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Suggested Readings in Continuum Mechanics and Earthquake Seismology

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Abstract Rotational seismology is a newly emerging field of interest to scientists from widely differing disciplines. We compile here some suggested readings for those wishing to become familiar with disciplines relevant to rotational seismology and its engineering applications. These readings are not exhaustive but contain a number of basic references with occasional annotations. We concentrate on two areas: continuum mechanics and earthquake seismology. Continuum mechanics is the branch of mechanics that deals with the deformation and motion of materials, including various elasticity theories. Although classical elasticity works well for studying earthquakes in the far field, there are alternate elasticity theories that may be more appropriate for studying earthquakes in the near field. The readings in continuum mechanics are intended for seismologists and include (1) fundamental textbooks, (2) intermediate to advanced books, (3) heteromodular theories, and (4) Cosserat theory. A brief introduction to continuum mechanics for seismologists is included as an appendix. In addition, because we wish to attract workers in other disciplines to rotational seismology, we include a short list of suggested readings in earthquake seismology.

Introduction

The *BSSA* special issue on rotational seismology and engineering applications contains 51 papers authored by workers in many disciplines (see Lee *et al.*, 2009). Here, we compile some suggested readings for readers who wish to become familiar with disciplines relevant to rotational seismology and its engineering applications. These suggested readings are by no means exhaustive but rather represent introductory materials and basic references with our own occasional annotations. We concentrate on two areas: continuum mechanics and earthquake seismology. The readings in continuum mechanics are intended for seismologists, especially for those who are not familiar with recent developments in continuum mechanics. In addition, because we wish to attract workers in other disciplines to rotational seismology, we also include a short list of suggested readings on earthquake seismology.

Continuum Mechanics

Continuum mechanics is a branch of mechanics that deals with the deformation and motion of materials. There are a variety of models for different materials, but there is a general scheme into which all of these models must fit. Often, and particularly for linear materials, an intuitive model of a particle, together with basic physical principles, determines the governing equation of motion. Classical continua, consisting of point masses, describe a wide range of phenomena well enough for most purposes, even when using a linear approximation. However, sometimes this model is not adequate. For example, at large strains we must consider the nonlinear theory, and when the microstructure participates actively in the dynamics of the medium, we need to change the model of the particles. Porous media, soils, magnetoelastic materials, granular materials, and composites, for example, need complex models to describe their behavior accurately.

In this section, we give several basic references on continuum mechanics that may help seismologists gain some knowledge of this field, especially regarding its modern developments. In addition, a brief introduction to continuum mechanics for seismologists is included as an appendix at the end of this article.

Fundamental Textbooks

To begin, seismologists are encouraged to learn the basics of solid mechanics, including classical elasticity theory, which has been very successful in traditional seismology. An appropriate textbook is

Fung, Y. C. (1965). *Foundations of Solid Mechanics*, Prentice Hall, Englewood Cliffs, New Jersey, 525 pp.

A reissue of this classic, including a major revision of the theory of large elastic deformations with finite strains and

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five new chapters on computational aspects, focusing on numerical methods, is

Fung, Y. C., and P. Tong (2001). *Classical and Computational Solid Mechanics*, World Scientific, Singapore, 930 pp.

Another classic introduction is

Jaeger, J., N. G. W. Cook, and R. Zimmerman (2007). *Fundamentals of Rock Mechanics*, Fourth Ed., Wiley–Blackwell, New York, 488 pp.

Intermediate to Advanced Books

One of the best books on continuum mechanics in the twentieth century is

Truesdell, C. A. (1991). *A First Course in Rational Continuum Mechanics*, Second Ed., Academic, New York, 352 pp.,

which introduces the basic concepts of classical mechanics, starting from reference systems, concepts of mass and force, etc. It then gives the nonlinear theory of continuous media, including kinematics, the concept of stress tensors, principles for deducing constitutive equations, and examples of various media so described. This book contains a section devoted to hydrodynamics, and a larger portion of the book describes nonlinear elasticity, including thermoelasticity.

Three classic books on elasticity are

Lurie, A. I. (1990). *Nonlinear Theory of Elasticity*, North-Holland, Amsterdam, 618 pp. (translated from Russian by K. A. Lurie),

Lurie, A. I. (2005). *Theory of Elasticity*, Springer, Berlin, 1050 pp. (translated from Russian by A. K. Belyaev),

and

Ogden, R. (1997). *Non-Linear Elastic Deformations*, Dover, New York, 544 pp.

Each of these books includes the necessary mathematical and physical basis for deducing the equations of classical nonlinear elastic media: tensor theory, the strain tensors in Eulerian and Lagrangean form, basic physical laws, the principle of material objectivity (see the appendix), examples of constitutive equations for various elastic materials, and selected boundary-value problems.

Nonlinear theories of various classical continua are presented together with some model problems in

Biot, M. A. (1965). *Mechanics of Incremental Deformation*, Wiley, New York, 504 pp.

and

Palmov, V. (1998). *Vibrations of Elasto-Plastic Bodies*, Springer, Berlin, 311 pp.

A thorough course on modern rational mechanics, including enriched continuum theories, can be found in books by P. A. Zhilin. Unfortunately, these have not yet been translated from Russian to English, and thus, they are not cited here.

Heteromodular Theories

There is a branch of nonlinear continuum theory called heteromodular theory in which stress is not a smooth function of strain and the elastic properties of the medium depend upon the stress. This theory can be, for instance, piecewise linear (the moduli depend upon loading), and this type of theory is used to describe granular materials, such as soils and rocks, whose defects make a medium react in different ways to different types of loads. The moduli vary more for loose soils than for consolidated ones, but for media with defects, the moduli are never equal for tension and compression, and this variability may influence wave propagation at large times (inversely proportional to the difference between the moduli) even for consolidated soil and rock.

Heteromodular elastic theories were proposed in the 1960s and were developed by Ambartsumyan, Hachatrian, Myasnikov, Oleinikov, and other authors. Unfortunately, most of the original articles are available only in Russian, as is the detailed classic book,

Ambartsumyan, S. A. (1982). *Different-Modulus Elasticity Theory*, Nauka. Fizmatlit Publ. Co., Moscow, 320 pp. (in Russian).

The development of heteromodular elastic theories and references has been published in English as

Curnier, A., Q. He, and P. Zysset (1994). Conewise linear elastic materials, *J. Elast.* **37,** no. 1, 1–38

and

Tsvelodub, I. (2008). Multimodulus elasticity theory, *J. Appl. Mech. Tech. Phys.* **49,** no. 1, 129–135.

Wave propagation in one-dimensional heteromodular media with slightly varying moduli is investigated analytically in

Gavrilov, S. N. (2006). Wave propagation in a heteromodular elastic medium, in *Proc. 68th EAGE Meeting*, Vienna, Austria. The poster is available at http://www.pdmi.ras.ru/~elgreco/pdf/poster-Serge-Gavrilov-EAGE2006-Vienna.pdf (last accessed January 2009).

Elastoplastic heteromodular theory is developed by Lomakin. See, for instance,

Lomakin, E. (2007). Mechanics of media with stress-state dependent properties, *Phys. Mesomech.* **10,** no. 5–6, 255–264, doi 10.1016/j.physme.2007.11.004.

Cosserat Theory

Apart from nonlinearity, the rotational dynamics of particles are important in some media, for example, in soils. Rotational dynamics have been formulated in the Cosserat theory, published by two brothers a century ago:

Cosserat, E., and F. Cosserat (1909). *Théorie des Corps Déformables*, A. Hermann et Fils, Paris, 226 pp.; available from the Cornell University Library Digital Collections.

One may read a summary of the development of Cosserat theory and some discussion of the theory itself at

Neff, P. (2008). Cosserat theory: the Cosserat or micropolar model, http://www.mathematik.tu-darmstadt.de/fbereiche/analysis/pde/staff/neff/patrizio/Cosserat.html (last accessed January 2009)

and

Forest, S. (2008). Mechanics of Cosserat media: an introduction, http://www.mathematik.uni-karlsruhe.de/iwrmm/media/cosserat.pdf (last accessed January 2009).

The EuroMech Colloquium Number 510, "Mechanics of Generalized Continua: A Hundred Years after the Cosserats" will be in Paris, 13–16 May 2009 (Maugin, 2008). An international conference on the legacy of Cosserat and Cosserat (1909) in the centenary of its publication will also take place in Paris, 15–17 July 2009 (http://cosserat2009.enpc.fr/, last accessed January 2009). Cosserat continua provide one example of such generalized continua and may provide a useful model for understanding rotational ground motions in the near field of earthquakes and the effects of voids at depth and of soils in general.

Perhaps the first example of a complete theory for a three-dimensional Cosserat continuum is proposed in

Kafadar, C., and A. C. Eringen (1971). Micropolar media, *Int. J. Eng. Sci.* **9,** 271–305.

The following books describe the theory of elastic Cosserat continua and some dynamic problems:

Nowacki, W. (1986). *Theory of Asymmetric Elasticity*, Pergamon, Oxford, U.K., 383 pp.

and

Eringen, A. C. (1999). *Microcontinuum Field Theories, I: Foundations and Solids*, Springer, New York, 325 pp. [This is an advanced monograph written by a pioneer of modern continuum mechanics.]

Cosserat theory is a particular case of micromorphic theory wherein a particle may deform in addition to rotating, and in the Eringen book listed previously the general micromorphic theory is presented. The particle's capacity for deformation may be important, for instance, if we wish to consider porous media and take into account how the pores breathe as well as the dynamic influences of such behavior on wave propagation. In the article

Pietraszkiewicz, W., and V. Eremeyev (2009). On natural strain measures of the non-linear micropolar continuum, *Int. J. Solids Struct.* **46,** no. 3–4, 774–787,

the authors compare the strain tensors used by various other authors for the Cosserat continua. Some of these tensors naturally appear in the law of balance of energy (see the appendix). This article also will help readers introduce themselves to the Cosserat world.

The following work deals with a three-dimensional nonlinear elastoplastic Cosserat continuum in an Eulerian formulation:

Zhilin, P. (2006). Phase transitions and general theory of elastoplastic bodies, in *Advanced Problems in Mechanics 2006*, Institute for Problems in Mechanical Engineering of Russian Academy of Sciences, 140–152, available at http://www.pdmi.ras.ru/~elgreco/pdf/Zhilin-APM2002.pdf (last accessed January 2009).

The author hypothesized that such a theory would be able to describe rearrangements of the contacts within a granular medium, which leads to the formation of a new material under loading, much as in a phase transition. This article began work that sadly remained unfinished upon the death of Zhilin.

Many researchers consider soils and granular materials to be complete Cosserat continua—that is, where rotation and translation are independent and the medium reacts to the gradient of rotation. This continuum's reaction is described by the couple-stress tensor. It plays the same role for torque as the stress tensor plays for force. A tensor $\boldsymbol{\mu}$ of second rank is called the couple-stress tensor if, for each unit vector $\mathbf{n}$ corresponding to the normal to an elementary material surface, $\mathbf{n} \cdot \boldsymbol{\mu}$ gives the torque acting upon this surface. Among others, this research includes

Vardoulakis, I. (1989). Shear-banding and liquefaction in granular materials on the basis of a Cosserat continuum theory, *Arch. Appl. Mech.* **59,** no. 2, 106–113,

Besdo, D. (1985). Inelastic behavior of plane frictionless block systems described as Cosserat media, *Arch. Mech.* **37,** no. 6, 603–619,

and

Latzel, M., S. Luding, and H. Herrmann (2001). *From Discontinuous Models: Towards a Continuum Description*, in Lecture Notes in Physics, Springer, New York and Berlin, 215–230.

Some authors suggest a reduced Cosserat model for the description of soils and granular materials, and in this reduced Cosserat continuum, translations and rotations are independent but the medium does not react on the gradient of rotation. As a consequence, the couple-stress tensor is zero. Such models can be used for media that do not react by trying to minimize the gradient in the rotation of neighboring particles but instead resist the rotation of each particle relative to the background.

An elastic isotropic reduced Cosserat model was suggested for the description of granular materials in

Schwartz, L. M., D. L. Johnson, and S. Feng (1984). Vibrational modes in granular materials, *Phys. Rev. Lett.* **52,** no. 10, 831–834.

The plastic reduced Cosserat model for granular materials is discussed in

Harris, D. (2006). Double-slip and spin: dilatant shear in a reduced Cosserat model, *Modern Trends in Geomechanics*, Springer Proc. in Physics, No. **106,** *International Workshop on Modern Trends in Geomechanics*, 27–29 June 2005, Vienna, 329–346.

Harris proposes a constitutive equation for three-dimensional deformation of granular materials. The equations generalize the classical plastic potential model by the addition of a noncoaxial term of the same form as the double-shearing model but in which the stress rate is replaced by the intrinsic spin of a reduced Cosserat continuum. The paper demonstrates that, in the context of dilatant shear, an initially noncoaxial flow may approach asymptotic coaxiality between the stress and deformation-rate tensors.

Porous media are multiphase materials in which two or three continua (solid, liquid, or/and gas) are intermixed. The seminal work on porous media is

Biot, M. (1955). Theory of elasticity and consolidation for a porous anisotropic solid, *J. Appl. Phys.* **26,** 182–185.

Many works and even entire scientific journals are devoted to this branch of mechanical science. The survey of this enormous field is beyond the scope of this paper, so we will limit ourselves to simply mentioning the following:

Molotkov, L., and A. Bakulin (1998). On attenuation in layered porous Biot media and their effective models, *Zap. Nauchnykh Semin. POMI* **250,** 244–262,

which deals with layered Biot-type media,

Tolstoy, I. (1992). *Acoustics, Elasticity, and Thermodynamics of Porous Media: Twenty-One Papers by M. A. Biot*, I. Tolstoy

(Editor), Acoustical Society of America, AIP Press, Melville, New York,

and

Biot, M. A. (1970). *Variational Principal in Heat Transfer*, Clarendon, Oxford, U.K., 185 pp.

Suggested Readings in Earthquake Seismology

The literature on earthquakes is vast and consists of millions of items in the form of books, journal papers, reports, and station bulletins. A good short introduction to the physics of earthquakes may be found in

Kanamori, H., and E. E. Brodsky (2001). The physics of earthquakes, *Phys. Today*, June 2001, 34–40, available from http://ptonline.aip.org/journals/doc/PHTOAD-ft/vol_54/iss_6/34_1.shtml (last accessed January 2009).

A more detailed review article with the same title is

Kanamori, H., and E. Brodsky (2004). The physics of earthquakes, *Rep. Prog. Phys.* **67,** no. 8, 1429–1496.

For nonseismologists, we provide the following suggested readings [mostly of recent publications] grouped into four subsections: (1) elementary books, (2) intermediate-level textbooks, (3) advanced books, and (4) general reference volumes.

Elementary Books

Bolt, B. A. (2005). *Earthquakes: 2006 Centennial Update*, W. H. Freeman, New York, 320 pp.

Hough, S. E. (2002). *Earthshaking Science: What We Know (and Don't Know) About Earthquakes*, Princeton Univ. Press, Princeton, New Jersey, 238 pp.

National Research Council (2003). *Living on an Active Earth: Perspectives on Earthquake Science*, National Academies Press, Washington, D.C., 418 pp.

Richter, C. F. (1958). *Elementary Seismology*, W. H. Freeman, San Francisco, California, 768 pp. [This is a classic work and should be read by anyone who is interested in earthquakes. It has long been out of print, but many libraries have it.]

Shearer, P. (1999). *Introduction to Seismology*, Cambridge Univ. Press, Cambridge, U.K., 260 pp.

Intermediate-Level Textbooks

Havskov, J., and G. Alguacil (2004). *Instrumentation in Earthquake Seismology*, Springer, Dordrecht, the Netherlands, 358 pp., with 1 CD-ROM attached.

Kennett, B. L. N. (2001). *Introduction and Theoretical Development*, in The Seismic Wavefield, Vol. **1**, Cambridge Univ. Press, Cambridge, U.K., 370 pp.

Kennett, B. L. N. (2002). *Interpretation of Seismograms on Regional and Global Scales*, in The Seismic Wavefield, Vol. **2**, Cambridge Univ. Press, Cambridge, U.K., 534 pp.

Lay, T., and T. C. Wallace (1995). *Modern Global Seismology*, Academic, San Diego, California, 521 pp. [This is a popular textbook.]

Newmark, N. M., and E. Rosenblueth (1971). *Fundamentals of Earthquake Engineering*, Prentice Hall, Englewood Cliffs, New Jersey, 640 pp. [This is a basic textbook.]

Pujol, J. (2003). *Elastic Wave Propagation and Generation in Seismology*, Cambridge Univ. Press, Cambridge, U.K., 444 pp.

Scholz, C. H. (2002). *The Mechanics of Earthquakes and Faulting*, Second Ed., Cambridge Univ. Press, Cambridge, U.K., 471 pp.

Stein, S., and M. Wysession (2003). *An Introduction to Seismology: Earthquakes and Earth Structure*, Blackwell, Malden, Massachusetts, 498 pp.

Yeats, R. S., K. Sieh, and C. R. Allen (1997). *The Geology of Earthquakes*, Oxford Univ. Press, New York, 568 pp. [This is an up-to-date textbook written in the spirit of Richter's *Elementary Seismology* (Richter, 1958)].

Advanced Books

Abercrombie, R., A. McGarr, G. di Toro, and H. Kanamori (Editors) (2006). *Earthquakes: Radiated Energy and the Physics of Faulting*, American Geophysical Monograph **170**, 327 pp.

Aki, K., and P. G. Richards (2002). *Quantitative Seismology*, Second Ed., University Science Books, Sausalito, California, 700 pp. [This is a classic advanced-level textbook that has become the most frequently cited reference in seismological research papers since its first edition in two volumes in 1980.]

Ben-Menahem, A., and S. J. Singh (1998). *Seismic Waves and Sources*, Second Ed., Dover, New York, 1136 pp.

Eringen, A. C., and E. S. Suhubi (1975). *Elastodynamics*, Vol. **2**, Academic, New York, 1015 pp.

Teisseyre, R., H. Nagahama, and E. Majewski (Editors) (2008). *Physics of Asymmetric Continuum: Extreme and Fracture Processes. Earthquake Rotation and Soliton Waves*, Springer, Berlin, 293 pp. [This is a monograph on rotational seismology.]

Teisseyre, R., M. Takeo, and E. Majewski (Editors) (2006). *Earthquake Source Asymmetry, Structural Media and Rotation Effects*, Springer, Berlin, 582 pp. [This is the first monograph on rotational seismology.]

General Reference Volumes

Lee, W. H. K., H. Kanamori, P. C. Jennings, and C. Kisslinger (Editors) (2002). *International Handbook of Earthquake and Engineering Seismology, Part A*, Academic, Amsterdam, 933 pp., with 1 CD-ROM attached. [This is the first part of the Centennial Volume of the International Association of Seismology and Physics of the Earth's Interior (IASPEI), and it serves as an encyclopedic source for earthquake seismology and its applications.]

Lee, W. H. K., H. Kanamori, P. C. Jennings, and C. Kisslinger (Editors) (2003). *International Handbook of Earthquake and Engineering Seismology, Part B*, Academic, Amsterdam, 1006 pp., with 2 CD-ROMs attached. [This is the second part of the Centennial Volume of IASPEI, and it serves as an encyclopedic source for earthquake seismology and its applications. An extensive subject index (combined for both parts A and B) is included at the end of this volume—a good starting point to find earthquake information.]

Meyers, R. A. (Editor-in-Chief) (2009). *Encyclopedia of Complexity and Systems Science*, Springer, New York, about 10,000 pp. [This Encyclopedia includes 42 articles on earthquakes, tsunamis, and volcanoes, edited by W. H. K. Lee, with an introduction that serves as a road map.]

Schubert, G. (Editor-in-Chief) (2007). *Treatise on Geophysics*, Elsevier, Amsterdam, 7000 pp. [Volume 4 is *Earthquake Seismology*, edited by Hiroo Kanamori.]

Data and Resources

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References

Cosserat, E., and F. Cosserat (1909). Théorie des Corps Déformables, A. Hermann et Fils, Paris, 226 pp., available from the Cornell University Library Digital Collections.

Lee, W. H. K., M. Çelebi, M. I. Todorovska, and H. Igel (2009). Introduction to the special issue on rotational seismology and engineering applications, *Bull. Seismol. Soc. Am.* **99,** no. 2B, 945–957.

Maugin, G. A. (Acting Chair) (2008). Mechanics of generalized continua: a hundred years after the Cosserats, First Announcement, *EuroMech Colloquium 510, UPMC*, Paris, France, May 13–16, 2009: available at http://www.euromech.org/colloquia/2009/510 (last accessed January 2009).

Appendix

Introductory Notes on Advanced Continuum Mechanics

We describe briefly a general scheme for deducing the basic equations representing any material. In each such theory there are two parts: mathematics and physical arguments. First, let us consider the mathematics. To understand the physical meaning of a theory it is necessary to know the applicable mathematics—tensor analysis, also called tensor calculus. The necessary material can be found in chapter 2 of Fung (1965) or of Fung and Tong (2001), cited in the section Fundamental Textbooks; in the appendices to the books by Lurie (1990, 2005), cited in the section Intermediate to Advanced Books.

Generally speaking, all natural phenomena are nonlinear, although in some cases one may apply a linear approximation and linearized theories have been used successfully in many applications for centuries. However, in the past few decades, many scientists have recognized the limitations of linear and linearized theories, and many new tools have been developed for solving nonlinear problems (see, e.g., Meyers, 2009, cited in the section General Reference Volumes).

To describe mathematically the deformations in nonlinear materials, one needs to understand nonlinear kinematics. The necessary concepts are the rotation tensor and nonlinear strain tensors, which are explained, for example, in the books cited previously and by Truesdell (1991) and Palmov (1998) cited in the section Intermediate to Advanced Books.

To deduce the basic equations for any medium, we must choose the model of a particle of the material. A particle in continuum mechanics is a point mass, an infinitesimal rigid or deformable body, or a more complex object. Sometimes, a particle consists of two point masses of different kinds and a bond between them (multiphase models, e.g., for porous media saturated with liquid). A particle (or point body) is always infinitesimal. We can consider it as an enriched point characterized, apart from mass density and translational displacement, by some other scalar, vector, and tensor fields—for instance, density of the moment of inertia and rotation. Next, we apply the following physical principles (each to a small volume of material or to a small volume of space containing the material):

1. The law of conservation of mass. If we follow a small material volume (Lagrangean or material description), this law holds identically. If we use the Eulerian description (considering a volume in space), it provides an independent relation. In the Lagrangean description of mechanical evolution, we use the material point and time as independent variables, whereas for the Eulerian description we use the spatial location and time as independent variables. See Fung (1965, pp. 119).
2. The laws of dynamics. These were written in their final form by L. Euler (1707–1783): (1) The law of balance of linear momentum states that in an inertial system of reference, the rate of change of linear momentum of a system of bodies is equal to the total applied force acting on the system plus the income of linear momentum, and (2) the law of balance of angular momentum (or kinetic moment) states that in an inertial system of reference, the rate of change of kinetic moment is equal to the total applied torque acting on the body, calculated relative to the same point, plus the income of the kinetic moment.
3. The first law of thermodynamics, which is the law of the balance of energy.
4. The second law of thermodynamics, which stipulates that the entropy never decreases. This principle holds true identically in the case of adiabatic processes in elastic media.

If we write down all of these principles, we find that the system of equations is not complete. To complete it, we must know how the internal forces and torques depend upon the relative positions and rotations of material particles and some other state-defining characteristics (for instance, temperature). In terms of a continuum, we need a relation between stress tensors on one side of the equation and strain tensors and other defining state parameters on the other side. These relations are called constitutive equations (a term introduced by W. Noll in 1954), and they are different for different materials. When we substitute the constitutive equations into the laws of dynamics, we obtain equations for the displacements in the medium.

Strain energy and entropy depend upon the deformations and temperature. The work of forces and torques upon translational and angular (or rotational) velocities and the increase of heat enter into these balance laws. In this way, the basic principles stated previously give us guidance for the constitutive equations, coupling the various basic physical quantities. As a result, we obtain stress tensor(s) expressed via derivatives of the free energy with respect to strain tensors, the tensor of rotation (for polar media), and the gradient of displacement. However, these principles do not determine the complete constitutive equations unless we also know how the free energy depends upon the defining parameters.

Another restriction narrows the range of possible constitutive equations. Intuitively, it is clear that a material's behavior does not depend upon the reference frame of an observer. Specifically, if a part of the material is subjected to a rigid motion, the stress and couple-stress tensors will be rotated together with the material, as if they were embedded in it, and the translational part of the rigid motion will not influence them. In other words, if an observer walks around a sample of the material, the material's behavior will not change, but the observer will see the stress and strain tensors rotated according to his motion. This constraint, though obvious, is very important and excludes many constitutive equations that otherwise might seem plausible. Because of the importance of this constraint, it has been given a special name: the principle of material objectivity, also called material indifference.

All of the constitutive equations that satisfy the principles stated previously describe materials that can exist in nature. Despite the fact that these principles seem very general, they limit the range of possible equations to realizable materials.

Thus, the principles given previously must be applied to any real material. In some cases, we may use additional principles, for example, the Curie–Neumann principle, which applies symmetry considerations to the constitutive equations, if the material exhibits some symmetry. Often, we may consider a material as isotropic, or transversely isotropic, orthotropic, etc. A widely used special case is a linear material in which nonlinear equations are linearized, yielding many simplifications. If the material is stable (i.e., not likely to undergo a chemical change, fracturing, or other change of condition in response to stresses), then one must write equations for this stability, a difficult problem in the nonlinear case. In linear elasticity, stability implies that the strain energy has to be positively definite. However, not all of the materials are stable. Also, there are models that allow phase transitions, breaking, and some discontinuities, and if we want to describe an explosion, we must use an unstable model.

If we apply all of these principles to deduce all possible equations for a linear isotropic elastic material consisting of mass points, we obtain the Lamé equations relating stress and strain (Fung, 1965, equations 7 and 8 on p. 129), and the only freedom we have is the choice of the Lamé constants or equivalent. The same formalism applied to a linear isotropic elastic material consisting of point bodies with rotational degrees of freedom leads us to the equations of a linear, elastic, isotropic Cosserat continuum, defined by six material parameters.

Physical arguments indicate which mathematical description is more appropriate. Indeed, the scheme described previously gives us the expression for stresses via derivatives of the free energy with respect to strains and kinematic characteristics. If we use strain tensors that are materially objective (the tensors rotate with the material if it is subjected to rigid whole-body motion) and define the free energy as a function of these strain tensors, then any constitutive equations obtained via this scheme satisfy automatically the principle of material objectivity.

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