ICIAM'07

Minisymposium on

Multigrid Methods for Systems of PDEs

Motivation

The last two decades have seen a steady increase in the interest in so called multigrid and mutilevel approaches for the solution of linear systems of equations arising from the discretisation of models described by partial differential equations. This phenomenon is primarily based on two factors. On the one hand side there is a constant growth in size and complexity of such models. On the other side optimally tuned multigrid methods can solve the associated linear systems with an amount of work that scales linearly with the number of unknowns. Making them especially interesting not only for stationary linear problems, but also for non-linear and time-dependent ones. Thus, multilevel methods have long transcended the realm of symmetric and positive definite problems for which they were originally designed.

However, in order to reach optimal convergence rates the individual components of a multilevel algorithm must always be adapted to the problem at hand. This can either be done manually for a concrete application or by automatic techniques as in blackbox and algebraic multigrid approaches. Systems of PDEs pose their very own challenges in this respect. Be it that many such applications are formulated as saddle point problems and the resulting indefinite linear algebraic problems require the classical point smoothers to be replaced by coupled or distributive relaxation approaches to reduce high-frequency error components. Be it that new methods are required in order to identify coarse error components and to construct coarse 'grids' in algebraic approaches for systems. Be it that schemes for the construction of operator dependent grid transfers in the case of strongly varying or jumping coefficients must be adapted from the scalar to the system setting.

This minisymposium aims at bringing together researchers from the multilevel community and from application areas where these methods are applied to systems of PDEs in order to present and discuss new concepts and ongoing research in this field. The talks will cover a broad range of applications and approaches ranging from geometric methods for systems resulting from the Finite Element discretisation of the generalised Stokes problem of geophysical mantle convection over multigrid methods applied to the poro-elasticity problem used for Finite Differences on staggered and collocated grids up to new approaches for systems in algebraic multigrid methods and problems in the new application area of image processing and optical flow.

Talks

- **Multigrid methods for the poroelasticity system**
  Francisco Gaspar, Francisco Lisbona and Cornelis Oosterlee

  In this talk, we present robust and efficient multigrid solvers for the poroelasticity system. We introduce a reformulation of the problem which enables us to treat the system in a decoupled fashion. This permits us to choose a highly efficient multigrid method for a scalar Poisson type equation for the overall solution of this poroelasticity system. With standard geometric transfer operators, a direct coarse grid discretization and a point-wise red-black Gauss-Seidel smoother, an efficient multigrid method is developed for all relevant choices of the problem parameters. The transformation boils down to a stabilization term in the original formulation for which a highly efficient multigrid solver also can be developed. A relation between the multigrid treatment of poro-elasticity and the Stokes equations is also included in this talk.

- **Generation of Coarse Grid Approximations for Systems of PDEs with Discontinuous Coefficients**
  Marcus Mohr, Harald Köstler and Ulrich Rüde

  Multigrid methods are well-known for combining fast convergence speeds with low costs per individual iteration step when applied for the solution of linear systems arising from elliptic partial differential equations (PDEs). However, in order to reach the ultimate goal of linear $\mathcal{O}(n)$ efficiency for solving such problems the interplay of the individual components that make up a multigrid method must be optimal. In detail these components are smoother, grid coarsening strategy, inter-grid transfer operators and the coarse grid approximation of the fine grid operator.

  This talk considers the construction of the latter component for the special situation that the linear system under consideration stems from the discretisation of a system of PDEs and features strongly varying or discontinuous coefficients. This situation arises in many practical applications, e.g. optical flow reconstruction in image processing or simulation of mantle convection in geophysics. We report on different approaches that extend the classical techniques for the case of scalar-valued stencil coefficients to the case of tensor-valued entries and present examples from the two application fields mentioned above.

- **Automatic Coarse-Grid Approximation for Elliptic Systems**
  Irad Yavneh

  A new method for constructing coarse-grid operators for the multigrid solution of discretized elliptic equations and systems is described. The coarse-grid operators (and, optionally, the restriction operators) are designed automatically, but not in a Galerkin or Petrov-Galerkin framework. The components are chosen so as to ensure exact coarse-grid correction of all errors belonging to the subspace spanned by a predetermined set of basis functions. The resulting solvers are robust, and the freedom of choosing the stencils of the restriction and coarse-grid operator yields great flexibility and control of complexity. Joint work with Roman Wienands.

- **Coarse Grid Selection for Systems of PDEs Based on Compatible Relaxation**
  Robert Falgout

  Algebraic multigrid is an important method for solving the large sparse linear systems that arise in many PDE-based scientific simulation codes. A major component of algebraic multigrid methods is the selection of coarse grids.

  The notion of *compatible relaxation* (CR) was introduced by Brandt in [1] as a modified relaxation scheme that keeps the coarse-level variables invariant. Brandt states that the convergence rate of CR is a general measure for the quality of the set of coarse variables, and in [2], we developed a supporting theory for this idea. We have since developed an algebraic coarsening algorithm based on compatible relaxation that has several nice properties over the classical coarsening schemes. One such characteristic is its ability to ensure the quality of the coarse grid. However, our original method has the potential to overselect coarse-grid points when applied to PDE systems problems.

  In this talk, we will describe recent progress developing a systems version of our coarsening algorithm, point out its current strengths and weaknesses, and discuss open questions and future directions.

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References

[1] A. Brandt, *General highly accurate algebraic coarsening*, Electronic Transactions on Numerical Analysis, 10 (2000), pp. 1–20.

[2] R. D. Falgout and P. S. Vassilevski, *On generalizing the AMG framework*, SIAM J. Numer. Anal., 42 (2004), pp. 1669–1693. UCRL-JC-150807.