein-Cartan theory of gravitation, in *Physics of Asymmetric Continuum: Extreme and Fracture Processes—Earthquake Rotation and Soliton Waves*, R. Teisseyre, H. Nagahama, and E. Majewski (Editors), Springer, Berlin, 249–272.

Majewski, E. (2008d). Twistors as spin and twist solitons, in *Physics of Asymmetric Continuum: Extreme and Fracture Processes—Earthquake Rotation and Soliton Waves*, R. Teisseyre, H. Nagahama, and E. Majewski (Editors), Springer, Berlin, 273–284.

Mason, L. J., L. P. Hugston, and P. Z. Kobak (Editors), (1995). *Further Advances in Twistor Theory: Vol 2: Integrable Systems, Conformal Geometry and Gravitation*, Longman Scientific & Technical, Essex, U.K. and John Wiley & Sons, New York.

Newman, E., and R. Penrose (1962). An approach to gravitational radiation by a method of spin coefficients, *J. Math. Phys.* **3,** 566–587.

Penrose, R. (1968). Twistor quantization and curved space-time, *Int. J. Theor. Phys.* **1,** 61–99.

Penrose, R. (1983). Spinors and torsion in general relativity, *Found. Phys.* **13,** 325–339.

Penrose, R., and W. Rindler (1984). *Spinors and Space-Time, Vol 1: Two-Spinor Calculus and Relativistic Fields*, Cambridge U Press, Cambridge.

Penrose, R., and W. Rindler (1986). Spinors and Space-Time, Vol 2: spinor and twistor methods in space-time geometry, Cambridge U Press, Cambridge.

Schreiber, K. U., G. E. Stedman, H. Igel, and A. Flaws (2006). Ring laser gyroscopes as rotation sensors for seismic wave studies, in *Earthquake Source Asymmetry, Structural Media and Rotation Effects*, R. Teisseyre, M. Takeo, and E. Majewski (Editors), Springer, Berlin, 377–389.

Stedman, G. E. (1997). Ring laser tests of fundamental physics and geophysics, *Rep. Prog. Phys.* **60,** 615–688.

Stedman, G. E., Z. Li, and H. R. Bilger. (1995). Sideband analysis and seismic detection in large ring lasers, *Appl. Opt.* **34,** 7390–7396.

Takeo, M. (2006). Ground rotational motions recorded in near-source region of earthquakes, in *Earthquake Source Asymmetry, Structural Media and Rotation Effects*, R. Teisseyre, M. Takeo, and E. Majewski (Editors), Springer, Berlin, 157–167.

Teisseyre, K. P. (2007). Analysis of a group of seismic events using rotational components, *Acta Geophys.* **55,** 535–553.

Teisseyre, K. P., and J. Suchcicki (2006). Rotation motions: recording and analysis, in *Earthquake Source Asymmetry, Structural Media and Rotation Effects*, R. Teisseyre, M. Takeo, and E. Majewski (Editors), Springer, Berlin, 186–197.

Teisseyre, R. (1973). Earthquake processes in a micromorphic continuum, *Pure Appl. Geophys.* **102,** 15–28.

Teisseyre, R. (1974). Symmetric micromorphic continuum: wave propagation, point source solutions and some applications to earthquake processes, in *Continuum Mechanics Aspects of Geodynamics and Rock Fracture Mechanics*, P. Thoft-Christensen (Editor), Reidel, Dordrecht, 201–244.

Teisseyre, R. (2009). Tutorial on new developments in the physics of rotational motions, *Bull. Seismol. Soc. Am.* **99,** no. 2B, 1028–1039.

Teisseyre, R., and M. Górski (2009). Fundamental deformations in asymmetric continuum, *Bull. Seismol. Soc. Am.* **99,** no. 2B, 1132–1136.

Teisseyre, R., M. Białecki, and M. Górski (2006). Degenerated asymmetric continuum theory, in *Earthquake Source Asymmetry, Structural Media and Rotation Effects*, R. Teisseyre, M. Takeo, and E. Majewski (Editors), Springer, Berlin, 43–55.

Teisseyre, R., J. Suchcicki, K. P. Teissyre, J. Wiszniowski, and P. Palangio (2003). Seismic rotation waves: basic elements of theory and recording, *Ann. Geophys.* **46,** no. 4, 671–685.

Ward, R. S. (1986). Multidimensional integrable systems, In de Vega, H. J., and N. Sanchez (Editors), *Field theory, Quantum Gravity and Strings*, Vol. **2,** p. 106.

Institute of Geophysics
Polish Academy of Sciences
ul. Księcia Janusza 64
01-452 Warsaw, Poland
emaj@igf.edu.pl

Manuscript received 3 July 2008