Bulletin of the Seismological Society of America, Vol. 99, No. 2B, pp. 1480–1482, May 2009, doi: 10.1785/0120080230

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Short Note

Software for Inference of Dynamic Ground Strains and Rotations and Their Errors from Short Baseline Array Observations of Ground Motions

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Abstract In two previous articles we presented a formulation for inferring the strains and rotations of the ground beneath a seismic array having a finite footprint. In this article we derive expressions for the error covariance matrices of the inferred strains and rotations, and we present software for the calculation of ground strains, rotations, and their variances from short baseline array ground-motion data.

Online Material: MATLAB scripts to calculate strain and rotation time series and errors, given array measurements of ground motions.

Introduction

In Spudich *et al.* (1995) and Spudich and Fletcher (2008) we presented a formulation for inferring the strains and rotations of the ground beneath a seismic array having a finite footprint. This formulation assumed linear elasticity and infinitesimal rotations, and it assumed that the displacement gradient tensor was spatially uniform beneath the array. In this short note and associated electronic supplement (ⒺMATLAB scripts to calculate strain and rotation time series and errors, given array measurements of ground motions, are available in the electronic edition of *BSSA*) we provide software for performing the calculations prescribed by that formulation, and we present some additional theory, implemented in the software of Spudich *et al.* (1995) and Spudich and Fletcher (2008) but not previously described, for estimating the error covariance matrices of the various strains and rotations. The provided software is designed for use with MATLAB, which is a high-level language and interactive environment produced by The MathWorks™. Because the total theory is distributed among this and two other articles, in this article we adhere to the terminology and notation of our two previously published articles as much as possible. We will occasionally associate our terms with the more favored terminology of Evans *et al.* (2009). Mathematical terms not defined here are presented in our previous two articles.

Error Covariance Matrices and Formal Error Estimates

Because the inferred strains and rotations are the result of a least-squares fit of noisy translational ground-motion data, we are obliged to estimate the errors in the inferred quantities based on the estimated errors in the ground-motion data. Because noise levels can vary from seismometer to seismometer, this variation can be accommodated by formulating the problem as one of weighted least squares, as follows. Let v_c^i be the variance (square of the standard deviation) of the $c = 1, 2, 3$ component of the station $i = 0, 1, \ldots, N$ ground displacement. (Note—throughout this article we will invert ground displacements to infer strains and rotations. Exactly the same formulation applied to ground velocities will yield strain rates and rotation rates, etc.) Generalizing from Spudich *et al.* (1995), here we assume the data noise covariance matrix is $C_u = \mathrm{diag}([\, v_1^0 \quad v_2^0 \quad v_3^0 \quad v_1^1 \quad v_2^1 \quad \ldots v_3^N \,])$. Here we use the MATLAB bracket notation to denote a row vector.

Spudich and Fletcher (2008) noted that two different types of digitizers having different noise levels were used in the U.S. Geological Survey Parkfield seismic array (UPSAR), and they adjusted their noise variances v_c^i appropriately, yielding formal error estimates on their inferred rotations. Examples include quoted standard deviations of tilt (rotation around a horizontal axis) and torsion (rotation around a vertical axis) in their figure 4. (Spudich *et al.*, 1995 also used different noise variances for different digitizers but neglected to mention this in their article.)

To get the covariance matrices of the various strains and rotations we will make relentless use of the following property. If vector $\mathbf{b}$ has covariance matrix C_b, and if $\mathbf{a} = B\mathbf{b}$, where B is some matrix, then the covariance matrix of $\mathbf{a}$ is

$$C_a = B C_b B^T \tag{1}$$

(Menke, 1984, from his equation 2.7). In our specific application the solution vector $\tilde{\mathbf{p}}$ (defined in the next section) and its covariance C_p will be used for $\mathbf{a}$ and C_a. For each quantity of interest we will present the appropriate B matrix, apply (1), and evaluate algebraically where easy. Note that standard deviations of the quantities of interest are the square roots of the corresponding diagonal elements of the derived covariance matrices.

Covariance of the Strain Tensor Elements

Recall that the free-surface boundary condition implies that strain tensor elements $e_{i3} = 0$ for $i = 1, 2$. Thus, a vector $\mathbf{e}$ made from the four independent nonzero elements of the strain tensor satisfies $\mathbf{e} = [e_{11}\ e_{21}\ e_{22}\ e_{33}\ ]^T = B_e\tilde{\mathbf{p}}$, where the unknown displacement gradient vector $\tilde{\mathbf{p}} = [u_{1,1}\ u_{1,2}\ u_{1,3}\ u_{2,1}\ u_{2,2}\ u_{2,3}]^T$ is a column vector of the six independent elements of the displacement gradient matrix, and the missing three elements are constrained by the free-surface boundary condition. We have then that

$$B_e = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ -2\eta & 0 & 0 & 0 & -2\eta & 0 \end{pmatrix},$$

where $\eta = \lambda/(\lambda + 2\mu)$, and the covariance matrix of $\mathbf{e}$ is $C_e = B_e C_p B_e^T$. The top four rows of B_e result from $e_{ij} = (u_{i,j} + u_{j,i})/2$, and the final row results from the free-surface boundary condition $u_{3,3} = \eta(u_{1,1} + u_{2,2})$.

Horizontal and Total Dilatation

Horizontal dilatation $d^h = B_{dh}\tilde{\mathbf{p}}$, where $B_{dh} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$. Therefore, the covariance matrix (which is a scalar variance in this case) of d^h is $C_{dh} = B_{dh}C_pB_{dh}^T = (C_p)_{11} + 2(C_p)_{15} + (C_p)_{55}$, and the standard deviation of d^h is $\sqrt{C_{dh}}$. Because the total dilatation $d = (1-\eta)d_h$, the variance of total dilatation is $\sigma_d^2 = (1-\eta)^2 C_{dh}$.

Covariances of Horizontal and Total Shear-Strain Tensor Elements

The shear-strain tensor is $\gamma = e - \frac{1}{3}\,\mathrm{Tr}(e)I$. Using $a = (2+\eta)/3$, $b = (1-\eta)/3$, and $c = (1+2\eta)/3$, we have $\begin{bmatrix} \gamma_{11} & \gamma_{21} & \gamma_{22} & \gamma_{33} \end{bmatrix}^T = B_\gamma\tilde{\mathbf{p}}$, where

$$B_\gamma = \begin{pmatrix} a & 0 & 0 & 0 & -b & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ -b & 0 & 0 & 0 & a & 0 \\ -c & 0 & 0 & 0 & -c & 0 \end{pmatrix},$$

and $C_\gamma = B_\gamma C_p B_\gamma^T$. Defining strain in the horizontal plane e^h as the first two rows and columns of strain tensor e, then shear strains across vertical planes are $\gamma^h = e^h - \frac{1}{2}\,\mathrm{Tr}(e^h)I$, $\begin{bmatrix} \gamma_{11}^h & \gamma_{12}^h & \gamma_{22}^h \end{bmatrix}^T = B_{\gamma h}\,\tilde{\mathbf{p}}$, and $C_{\gamma h} = B_{\gamma h}C_pB_{\gamma h}^T$, where

$$B_{\gamma h} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Covariances of the Maximum Principal Strains

Because the principal strains are eigenvalues of the strain tensor, we cannot apply our simple procedure and, thus, have not derived covariance matrices for these quantities.

Covariance of Rotation

Let the rotation tensor be ω. Then we have $\begin{bmatrix} \omega_{21} & \omega_{31} & \omega_{32} \end{bmatrix}^T = B_\omega\tilde{\mathbf{p}}$, where

$$B_\omega = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Not surprisingly, $C_\omega = B_\omega C_p B_\omega^T$, and the variances of torsion and tilt can be easily expressed algebraically.

Torsion, rotation around the x_3 axis, is $\omega_3 = B_{\omega_3}\tilde{\mathbf{p}}$, where $B_{\omega_3} = \begin{bmatrix} 0 & -1/2 & 0 & 1/2 & 0 & 0 \end{bmatrix}$, where we follow Fung (1965) who defines $\omega_{ij} = (u_{j,i} - u_{i,j})/2$. As usual $C_{\omega_3} = B_{\omega_3}C_pB_{\omega_3}^T$, yielding the expression found in Spudich and Fletcher (2008), $\sigma_{\omega_3} = \frac{1}{2}\sqrt{(C_p)_{22} - 2(C_p)_{24} + (C_p)_{44}}$. Expressions for σ_{ω_1} and σ_{ω_2} are also given in Spudich and Fletcher (2008).

Tilt, as defined in Spudich *et al.* (1995) and Spudich and Fletcher (2008), is $\sqrt{\omega_1^2 + \omega_2^2}$, rotation about a horizontal axis. There is no comparable quantity defined in Evans *et al.* (2009). Because tilt is a nonlinear combination of rotations, its variance cannot be determined using our simple technique. However, following Papoulis (1965, p. 195), if $\sigma_{\omega_1} = \sigma_{\omega_2} = \sigma_\omega$ and if the mean values of ω_1 and ω_2 are zero, then the standard deviation σ_t of tilt is $\sigma_t = \sigma_\omega\sqrt{2 - \pi/2}$. To be conservative we choose $\sigma_\omega = \max(\sigma_{\omega_1}, \sigma_{\omega_2})$. We have tested this expression for σ_t using numerical signals having known noise levels, and we have found that it is usually about 70% of the actual variance of the inferred tilt (i.e., it is a slight underestimate).

Data and Resources

No data were used in this article. A software package was developed for use with the MATLAB interactive environment produced by The MathWorks™ (www.mathworks.com).

Acknowledgments

We thank Fred Pollitz, W. H. K. Lee, John R. Evans, and an anonymous reviewer for their helpful comments.

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Manuscript received 11 August 2008