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About the Nonunique Sensitivity of Pendulum Seismometers to Translational, Angular, and Centripetal Acceleration

by Thomas Forbriger

Abstract The displacement of each part of a seismometer's frame is identical for a purely translational motion. However, in the presence of rotary motion the different parts of a seismometer's frame will undergo different displacements. The definition of the sensitivity of the seismometer then requires the selection of a reference location on the seismometer's frame to which the sensitivity is attributed. This location does not necessarily coincide with the hinge and can be selected arbitrarily. The appropriate choice is to attribute the output signal to the location of the point mass of the equivalent simple pendulum (or reduced pendulum), which usually lies within the seismometer's casing. Rotations of the sensor about this location produce no output signal due to angular or centripetal acceleration. The sensor then appears sensitive to linear acceleration only.

Introduction

In the context of rotational seismology possibilities for the observation of rotary components of seismic displacement are frequently discussed. Their observation requires appropriate sensors, which are sensitive to either angular velocity like ring laser gyroscopes (Stedman *et al.*, 1995) or to angular acceleration. In the present study I specifically discuss the sensitivity of pendulum seismometers to angular acceleration. Angular acceleration is occasionally considered in studies of strong-motion observations in the near field of earthquakes. In contrast to teleseismic observations in the far field, where angular acceleration can safely be ignored, contributions near earthquake faults can be observable. Also, structures can respond with significant torsional vibrations to any kind of ground shaking. For this reason angular acceleration should be a supplementary recording when monitoring the seismic response of buildings. Observation of angular acceleration is also discussed in the context of inferring horizontal ground displacement from seismic recordings. The output signal of horizontal component seismometers can be significantly contaminated by ground tilt induced gravity, which inhibits the deduction of displacement (e.g., Graizer, 2006). The record can be corrected for the tilt-induced contribution with an independent record of the tilt angle, which theoretically could be obtained from an observation of angular acceleration. With this background Bradner and Reichle (1973) and Graizer (2009b) discuss potential configurations of inertial sensors to observe angular acceleration.

Pendulum seismometers are primarily regarded as being sensitive to translational acceleration. However, exposing such an instrument to angular acceleration will in most cases also result in an output signal. Appropriate terms, which describe the sensitivity to angular and centripetal acceleration, appear in theoretical descriptions of pendulum seismometers. For example, Byerly (1952) derives the full equation of motion based on the work by early pioneers of seismometry. Rodgers (1968) extends this with a focus on the response to tilt-induced gravity. In the present issue general expressions for the response of pendulum seismometers are presented by Peters (2009) and Graizer (2009a,b).

While the meaning of the output signal in terms of a component of linear acceleration is unique in the absence of rotations, the partitioning into linear and angular contributions becomes ambiguous in the presence of rotations. The motions of the sensor must then be referred to a reference location in the seismometer's frame, and the instrument's sensitivity will depend on the choice made for this reference. I am not aware of any theoretical study of the response of pendulum seismometers that would point out this fact. It is common to refer all motion of the seismometer's frame to the location of the hinge, which, however, is an arbitrary choice. Rodgers (1969) and Pillet and Virieux (2007) explicitly discuss the sensitivity to angular acceleration in this way. Consequently, they miss the fact that this is ambiguous with respect to the reference location in the seismometer's frame and that it vanishes if the reference location is chosen at the point mass of the equivalent simple pendulum.

The equivalent simple pendulum has the whole mass of the suspended body concentrated in a point mass and has the same free period in a gravity field. This concept is known as the reduced pendulum in the theory of the reversible pendulum (Rodgers, 1969; Leybold, 2007). The location of the point mass of the equivalent simple pendulum is sometimes referred to as center of oscillation (Byerly, 1952).

The intention of my study is to illuminate this non-uniqueness and to show that pendulum seismometers can be regarded as insensitive to angular and centripetal acceleration as well. This is not only due to angular acceleration being small in practical applications, but it can also be shown in a mathematically rigorous way.

After two basic remarks regarding the problem under consideration, I introduce a simplified model for a single seismic pendulum sensor, which will be used for all subsequent investigations. This model can already be used for simple physical considerations that illustrate the inherent ambiguity when discussing the partitioning of the pendulum's deflection into contributions resulting from translational and angular acceleration. Subsequently, I derive this ambiguity quantitatively and discuss aspects such as the customary perception of seismic sensors, the effect of centripetal acceleration, the behavior of linearly suspended sensors (geophones), and the possibility to distinguish all six translational and rotary degrees of freedom when using at least six sensors that are suitably arranged within one rigid frame. Finally, the demands for the design of a sensor for angular acceleration are discussed.

Rigid-Body Motion

The motion of the seismometer's frame with the ground is the motion of a rigid body. In the absence of rotations (as is customarily considered) all points on the frame undergo the same motion (Fig. 1). In this case there is no difference as to which point we attribute the acceleration derived from the seismometer's output signal. However, in the presence of rotations the motion must be expressed by the translational displacement of a reference point on the seismometer's frame and the rotation of the frame about this moving reference point. The reference point can be understood as the origin in a coordinate system that moves with the frame, and the choice of its location within this system is entirely arbitrary. Using a different reference point just equals the choice of a different origin to express the kinematic motion, while the motion itself remains the same. The rotary component is not altered by the choice of a different origin, while the translational component can even vanish if the reference point is appropriately chosen (Fig. 2). Aki and Richards (2002, fig. 12.7) define the reference location (i.e., the origin of the seismometer's coordinate system) at the Earth's center to express all horizontal motions of the sensor along the Earth's surface by a pure rotation about the center of the Earth.

Remarks on Ground Tilt

The sensitivity to angular acceleration must not be confused with the sensitivity to ground tilt. Ground tilt couples gravity into the horizontal components of seismometers (e.g., Wielandt, 2002, sec. 2.4). Gravity is a linear acceleration and cannot be distinguished from translational linear acceleration due to fundamental physics (i.e., the equivalence of gravitational and inertial mass.) The theory of general relativity was developed to unify both within one physical concept. In the presence of gravity all terms that represent a component of translational linear acceleration in the mathematical description given subsequently can also describe the effect of the corresponding component of gravity if it is not compensated by a suspension.

While the acceleration resulting from a periodic displacement at a given amplitude decreases with the square of the signal's period (Forbriger, 2007), the tilt-induced acceleration remains constant. For this reason it becomes a necessity to account for ground tilt when inferring true horizontal ground displacement from long-period seismic records (e.g., Dahlen and Tromp, 1998, table 10.1; Yuan *et al.*, 2005; Graizer, 2006). This process can be successful only if the displacement's time history is known from observations of the vertical component (Wielandt and Forbriger, 1999) or if the tilt can be obtained reliably from independent observations

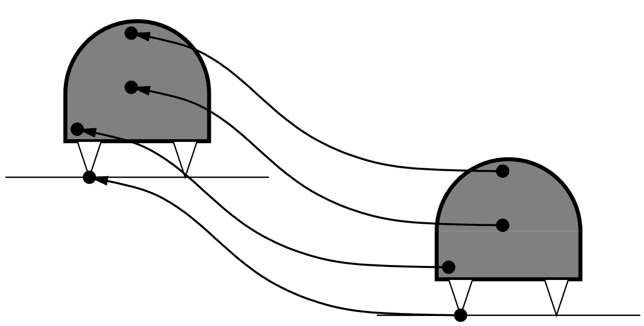

Figure 1. In the absence of rotations all locations in the seismometer's reference frame undergo the same displacement. It is not necessary to define a reference location to which the motion is attributed.

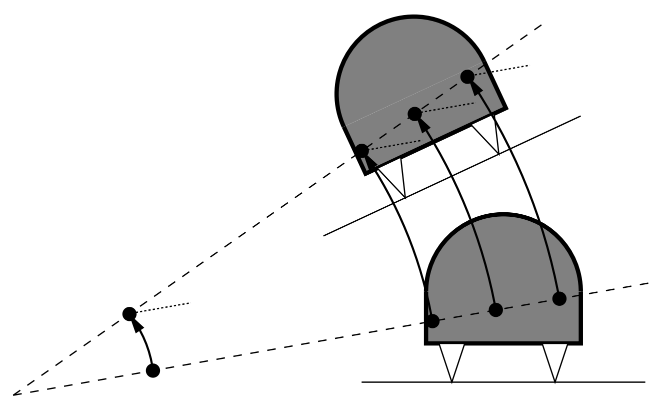

Figure 2. In the presence of rotations all locations in the seismometer's reference frame undergo a different displacement. To quantify displacement, a reference location must be defined. The angle of rotation about this reference location is independent of the choice made. Translational displacement depends on the reference location and vanishes if the center of rotation is selected as the reference.

of angular acceleration. The latter is not possible with any sensors for translational acceleration.

A Simple Model of the Pendulum Seismometer

In the present study I use a simplified model of the seismometer that is appropriate to focus on the partitioning between linear, angular, and centripetal acceleration. Simplification takes place in four respects: (1) Motion takes place in only one plane (two translational degrees and one rotary degree of freedom). (2) Gravity is ignored because it is indistinguishable from translational acceleration due to fundamental physics. (3) The pendulum experiences no restoring force and no damping. Both would influence the frequency response but not the partitioning of sensitivity at a given frequency. (4) The pendulum will be held in its reference location by a feedback mechanism, which is usual in modern broadband seismometers. Hence, the orientation of the pendulum with respect to the seismometer's frame is constant.

Consider a simple single-component seismic sensor. A sketch is displayed in Figure 3. The large box is the seismometer's frame. The sensor's mass m is represented by the gray pendulum body. It is an extended mass (not a point mass) and, therefore, has a finite moment of inertia J_S for rotations centered on S (the center of mass). The pendulum is supported by a hinge at location H, which constrains the motion of the pendulum with respect to the seismometer's frame to one degree of freedom. The motion along this degree of freedom (rotation centered on H) is measured by the angle φ with respect to inertial space. The hinge is attached to the seismometer's frame and, thus, moves with the ground. Without loss of generality all motions are restricted to translational displacement in the (x, y)-plane and to rotations centered on an axis perpendicular to the drawing plane, because most forces due to motions in other directions are entirely compensated by the hinge of the pendulum. Only centripetal forces due to rotations about an axis that is parallel to the (x, y)-plane can additionally contribute to the signal produced by the seismometer, as will be discussed in the following section. The motion of the frame is expressed by the displacement of the reference location R and rotations about R. The location of R on the frame may be chosen arbitrarily.

The initial orientation of the sensor does not matter, because gravity does not appear explicitly in this model. The results are valid for horizontal as well as vertical or any oblique pendulum seismometer component. Gravity can easily be incorporated in the model as part of the linear acceleration, as will be shown in the mathematical derivation in the Translational Acceleration section.

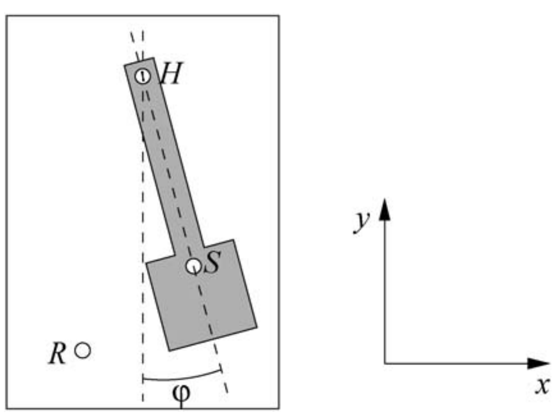

Figure 3. A simple model for a single seismic sensor. The box represents the frame of the seismometer, which moves rigidly with the ground. The gray body represents the seismometer's pendulum, which constitutes the seismic mass with finite moment of inertia. The center of mass is located at S. The pendulum is attached to the frame by a hinge at H, which constrains the motion of the pendulum to a single degree of freedom, that is, a rotation centered on H. R defines the reference location to which motions of the seismometer's frame are referred. Without loss of generality all motions are restricted to the (x, y) plane.

A Qualitative Consideration of Inertia

The effect to be discussed by mathematical derivation can already be understood from simple physical considerations of inertia. In Figure 4a the seismometer is displayed in its rest position. Now consider a motion of the seismometer's frame due to a ground motion. The pendulum body itself will sense this due to a motion of the hinge H. If the hinge is linearly displaced to the right (in the positive x direction), the pendulum will be deflected clockwise relative to the frame due to the inertia of the pendulum. This is the way we usually understand a seismometer.

If the motion is such that the frame is rotated about the hinge axis at H, the pendulum will rest at its position in inertial space because the hinge is assumed to exert no torque on the pendulum. We will observe a deflection relative to the seismometer's frame that equals the rotation of the frame.

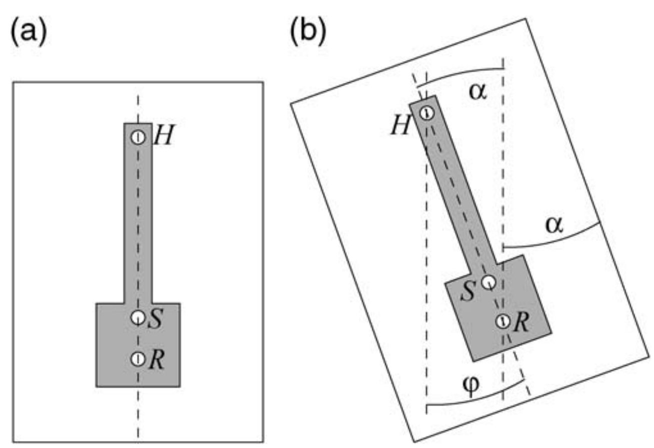

Figure 4. Rotary motion of the seismometer. (a) The seismometer in its reference position with the pendulum in its rest position. (b) Motion of the seismometer and pendulum in the special case of a rotation of the frame by an angle α centered on the location of the point mass of the equivalent simple pendulum. It turns out that $\ddot{\varphi} = \ddot{\alpha}$ if the distance $\overline{HR}$ equals the length of the equivalent simple pendulum.

This is a prevalent way to discuss sensitivity to angular acceleration.

Consider now the seismometer's frame being rotated about the initial location of S (center of mass) by an angle α. If the seismic mass were a point mass with vanishing moment of inertia ($J_S = 0$), the pendulum would easily follow the motion of the frame such that it preserves the location of its center of mass, thus preserving its linear momentum. No deflection would be observable with respect to the frame. However, if the moment of inertia is finite, the pendulum shows the desire to preserve its angular momentum too. Because of its translational acceleration the hinge must exert forces at H to accelerate the pendulum's body. The pendulum's body will try to resist the change in angular momentum by keeping its orientation in inertial space but will be slightly deflected with respect to inertial space in order to preserve linear momentum at the same time. The motion of the pendulum will be such that the torques due to angular and linear momentum with respect to an axis at H will cancel. $\ddot{\varphi}$ and $\ddot{\alpha}$ will not be equal in this case. If the frame is accelerated counterclockwise, the pendulum will be deflected clockwise with respect to the frame.

Now consider a rotation about a center R that is moved farther away from H along the line from H to S. The pendulum will still try to preserve its angular momentum by keeping its orientation in space. Because of the conservation of angular momentum only, a counterclockwise rotation of the frame would deflect the pendulum clockwise with respect to the frame. However, the linear motion of H to the left becomes larger the larger the distance $\overline{HR}$ is. The linear acceleration of the center of mass of a pendulum that preserves its orientation in space would, hence, become larger too. The torques on the pendulum due to angular and linear momentum with respect to an axis at H will cancel only for a larger $\ddot{\varphi}$. The distance between H and location R for the axis of rotation, where both effects cancel in the way that $\ddot{\varphi} = \ddot{\alpha}$ and such that there is no resulting deflection with respect to the frame (Fig. 4b), equals the length of the equivalent simple pendulum. The seismometer is insensitive to rotations centered on this axis. Because any motion can be geometrically decomposed into a translational displacement of this axis and a rotation centered on this axis, the seismometer apparently senses only the translational part of motion.

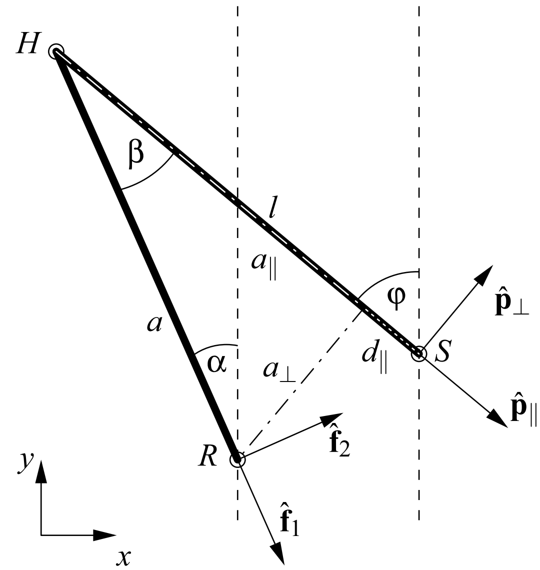

Figure 5. The geometry of the seismometer is defined by the locations of the center of mass of the pendulum's body at S, the hinge at H, and the reference point R on the frame. The response of the seismometer's pendulum to a translational displacement of R and a rotation of the frame centered on R is discussed. With respect to inertial space, φ defines the orientation of the pendulum and α defines that of the seismometer's frame. $\beta = \varphi - \alpha$ is the angle that is observed by the transducer in the seismometer. Distances $l = \overline{HS}$ and $a = \overline{HR}$ are constant and are displayed by thick lines. The unit vector in direction of the line connecting H and R is $\hat{\mathbf{f}}_1$, and $\hat{\mathbf{f}}_2$ is the unit vector perpendicular to it. They are base vectors of a coordinate system that moves and turns with the frame. Similarly, $\hat{\mathbf{p}}_{\|}$ and $\hat{\mathbf{p}}_{\perp}$ are unit vectors parallel and perpendicular to the line connecting H and S. They are base vectors of a coordinate system that moves and turns with the pendulum. Components $a_{\|}$ and $a_{\perp}$ of the vector from H to R in the pendulum's coordinate system are displayed by dash-dotted lines and are parallel and perpendicular to $\hat{\mathbf{p}}_{\|}$, respectively. Similarly, the component $d_{\|}$ of the vector from R to S is parallel to $\hat{\mathbf{p}}_{\|}$ and is displayed by white dots. Its component parallel to $\hat{\mathbf{p}}_{\perp}$ is $d_{\perp} = a_{\perp}$.

Quantitative Solution

Geometry of the Problem

The geometry that I use for the derivation of the equation of motion is defined in Figure 5. S is the location of the pendulum's center of mass, H is the location of the hinge of the seismometer's pendulum, and R is the reference point on the seismometer's frame. Their location vectors are $\mathbf{s}$, $\mathbf{h}$, and $\mathbf{r}$, respectively. They are defined in a coordinate system of inertial space. R can be understood as the origin of a coordinate system that moves and turns with the frame. $\hat{\mathbf{f}}_1$ and $\hat{\mathbf{f}}_2$ are base vectors within this coordinate system and turn with the frame. $\hat{\mathbf{f}}_1$ is in the direction of the line connecting H and R, while $\hat{\mathbf{f}}_2$ is perpendicular to it. Similarly, H may be understood as the origin of the pendulum's coordinate system with the base vectors being $\hat{\mathbf{p}}_{\|}$ and $\hat{\mathbf{p}}_{\perp}$, which turn with the pendulum. $\hat{\mathbf{p}}_{\|}$ is in the direction of the line connecting H and S, while $\hat{\mathbf{p}}_{\perp}$ is perpendicular to it. Vector components parallel to $\hat{\mathbf{p}}_{\|}$ and $\hat{\mathbf{p}}_{\perp}$ control the sensitivity to different kinds of acceleration as will be discussed in the following section.

Only translational motions of R in the (x, y) plane and rotations centered on an axis through R perpendicular to this plane are considered. Only centripetal forces due to rotations about an axis that is parallel to the (x, y) plane such that $a_{\perp}$ is finite can additionally contribute to the signal produced by the seismometer as will be discussed in the next section. Forces due to all other motions are entirely compensated by the hinge and will not be observable. While the locations S, H, and R may change with time t, the distances $a = \overline{HR}$

and $l = \overline{HS}$ remain constant. The hinge at H and the reference location at R are fixed to the seismometer's frame. The angle $\alpha(t)$ thus defines the orientation of the seismometer's frame with respect to the y direction in inertial space. Similarly, the center of mass at S and the hinge at H are fixed to the pendulum's body. The angle $\varphi(t)$ thus defines the orientation of the seismometer's pendulum with respect to inertial space. The seismometer senses the deflection of the pendulum with respect to the frame. This is expressed by the angle $\beta(t) = \varphi(t) - \alpha(t)$.

Equation of Motion

The Lagrangian for the pendulum is

$$L = \frac{1}{2} m(\dot{s}_x^2 + \dot{s}_y^2) + \frac{1}{2} J_S \dot{\varphi}^2, \tag{1}$$

where m is the total mass of the pendulum, J_S is its moment of inertia for angular acceleration centered on S, and a dot means derivation with respect to time. With the components

$$s_x(t) = r_x(t) - a \sin\alpha(t) + l \sin\varphi(t) \tag{2}$$

and

$$s_y(t) = r_y(t) + a \cos\alpha(t) - l \cos\varphi(t) \tag{3}$$

of the vector $\mathbf{s}(t) = \mathbf{r}(t) - a\hat{\mathbf{f}}_1[\alpha(t)] + l\hat{\mathbf{p}}_\parallel[\varphi(t)]$, the motion of the center of mass is expressed by the motion of the reference location R and the rotation of the seismometer's frame centered on R as well as the rotation of the pendulum centered on the hinge at H. This results in

$$\dot{\mathbf{s}}(t) = \dot{\mathbf{r}}(t) - a\dot{\alpha}(t)\hat{\mathbf{f}}_2[\alpha(t)] + l\dot{\varphi}(t)\hat{\mathbf{p}}_\perp[\varphi(t)] \tag{4}$$

and

$$\begin{aligned}\ddot{\mathbf{s}}(t) = \ddot{\mathbf{r}}(t) - a\ddot{\alpha}(t)\hat{\mathbf{f}}_2[\alpha(t)] + a\dot{\alpha}^2(t)\hat{\mathbf{f}}_1[\alpha(t)] \\ + l\ddot{\varphi}(t)\hat{\mathbf{p}}_\perp[\varphi(t)] - l\dot{\varphi}^2(t)\hat{\mathbf{p}}_\parallel[\varphi(t)].\end{aligned} \tag{5}$$

With Lagrange's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0, \tag{6}$$

the equation of motion is

$$ml(\ddot{\mathbf{r}}\hat{\mathbf{p}}_\perp - a\ddot{\alpha}\hat{\mathbf{f}}_2\hat{\mathbf{p}}_\perp + a\dot{\alpha}^2\hat{\mathbf{f}}_1\hat{\mathbf{p}}_\perp) + (ml^2 + J_S)\ddot{\varphi} = 0 \tag{7}$$

after a few pages of calculus. For simplicity, the explicit dependency on time and angles α and φ is dropped from here on.

With

$$\beta = \varphi - \alpha, \tag{8}$$

$$\hat{\mathbf{f}}_2\hat{\mathbf{p}}_\perp = \cos\beta, \tag{9}$$

and

$$\hat{\mathbf{f}}_1\hat{\mathbf{p}}_\perp = -\sin\beta, \tag{10}$$

the equation of motion is expressed by the angle β that is observed by the transducer of the seismometer as well as $\mathbf{r}$ and α, which define the motion of the seismometer's frame. The resulting differential equation is

$$\begin{aligned}\underbrace{(ml^2 + J_S)}_{=J_H}\ddot{\beta} = -ml\ddot{\mathbf{r}}\hat{\mathbf{p}}_\perp + ml\ddot{\alpha}\Bigg[\underbrace{a\cos\beta}_{=a_\parallel} - \Bigg(\underbrace{l + \frac{J_S}{ml}}_{=l_{\text{esp}}}\Bigg)\Bigg] \\ + ml\dot{\alpha}^2\underbrace{a\sin\beta}_{=a_\perp},\end{aligned} \tag{11}$$

where J_H is the moment of inertia of the pendulum for angular acceleration centered on the hinge at H and l_{esp} is the length of the equivalent simple pendulum. $a_\parallel$ and $a_\perp$ are components of the vector

$$\mathbf{r} - \mathbf{h} = a_\parallel\hat{\mathbf{p}}_\parallel + a_\perp\hat{\mathbf{p}}_\perp \tag{12}$$

from the hinge at H to the reference location R given in the coordinate system of the pendulum. They are the components parallel and perpendicular to the pendulum's direction $\hat{\mathbf{p}}_\parallel$ from H to S and, therefore, depend on β.

Sensitivity of the Pendulum

The sensitivity of the pendulum to the different constituents of motion as derived from equation (11) becomes apparent by

$$\ddot{\beta} = -\frac{1}{l + \frac{J_S}{ml}}\left[\ddot{r}_\perp + \ddot{\alpha}\left(l + \frac{J_S}{ml} - a_\parallel\right) - \dot{\alpha}^2 a_\perp\right]. \tag{13}$$

There, $\ddot{r}_\perp = \ddot{\mathbf{r}}\hat{\mathbf{p}}_\perp$ is the component of the translational acceleration of R perpendicular to the pendulum's direction from H to S (i.e., the acceleration in the sensitive direction $\hat{\mathbf{p}}_\perp$ of the seismometer).

Upon excitation of the pendulum, β will vary, and hence, $a_\parallel$ and $a_\perp$ may vary too. Therefore, we must understand equation (13) as defining the sensitivity of the pendulum in its reference position. Alternatively, the pendulum can be considered within a feedback loop that keeps variations of β generally small and vanishing in the temporal average. In this way equation (13) becomes a more general description of the sensitivity. $a_\parallel$ and $a_\perp$ are then defined by the location of R with respect to the seismometer's frame only and, thus, remain constant. Both are controlling the partitioning of the different contributions to the seismometer's sensitivity.

Translational Acceleration. The contribution to equation (13) due to $\ddot{r}_\perp$ is the sensitivity to translational acceleration. $\hat{\mathbf{p}}_\perp$ defines the sensitive direction of the seismometer, and the pendulum is entirely insensitive to the component $\ddot{r}_\| = \ddot{\mathbf{r}}\hat{\mathbf{p}}_\|$ of translational acceleration. This is customarily considered when discussing the properties of inertial seismometers. In the absence of rotary components ($\dot{\alpha} = 0$ and $\ddot{\alpha} = 0$) all locations on the seismometer's frame undergo the same motion. Then it is not necessary to distinguish between possible reference points, and $\ddot{r}_\perp$ may be replaced by $\ddot{h}_\perp$, for example.

If $\mathbf{g}$ is the vector of gravity acting on the pendulum's mass, $\ddot{r}_\perp$ may be replaced by $\ddot{r}_\perp - \mathbf{g}\hat{\mathbf{p}}_\perp$ if $\mathbf{g}\hat{\mathbf{p}}_\perp$ is not compensated by a suspension. Equation (13) then describes the response to ground tilt also. For a horizontal seismometer on leveled ground $\mathbf{g}\hat{\mathbf{p}}_\perp = 0$.

Angular Acceleration. The second contribution is due to angular acceleration $\ddot{\alpha}$. It obviously vanishes for

$$a_\| = l_{\mathrm{esp}} = l + \frac{J_S}{ml}. \tag{14}$$

This is the case if $a_\|$ equals the length of the equivalent simple pendulum l_{esp}.

Centripetal Acceleration. The last contribution proportional to $\dot{\alpha}^2$ is due to the centripetal acceleration. It vanishes if the center of rotation (i.e., the reference location on the frame) is in line with H and S, and thus,

$$a_\perp = 0. \tag{15}$$

The motion of the seismometer was restricted to translations in the (x, y) plane and to rotations about an axis perpendicular to this plane. Rotations about an axis that is parallel to the (x, y) plane will also cause centripetal acceleration as long as the distance vector between the axis of rotation and the center of mass has a finite component parallel to $\hat{\mathbf{p}}_\perp$ in the (x, y) plane. However, angular acceleration centered on such an axis will have no contribution to equation (13), because the resulting forces are compensated by the hinge.

In a Streckeisen STS-2 seismometer with its homogeneous triaxial configuration (Wielandt, 2002), $a_\perp$ is finite for rotations centered on the vertical axis of symmetry of the instrument. This motion will exert the same centripetal acceleration on each of the internal U, V, and W component sensors. Their signals are added to form the vertical component's signal. Centripetal acceleration due to rotation about the axis of symmetry, therefore, will produce an apparent vertical acceleration in the output signal. In practice, this can be neglected because $a_\perp$ is too small to produce a significant contribution to the total output signal in seismological applications.

Scaling the Contribution of Rotations

Although α is independent of the selected reference location, its contribution to the sensitivity as defined by equation (13) can be controlled by the choice of R because this selection will define the factors $a_\|$ and $a_\perp$. At the same time the amount of displacement of R depends on the selection of R. This way the contributions to the output signal (which is independent of R naturally) can be seemingly shifted from translational to angular acceleration and vice versa. Choosing the reference location on the seismometer's frame according to equations (14) and (15) will place the reference at the location of the point mass of the equivalent simple pendulum. In this case

$$\ddot{\beta} = -\frac{1}{l + \frac{J_S}{ml}} \ddot{r}_\perp \tag{16}$$

even in the presence of rotations and angular acceleration. The pendulum appears entirely insensitive to the rotary component of motion. Notice that $\ddot{r}_\perp$ may not be replaced by $\ddot{h}_\perp$ (i.e., translational acceleration of the hinge) in this case. We can release the restriction to rotations about an axis perpendicular to the (x, y) plane here. As long as the axis of rotation passes through the point mass of the equivalent simple pendulum, no centripetal acceleration will contribute to the observation, and equation (16) is still valid.

On the other hand, the sensitivity to angular acceleration can be made arbitrarily large by $a_\| \to \infty$, thus moving R to infinity. In fact, all acceleration then will be expressed in terms of angular acceleration.

Linear Suspension

To discuss the sensitivity of a pendulum with linear suspension as if it is common in geophones, the deflection is more appropriately expressed by the component

$$b_\perp = l\beta \tag{17}$$

of the displacement of the center of mass at S in the sensitive direction of the pendulum with respect to the seismometer's frame. Further, the vector $\mathbf{d}$ from the reference location at R to that location on the frame where the center of mass (S) initially was located is used to express the sensitivity to rotation. With the components

$$d_\| = \mathbf{d}\hat{\mathbf{p}}_\| = l - a_\| \tag{18}$$

(Fig. 5) and

$$d_\perp = \mathbf{d}\hat{\mathbf{p}}_\perp = a_\perp \tag{19}$$

of $\mathbf{d}$ and the deflection expressed by $b_\perp$, the sensitivity becomes

$$\ddot{b}_\perp = -\frac{1}{1 + \frac{J_S}{ml^2}} \left[\ddot{r}_\perp + \ddot{\alpha}\left(\frac{J_S}{ml} + d_\|\right) - \dot{\alpha}^2 d_\perp \right]. \tag{20}$$

The case of a linear suspension is obtained by moving the hinge H to infinity while preserving R. This is equivalent to $l \to \infty$ and provides the result

$$\ddot{b}_\perp = -\ddot{r}_\perp - \ddot{\alpha} d_\parallel + \dot{\alpha}^2 d_{\perp,} \tag{21}$$

which is also obtained for a point mass with $J_S = 0$. As is expected the geophone is insensitive to rotations centered on its center of mass ($d_\parallel = 0$ and $d_\perp = 0$). Referring the motion to this location results in

$$\ddot{b}_\perp = -\ddot{r}_{\perp,} \tag{22}$$

which is customarily considered when discussing geophones.

More Than One Sensor in a Frame

The ambiguity regarding the decomposition of the seismometer's output signal into two translational components and one rotary component results from the observation of a motion with three degrees of freedom using a sensor that has only one degree of freedom. In the case of translational displacements in the (x, y) plane together with a rotation about an axis perpendicular to that plane, three appropriately aligned pendulums will allow a unique decomposition of the output signals into translational and rotary contributions. However, the deduction of the translational displacement in the presence of rotations still suffers from the nonuniqueness due to the freely selectable reference location for the kinematic description of the motion.

The pendulums must be aligned such that the locations of the point masses of the equivalent simple pendulums do not coincide. The pendulums will most appropriately be arranged star-like with angles of 120° between them. The distance to the center of the arrangement should be as large as possible, while all three must be attached to the same rigid frame (i.e., the seismometer's housing or the pier on which instruments are deployed). Choosing the point of symmetry in the center of the arrangement as the reference location R for all three pendulums will make the centripetal contribution to their output signal vanish due to $a_\perp = 0$.

Considering a general motion in space with six degrees of freedom, three seismic components are not sufficient for a unique decomposition. With conventional seismometers it is, therefore, impossible to distinguish translational and angular acceleration. Six appropriately aligned sensors are required in this case. However, this can be difficult or even impossible in practice. Using six conventional seismic sensors, the rotary contribution is derived from differences between components. Gain errors will be amplified in the differences and can make the result useless or even alter its sign. Consequently, Nigbor (1994) recommends the use of a combination of three sensors for linear acceleration and three sensors for angular acceleration rather than six linear sensors. Suryanto *et al.* (2006), however, have demonstrated that this concept is applicable to the rigid motion of an extended array of seismometers with significantly larger baselines between individual sensors. Graizer (2009a) also studied the potential of a set of pendulum sensors to observe rotary components of motion.

A Seismological Sensor for Angular Acceleration

In the optimal sensor for angular accelerations, the hinge is placed at the center of mass ($H = S$). Then, from equation (13)

$$\ddot{\beta} = -\ddot{\alpha} \tag{23}$$

with $l \to 0$ (see also Peters, 2009). Because the hinge is supposed to exert no torque, the pendulum will maintain its original orientation in inertial space while rigidly following any translational motion of the frame. Readings of the angle between the pendulum and the frame directly provide a measure of rotary motion. The sensitivity is independent of the reference location. However, imperfections of the real sensor can displace S from H and will make l finite and, therefore, produce a sensitivity to translational motion.

A sensor of this kind can be constructed with precisely balanced masses on one pendulum. In practice it is difficult to build a mechanical system of this high degree of symmetry (Graizer, 2009a). Alternatively, the sensor can be designed as a circular tube as in the R–1 (eentec, 2008), which uses an electrolyte-filled toroid. The tube has to be filled completely with an incompressible fluid of high density and low viscosity. The fluid is constrained by the walls of the tube and rigidly follows all motions except rotations centered on the axis of symmetry. This sensor will be entirely insensitive to translational motion and to tilt-induced gravity as well. However, in a rotating reference frame (as on the Earth's surface) the sensor will be sensitive to translational motion and tilt that change its angular momentum with respect to its axis of symmetry. In seismological applications these contributions are small compared to high-frequency angular acceleration centered on a local axis. Wassermann *et al.* (2009) and Nigbor *et al.* (2009) recently tested the performance of the R–1.

An advantage of the fluid design is that the sensitivity could be increased comparatively easily. For an observatory installation the circular tube could be enlarged to a few meters diameter. As in the fluid tiltmeter that is operated at the Black Forest Observatory (Horsfall and King, 1978; Emter *et al.*, 1989), a membrane in the tube could be used as a transducer. The membrane would constrain the fluid to follow the rotations of the sensor too, while sensing the force that is needed to accelerate the fluid. This force is proportional to $\ddot{\alpha} J_S / \rho$, where ρ is the radius of the toroid and the moment of inertia J_S increases with the radius of the circular tube to the power of three. Thus, increasing the dimension of the sensor from 10 cm to 10 m would potentially increase the sensitivity by a factor of 10,000. This could be done in an observatory installation. These simple considerations certainly ignore possible difficulties of a technical realization.

Conclusions

Any motion of a rigid seismometer's frame can be expressed geometrically by a translational displacement of a reference point on the frame and a rotation centered on this point. This reference point can be arbitrarily chosen while describing the same motion with a different amount of linear displacement. Because a single pendulum seismometer senses only one degree of freedom, the translational and rotary constituents of the motion cannot be inferred uniquely from the output signal. In consequence, the sensitivity to rotation and the amount of translation depend on the chosen reference location, while the sensitivity to translation and the amount of rotation as well as the seismometer's output signal are independent of this choice. Using the freedom to purposefully select a reference point, the contribution of rotary motion to the output signal can be shifted to the contribution by translational motion, and vice versa. In particular, the pendulum can apparently be made insensitive to angular acceleration in general. In fact, a pendulum seismometer will produce no output signal due to rotations about the location of the point mass of the equivalent simple pendulum. The definition of a universally valid sensitivity to angular acceleration is impossible as well as the definition of a sensitivity to linear acceleration in the presence of rotations. A definition of sensitivity always requires a preceding definition of a reference location for the kinematic description of motion. It is most appropriate to refer motions of pendulum seismometers to the location of the point mass of the equivalent simple pendulum, because the sensor then appears sensitive to linear acceleration only.

A complete seismometer, in general, would need six appropriately aligned sensors in one rigid frame to distinguish all degrees of freedom of linear and angular acceleration. The freedom to select an arbitrary reference location for the kinematic description is a remaining nonuniqueness when inferring translational displacement in the presence of rotations. Only the deduction of rotation from a six-component sensor is unique without the definition of a reference location.

The sensitivity to translational acceleration can be controlled by the design of the sensor such that it vanishes. This happens for $l_{\mathrm{esp}} \to \infty$, which is the case if J_S dominates (i.e., $l = 0$, which is not the case in conventional seismometers). In this way, by placing the center of mass of the pendulum and the hinge at the same location, a sensor for angular acceleration could be constructed that is entirely insensitive to translational acceleration and gravity induced by ground tilt. Angular acceleration can be derived from this sensor's output signal uniquely without the definition of a reference location. Theoretically, such a sensor has the potential to provide observations of the angle of ground tilt that is required to remove tilt effects from long-period seismic observations. However, the twofold integration that is necessary to obtain the angle from angular acceleration strongly increases long-period noise. A result for translational displacement even requires a fourfold integration of the angular acceleration time series (Bradner and Reichle, 1973). This and the difficulty of precisely balancing the mass in practice such that the center of mass and the hinge are at the same spot (Graizer, 2009a) renders the success of this approach questionable. By contrast, laser gyroscopes observe angular velocity. The angle can be obtained from their output by a single integration. For this reason they can be superior in this context.

Usually, the possible contributions due to rotations (not tilt-induced gravity) are ignored in interpretations of seismic signals. The magnitude of the resulting error can be estimated. Because the output signal is the same for all reference locations, we may chose the location of the point mass of the equivalent simple pendulum as reference without loss of generality. Ignoring rotary motions then means ignoring the change in sensitive direction $\hat{\mathbf{p}}_\perp$ due to the rotation. Translational acceleration derived from the seismometer's output will be attributed to one direction in inertial space. Changing its orientation by $\Delta\alpha$, the sensor will observe only $\cos\Delta\alpha$ of the component that is expected to be observed, while $\sin\Delta\alpha$ of the component to which the instrument is expected to be insensitive contributes to the output signal. The relative error with respect to translational acceleration in this direction is

$$\frac{\Delta\ddot{r}_\perp}{\ddot{r}_\perp} = \cos\Delta\alpha - 1 - \frac{\ddot{r}_\parallel}{\ddot{r}_\perp}\sin\Delta\alpha. \tag{24}$$

For a plane Love wave of wavelength λ and amplitude u of horizontal displacement,

$$\Delta\alpha \approx \pi\frac{u}{\lambda} \ll 1. \tag{25}$$

This immediately results from the expressions provided by Widmer-Schnidrig and Zürn (2009) for *SH*-type waves. The effect of rotary motion can, therefore, be safely ignored in seismological applications except for tilt-induced gravity.

The considerations discussed in this article show that a pendulum seismometer must be understood as sensitive to linear acceleration only in seismological applications. The observation of angular acceleration with appropriately designed sensors can, however, be interesting in the context of strong-motion observations in the near field or the observation of motions that structures undergo due to seismic shaking where it can become comparatively large.

Data and Resources

No data were used in this study. Figures were prepared with Xfig. The manuscript was typeset with LaTeX2ε. The list of references was assembled with bibTeX.

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